Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (4) 1990, S. 313–318

On the Global Solvability of the Diffusion Equation of Satsuma and Mimura

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Dedicated to Prof. Dr. L. Berg on the occasion of his 60th birthday

Es wird eine Vermutung von Satsuma und Mimura über blow-up-Effekte von Lösungen einer nichtlinearen Diffusionsgleichung bewiesen.

Доказывается гипотеза Сатсумы и Мимуры об эффектах обострения решений одного нелинейного диффузионного уравнения.

A conjecture of Satsuma and Mimura concerning blow-up effects of solutions of a nonlinear diffusion equation is proved.

1. Introduction. In recent papers M. MIMURA and J. SATSUMA [1–3] proposed a class of nonlinear diffusion equations with singular integral terms for describing certain diffusion processes with nonlocal aggregation effects. They developed an exact linearization method and showed the occurence of blowing up in finite time for certain examples. In this connection Satsuma and Mimura conjectured that there is a critical value I_0 of the total population I determining whether the solution exists globally for all times or blows up in finite time.

In [4] L. v. WOLFERSDORF reduced the equation of Satsuma and Mimura to partial differential equations in a complex domain and the general initial value problem to a Hammerstein integral equation. In this way he showed the existence of a solution in a sufficiently small time interval for an arbitrary Hölder continuous initial function. Further, in [5], L. v. WOLFERSDORF proved the first part of Satsuma's and Mimura's conjecture that the solution exists globally if $I \leqq I_0$. The second part of the conjecture, namely that for $I > I_0$ the solution always blows up, remained open. In the paper at hand we present a new geometric approach which fits for proving both parts of the conjecture from a unifying point of view.

2. Statement of problem and main result. The problem under study is to find a real-valued function $u: \mathbb{R} \times [0, T) \to \mathbb{R}$ $(0 < T \leqq \infty)$ which satisfies the equation

$$\partial u/\partial t - d\, \partial^2 u/\partial x^2 + \partial(uSu)/\partial x = 0 \qquad \text{on} \quad \mathbb{R} \times (0, T) \tag{1}$$

and the initial condition

$$u(x, 0) = f(x), \qquad x \in \mathbb{R}. \tag{2}$$

Here S denotes the singular integral operator of Cauchy type

$$Su(x, \cdot) = \frac{1}{\pi} \int_{-\infty}^{+\infty} u(\xi, \cdot) \frac{d\xi}{\xi - x}, \qquad x \in \mathbb{R}.$$

The positive constant d and the nonnegative Hölder continuous function f are assumed to be given. We suppose that

$$0 \leqq f(x) \leqq C\,|x|^{-\alpha} \tag{3}$$

for some constants $C > 0$ and $\alpha > 1$. Consequently, $I := \int_{\mathbb{R}} f(x)\,dx < \infty$. It was already shown in [5] that the problem is uniquely solvable for small T and under the assumption $I \leqq I_0 := 2\pi d$ for $T = +\infty$. The following result was conjectured by Satsuma and Mimura.

Theorem: *Let the function f satisfy the above assumptions and let $I := \int_{\mathbb{R}} f(x)\,dx$, $I_0 := 2\pi d$.*

(i) *If $I \leqq I_0$, then the problem* (1), (2) *is uniquely solvable for $T = +\infty$.*

(ii) *If $I > I_0$, then there exists a number $T_0 > 0$ such that the problem* (1), (2) *is uniquely solvable for $T < T_0$. The solution u satisfies* $\lim_{t \to T_0} \sup_{x \in \mathbb{R}} |u(x, t)| = \infty$.

3. Preparations. For proving the theorem we follow L. v. WOLFERSDORF [5] who introduced the new unknown function $w = u - iSu$ which can be extended analytically onto the upper half plane Π of the complex plane $\mathbb{C}$. In this way each solution u of (1) gives rise to a solution w of the equation

$$\partial w/\partial t - d\, \partial^2 w/\partial z^2 + \mathrm{i} w\, \partial w/\partial z = 0 \quad \text{on} \quad \Pi \times (0, T) \tag{4}$$

which satisfies the initial condition

$$w(z, 0) = \varphi(z) := \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{+\infty} f(x) \frac{dx}{x - z}, \qquad z \in \Pi, \tag{5}$$

and conversely, $u = \operatorname{Re} w$ solves (1), (2) if w satisfies (4), (5). The transformation

$$W(z, t) := \exp\left(\frac{1}{2\mathrm{i}\, d} \int_0^z w(\zeta, t)\, d\zeta\right) \tag{6}$$

gives a one-to-one correspondence between the solutions w of (4), (5) and the nonvanishing solutions W of the complex heat equation

$$\partial W/\partial t - d\, \partial^2 W/\partial z^2 = 0 \quad \text{on} \quad \Pi \times (0, T) \tag{7}$$

with the initial condition

$$W(z, 0) = \Phi(z) := \exp\left(\frac{1}{2\mathrm{i}\, d} \int_0^z \varphi(\zeta)\, d\zeta\right), \qquad z \in \Pi, \tag{8}$$

cf. [5]. Thus the solution u of (1), (2) exists as long as the solution W of the heat equation (7), (8) does not vanish on $\Pi \cup \mathbb{R}$. The moment T_0 at which the solution u (possibly) blows up is the first moment at which $W(z_0, T_0)$ becomes zero for some $z_0 \in \Pi \cup \mathbb{R}$. It was shown in [5, § 4] that z_0 must lie on the real axis. Therefore it suffices to consider equations (7) and (8) on $\mathbb{R} \times (0, T)$ only.

From (8) we obtain

$$|\Phi(x)| = \exp\left(-\frac{1}{2d}\int_0^x Sf(\xi)\,d\xi\right), \tag{9}$$

$$\arg\Phi(x) = -\frac{1}{2d}\int_0^x f(\xi)\,d\xi. \tag{10}$$

The function $\arg\Phi$ is a monotone decreasing bounded function and the limits $a_\pm$ of $\arg\Phi(x)$ as $x \to \pm\infty$ exist because of (3). We remark that $a_- - a_+ = I/2d$. The asymptotics of φ and Φ for large real arguments are given in [5]:

$$\varphi(x) \sim \mathrm{i}\frac{I}{\pi}\frac{1}{x} \quad \text{as } x \to \pm\infty, \tag{11}$$

$$\Phi(x) \sim x^{I/2\pi d}\exp\left[-\frac{\mathrm{i}}{2d}\int_0^{+\infty} f(\xi)\,d\xi - \frac{1}{2\pi d}\int_{-\infty}^{+\infty} f(\xi)\ln|\xi|\,d\xi\right] \quad \text{as } x \to +\infty, \tag{12}$$

$$\Phi(x) \sim (-x)^{I/2\pi d}\exp\left[\frac{\mathrm{i}}{2d}\int_{-\infty}^{0} f(\xi)\,d\xi - \frac{1}{2\pi d}\int_{-\infty}^{+\infty} f(\xi)\ln|\xi|\,d\xi\right] \quad \text{as } x \to -\infty. \tag{13}$$

In particular,

$$|\Phi(x)| \sim |x|^{I/2\pi d}\exp\left(-\frac{1}{2\pi d}\int_{-\infty}^{+\infty} f(\xi)\ln|\xi|\,d\xi\right) \quad \text{as } |x| \to \infty. \tag{14}$$

For the convenience of the reader we list some properties of the solution to the heat equation

$$\partial W/\partial t - d\,\partial^2 W/\partial x^2 = 0, \qquad (x, t) \in \mathbb{R} \times (0, \infty), \tag{15}$$

which satisfies an initial condition

$$W(x, 0) = \Psi(x), \quad x \in \mathbb{R}. \tag{16}$$

If Ψ is piecewise continuous and subject to the estimate $|\Psi(x)| \leqq C(1 + |x|^\beta)$ for some $\beta \in \mathbb{R}$, then (15), (16) is uniquely solvable in the class of functions with at most polynomial growth in x. This solution is given by

$$W(x, t) = \int_{-\infty}^{+\infty} G(\xi, t)\,\Psi(x - \xi)\,d\xi, \tag{17}$$

where G is the fundamental solutions of the heat equation:

$$G(x, t) = (4d\pi t)^{-1/2}\exp\left(-x^2/4dt\right). \tag{18}$$

Lemma 1: *Let Ψ be complex valued and let ε be a positive number.*

(i) *If $\Psi(x) = 0$ for all x with $|x| \geqq R$, then there exists a $t_0 > 0$ such that the solution W to* (15), (16) *satisfies the estimate* $\sup_{x\in\mathbb{R}} |W(x,t)| \leqq \varepsilon$ *for all* $(x,t) \in \mathbb{R} \times [t_0, \infty)$.

(ii) *If* $\Psi(x) = 0$ *for all* x *with* $x \geqq R$, *then, for each* $t_0 \in \mathbb{R}_+$, *there exists a number* x_0 *such that the solution* W *to* (15), (16) *satisfies the estimate* $\sup_{x \geqq x_0} |W(x, t)| \leqq \varepsilon$ *for all* $(x, t) \in [x_0, \infty) \times [0, t_0]$.

The proof immediately follows from the representation (17), (18) ∎

Lemma 2: *Let* $K \subseteqq \mathbb{C}$ *be a closed convex set.*

(i) *If* $\Psi(x) \in K$ *on* $\mathbb{R}$, *then* $W(x, t) \in K$ *on* $\mathbb{R} \times \mathbb{R}_+$.

(ii) *If, in addition,* Ψ *is continuous on* $\mathbb{R}$ *and if* $W(x_0, t_0) \in \partial K$ *for some* $x_0 \in \mathbb{R}$, $t_0 \in \mathbb{R}_+$, *then* $\Psi(x) \in \partial K$ *for all* $x \in \mathbb{R}$.

Proof: 1. The maximum principle yields the implication

$$\operatorname{Re} \Psi(x) \geqq 0 \quad \forall\, x \in \mathbb{R} \Rightarrow \operatorname{Re} W(x, t) \geqq 0 \quad \forall\, x \in \mathbb{R}, \quad t \in \mathbb{R}_+.$$

2. Let E be an arbitrary closed half plane in $\mathbb{C}$. Since the solutions of the heat equation with the initial functions $\Psi + c$ $(c \in \mathbb{C})$ and $e^{i\alpha}\Psi$ $(\alpha \in \mathbb{R})$ are $W + c$ and $e^{i\alpha}W$, respectively, the implication

$$\Psi(x) \in E \quad \forall\, x \in \mathbb{R} \Rightarrow W(x, t) \in E \quad \forall\, x \in \mathbb{R}, \quad t \in \mathbb{R}_+$$

follows from the first step of the proof. 3. By representing the closed convex set K by an intersection of closed half planes one obtains the assertion (i). 4. Taking into account the implication $(\Psi \in C(\mathbb{R}))$

$$\left.\begin{array}{l}\operatorname{Re} \Psi(x) \geqq 0 \ \forall\, x \in \mathbb{R} \\ \operatorname{Re} \Psi(x_0) > 0\end{array}\right\} \Rightarrow \operatorname{Re} W(x, t) > 0 \quad \forall\, x \in \mathbb{R}, \quad t \in \mathbb{R}_+,$$

which follows immediately from (17), (18), the second assertion (ii) can be proved by similar means ∎

Lemma 3: *Let* $K \subseteqq \mathbb{C}$ *be a closed convex set and let* ε *be a fixed positive number. Put* $K_\varepsilon := \left\{z \in \mathbb{C} : \inf_{x \in K} |x - z| < \varepsilon\right\}$.

(i) *If* $\Psi(x) \in K$ *for each* $x \in \mathbb{R}$ *with* $|x| \geqq R$, *then there exists a positive number* t_0 *such that* $W(x, t) \in K_\varepsilon$ *for all* $x, t \in \mathbb{R}$ *with* $t \geqq t_0$.

(ii) *If* $\Psi(x) \in K$ *for each* $x \in \mathbb{R}$ *with* $x \geqq R$, *then, for each positive number* t_0, *there exists a number* x_0 *such that* $W(x, t) \in K_\varepsilon$ *for all* $x, t \in \mathbb{R}$ *with* $0 \leqq t \leqq t_0$ *and* $x \geqq x_0$.

Proof: We define the functions

$$\Psi_1(x) = \begin{cases} \Psi(x) & \text{if } |x| \geqq R, \\ \Psi(R) & \text{if } |x| < R, \end{cases} \qquad \Psi_2(x) = \begin{cases} 0 & \text{if } |x| \geqq R, \\ \Psi(x) - \Psi(R) & \text{if } |x| < R. \end{cases}$$

The related solutions W_1 and W_2 of (15), (16) satisfy the relations (cf. Lemma 2/(i) and Lemma 1/(i)) $W_1(x, t) \in K$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and $\sup \{|W_2(x, t)| : x \in \mathbb{R}\} < \infty$ for all $t \geqq t_0$. The assertion (i) follows from $W = W_1 + W_2$. In a similar manner one can prove (ii) applying Lemma 1/(ii) ∎

4. Proof of the theorem. As has already been remarked in Section 3, it suffices to investigate whether the solution $W(x, t)$ of (7), (8) vanishes for some $x = x_0$, $t = T_0$ or not.

1. $I < 2\pi d$. Because of $a_+ \leqq \arg \Phi(x) \leqq a_-$ and $0 < a_- - a_+ < \pi$, as well as $\inf |\Phi(x)| > 0$, the curve $L = \{\Phi(x) : x \in \mathbb{R}\}$ completely lies in the closed half plane

$$K_\delta = \left\{z \in \mathbb{C} : \operatorname{Re}\left(z \exp\left(-i2^{-1}(a_+ + a_-)\right)\right) \geqq \delta\right\} \tag{19}$$

if the positive number δ is sufficiently small. By applying Lemma 2/(i) we see that the curve $L(t) = \{W(x, t): x \in \mathbb{R}\}$ lies in K_δ for each $t \in \mathbb{R}_+$. This means $W(x, t) \neq 0$ on $\mathbb{R} \times \mathbb{R}_+$.

2. $I = 2\pi d$. Since $\arg \Phi$ is a monotone function, the curve L is contained in the closed half plane K_0 (see (19)). Now let us assume that $0 \in L(t_0)$ for some $t_0 > 0$. Since the origin lies on the boundary of K_0, it follows from Lemma 2/(ii) that L is completely contained in ∂K_0. But this is impossible since $a_+ < \arg \Phi(x) < a_-$ for some $x \in \mathbb{R}$. Hence $W(x, t) \neq 0$ on $\mathbb{R} \times \mathbb{R}_+$.

3. $I > 2\pi d$. Let δ be a positive number. We define

$$K_\delta^\pm = \text{clos conv}\,\{z \in \mathbb{C}: |\arg z - a_\pm| \leqq \delta, |z| \geqq 1\}.$$

The asymptotic formulas (12), (13) give the existence of $R > 0$ such that $\Phi(x) \in K_\delta^+$ for all $x \geqq R$ and $\Phi(x) \in K_\delta^-$ for all $x \leqq -R$. Now we take into account that δ is an arbitrary positive number and apply Lemma 3/(ii) to see that for each given $\varepsilon, t_0 \in \mathbb{R}_+$ there exists a positive number x_0 so that, for $0 \leqq t \leqq t_0$,

$$W(x, t) \in \{z \in \mathbb{C}: |\arg z - a_+| < \varepsilon\} \qquad \text{if } x \geqq x_0, \tag{20}$$

$$W(x, t) \in \{z \in \mathbb{C}: |\arg z - a_-| < \varepsilon\} \qquad \text{if } x \leqq -x_0. \tag{21}$$

If the solution u would not blow up in finite time, then we could continuously extend the branch of arg used in (10) to define $\arg W(x, t)$ $(x \in \mathbb{R}, t \in \mathbb{R}_+)$, since in this case W would never vanish (W continuously depends on x, t). Because of (20) and (21) we would have

$$\lim_{x \to \pm\infty} \arg W(x, t) = a_\pm \qquad \text{for all } t \in \mathbb{R}_+. \tag{22}$$

In the following we show that (22) cannot hold for sufficiently large t if $I > 2\pi d$.

3.1. Let $I = 2(2k + 1)\pi d$ with $k \in \mathbb{N}$. By rotating the complex plane it can be achieved that $a_+ = 0$ and $a_- = (2k + 1)\pi$. The relations (8), (11) and (14) imply

$$\frac{d}{dx}|\Phi(x)| = \frac{1}{2d}|\Phi(x)| \operatorname{Im} \varphi(x) \geqq C|x|^{I/2\pi d - 1} \geqq C > 0 \qquad \text{if } |x| \geqq R.$$

For sufficiently large x we have $0 \leqq \arg \Phi \leqq \pi/3$ and therefore

$$\begin{aligned} d \operatorname{Re} \Phi(x)/dx &= \cos \arg \Phi(x)\, d|\Phi(x)|/dx \\ &\quad - |\Phi(x)| \sin \arg \Phi(x)\, d \arg \Phi(x)/dx \\ &\geqq 2^{-1} d|\Phi(x)|/dx \geqq C/2 > 0 \qquad \text{for all } x \geqq R_0. \end{aligned} \tag{23}$$

Analogously one can show that

$$d \operatorname{Re} \Phi(x)/dx \geqq C/2 > 0 \qquad \text{for all } x \leqq -R_0.$$

Since $\partial \operatorname{Re} W/\partial x$ is a solution of the heat equation with the initial function $d \operatorname{Re} \Phi/dx$, the existence of a positive number t_0 with $\partial \operatorname{Re} W(x, t)/\partial x \geqq C/4 > 0$ for all $x \in \mathbb{R}$, $t \geqq t_0$ follows from Lemma 1/(i) because of (23), (24). This estimation shows that $\operatorname{Re} W(\cdot, t)$ is a monotone function (for $t \geqq t_0$) which contradicts

$$\lim_{x \to -\infty} \arg W(x, t_0) - \lim_{x \to +\infty} \arg W(x, t_0) = a_- - a_+ \geqq 3\pi.$$

3.2. Let finally I satisfy $I > 2\pi d$, $I \notin \{2(2k + 1)\pi d : k \in \mathbb{N}\}$. We define

$$a_0 = \begin{cases} 2^{-1}(a_+ + a_-) & \text{if } \cos 2^{-1}(a_- - a_+) > 0, \\ 2^{-1}(a_+ + a_-) + \pi & \text{if } \cos 2^{-1}(a_- - a_+) < 0. \end{cases}$$

If x is sufficiently large, the values $\Phi(x)$ lie in the half plane

$$K = \{z \in \mathbb{C}: \operatorname{Re}(z \exp(-i\alpha_0)) \geqq 1\}.$$

Lemma 3/(i) implies the existence of a $t_0 > 0$ such that

$$\operatorname{Re}(W(x, t) \exp(-i\alpha_0)) \geqq 1/2 \qquad \text{for all} \quad x \in \mathbb{R}, \qquad t \geqq t_0.$$

This leads to the estimate

$$\sup_{x \in \mathbb{R}} \arg W(x, t) - \inf_{x \in \mathbb{R}} \arg W(x, t) \leqq \pi,$$

in contradiction to $\alpha_- - \alpha_+ > \pi$ together with (22). Hence the assumption $W(x, t) \neq 0$ on $\mathbb{R} \times \mathbb{R}_+$ is wrong ∎

Remarks: 1. Grow-up of u, i.e. $\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t)| = \infty$, can but must not necessarily occur if $I = 2\pi d$. 2. The idea of the proof is closely related to the invariance of the total population $\int_{-\infty}^{+\infty} u(x, t)\, dx$ used by Satsuma and Mimura in other context, namely

$$\frac{1}{2d} \int_{-\infty}^{\infty} u(x, t)\, dx = -\int_{-\infty}^{\infty} d \arg W(x, t) = \arg W(-\infty, t) - \arg W(+\infty, t).$$

3. Careful calculation of the constants in Lemmas 1—3 would also lead to estimates (from above) for the blow-up time.

Acknowledgement: The author is grateful to Prof. L. v. Wolfersdorf for bringing the problem to his attention and for many stimulating discussions about the subject.

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Manuskripteingang: 17. 04. 1989

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