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On Uniform Limitability and Banach Limits

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Dedicated to Prof. L. Berg on the occasion of his 60th birthday

Es wird gezeigt, daß eine beschränkte Zahlenfolge genau dann H_∞-limitierbar zum Wert a ist, wenn alle starken Banach-Limites dieser Zahlenfolge denselben Wert a zuordnen. Ein starker Banach-Limit ist dabei ein lineares Funktional mit bestimmten Eigenschaften im Raum M aller beschränkten Zahlenfolgen. Allgemeinere gleichmäßige Limitierungsverfahren lassen sich in ähnlicher Weise charakterisieren.

Показывается, что ограниченная числовая последовательность может лимитироваться H_∞-методом к значению a тогда и только тогда когда все сильные Банахы пределы приписывают этой последовательности то же самое значение a. Сильный Банахов предел является при этом линейным функционалом с некоторыми свойствами в пространстве M всех ограниченных числовых последовательностей. Более общие методы равномерного лимитирования могут быть охарактеризованы аналогичным образом.

A bounded sequence is shown to be H_∞-limitable to the value a if and only if all strong Banach limits assign the same value a to this sequence. In this connection, a strong Banach limit means a linear functional with certain properties on the space M of bounded sequences. More general uniform limitation methods can be characterized in a similar manner.

1. Introduction

The concept of almost convergence was introduced and studied by LORENTZ [7], cf. also ZELLER and BEEKMANN [14: p. 12], and STIEGLITZ [13] for a generalization. Almost convergence has found its most important application, perhaps, in the theory of uniform distribution of sequences where it leads to the concept of well-distribution, cf. KUIPERS and NIEDERREITER [6: p. 40], SCHATTE [9, 10], DRMOTA and TICHY [3].

Almost convergence can be regarded as a uniform version of the H_1-method (arithmetic means). It is no matrix method and must be defined by a two-dimensional limiting process. On the other hand, a bounded sequence $\{a_n\}$ is almost convergent to the value a if and only if certain linear functionals, so-called *Banach limits*, assign the same value a to this sequence.

Another limitation method being defined by a two-dimensional limiting process is the H_∞-method, cf. ZELLER and BEEKMANN [14: p. 11], SCHATTE [8]. The H_∞-method possesses also applications in probability theory, for a survey cf. SCHATTE [12]. H_∞-limitation can be regarded as a uniform version of limitation by logarithmic means. Because of these similarities with almost convergence, the question arises whether the H_∞-method can also be characterized by the fact that certain linear functionals assign the same value a to a given sequence. This question can be answered in the affirmative. The corresponding functionals will be called *strong Banach limits*.

Further the similarities between almost convergence and H_∞-limitability induce the trial to fill the gap between both methods. Thus the concept of uniform n^{-d}-limitability is considered as a special case of the uniform p_n-limitability, cf. SCHATTE [9–11], DRMOTA and TICHY [3]. The uniform n^{-d}-limitability can also be characterized by linear functionals which will be called *n^{-d}-Banach limits.*

2. Almost convergence

The starting-point for our considerations is the concept of almost convergence as introduced and characterized in LORENTZ [7].

Definition 1: A sequence $\alpha = \{a_n\}_{n=1}^{\infty}$ of real numbers is called *almost convergent to the value a* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} a_n = a \text{ uniformly in } k = 0, 1, 2, \ldots \tag{1}$$

Remark 1: Obviously (1) is equivalent to

$$\lim_{N\to\infty} \sup_{k=0,1,\ldots} \left| \frac{1}{N} \sum_{n=k+1}^{k+N} a_n - a \right| = 0, \tag{2}$$

and this is again equivalent to

$$\lim_{N\to\infty} \limsup_{k\to\infty} \left| \frac{1}{N} \sum_{n=k+1}^{k+N} a_n - a \right| = 0, \tag{3}$$

since (3) implies the boundedness of $\{a_n\}$.

Remark 2: On account of (3) almost convergence is equivalent to the method H^* (verkürzte arithmetische Mittel) considered in SCHATTE [8]. From Satz 7 of that place we obtain the following assertion: *If the Fourier series of an integrable function is almost convergent, then it is even convergent.* This justifies further the denotation "almost" convergent.

We consider the set M of all bounded sequences $\alpha = \{a_n\}_{n=1}^{\infty}$. Then M is a Banach space if the linear operations are defined termwise and if the norm $\|\alpha\|$ is defined by $\|\alpha\| = \sup |a_n|$. We shall write $\alpha \geqq 0$ if $a_n \geqq 0$ for $n = 1, 2, \ldots$ Further we denote by S the shift operator, $S\alpha = \{a_{n+1}\}$.

Definition 2: A linear functional f on M is called a *Banach limit* if

(i) $f(\{1, 1, \ldots\}) = 1$,

(ii) $f(\alpha) \geqq 0$ for $\alpha \geqq 0$,

(iii) $f(S\alpha) = f(\alpha)$.

The existence of such functionals was shown by Banach and Mazur, cf. ZELLER und BEEKMANN [14: p. 12]. If the sequence α converges to a, then $f(\alpha) = a$ for every Banach limit f by (i) to (iii). But we can say much more.

Theorem A: *The sequence $\alpha \in M$ is almost convergent to the value a if and only if $f(\alpha) = a$ holds for every Banach limit f in M.*

This interesting characterization of almost convergence was used by LORENTZ [7] for the definition of almost convergence. Then the characterization in Definition 1 appears as a theorem which is proved in LORENTZ [7].

3. H_∞-limitability

In what follows we shall characterize the H_∞-limitability analogously to almost convergence by a class of linear functionals.

Definition 3: A sequence $\{a_n\}$ of real numbers is called *H_∞-limitable to the value* a if

$$\lim_{k\to\infty} \limsup_{n\to\infty} |H_k(a_n) - a| = 0, \tag{4}$$

where

$$H_0(a_n) = a_n, \qquad H_{k+1}(a_n) = \frac{1}{n} \sum_{j=1}^{n} H_k(a_j), \qquad k \geqq 0.$$

The H_∞-limitability of a bounded sequence $\{a_n\}$ can be defined analogously as almost convergence by replacing the arithmetic means by logarithmic means. The following theorem is identical with Satz 3 in SCHATTE [8].

Theorem B: *The sequence $\alpha \in M$ is H_∞-limitable to the value a if and only if*

$$\lim_{N\to\infty} \limsup_{k\to\infty} \left| \frac{1}{\log N} \sum_{n=k+1}^{kN} \frac{a_n}{n} - a \right| = 0. \tag{5}$$

In comparison to condition (4), the condition (5) can be analyzed more easily by asymptotic methods as represented in, e.g., BERG [1]. We intend to reduce H_∞-limitability to almost convergence. To this end we define $\beta = B\alpha = \{b_n\}$ by

$$b_n = \frac{1}{\log 2} \sum_{j=2^{n-1}}^{2^n-1} \frac{a_j}{j}, \qquad n = 1, 2, \ldots \tag{6}$$

If the given sequence $\alpha = \{a_n\}$ is bounded, then $B\alpha$ is bounded, too.

Corollary 1: *The sequence $\alpha \in M$ is H_∞-limitable to the value a if and only if the sequence $\beta = B\alpha$ is almost convergent to the value a.*

Proof: In (5) we can restrict ourselves to $k = 2^K$, $N = 2^r$ with integers K, r. Then we have

$$\left| \frac{1}{\log N} \sum_{n=k}^{kN-1} \frac{a_n}{n} - a \right| = \left| \frac{1}{r} \sum_{n=K+1}^{K+r} b_n - a \right| \blacksquare$$

Inspired by COHEN [2], we introduce in the Banach space M of bounded sequences $\alpha = \{a_n\}$ the operator $T\alpha = \gamma = \{c_n\}$, where $c_n = (a_{2n-1} + a_{2n})/2$. Then $2T\alpha = \{a_1 + a_2, a_3 + a_4, a_5 + a_6, \ldots\}$. We obtain

$$T^k\alpha = \left\{ 2^{-k} \sum_{j=(n-1)2^k+1}^{n2^k} a_j \right\}. \tag{7}$$

Definition 4: A linear functional g on M is called a *strong Banach limit* if

(i) $g(\{1, 1, \ldots\}) = 1$,

(ii) $g(\alpha) \geqq 0$ for $B\alpha \geqq 0$,

(iii) $g(T\alpha) = g(\alpha)$.

Remark 3: Let the sequence α be almost convergent to 0. Then from (7) we obtain $\|T^k\alpha\| \leqq \varepsilon$ for sufficiently large k and consequently $|g(\alpha)| = |g(T^k\alpha)| \leqq \varepsilon$ according to (ii), (i). It follows that $g(\alpha) = 0$ for each strong Banach limit g. Especially, *if the sequence α is convergent to 0, then $g(\alpha) = 0$ for each strong Banach limit g.*

Remark 4: Let $\alpha \in M$ and $\gamma = \alpha - S\alpha$. Then γ is almost convergent to 0 and $g(\gamma) = 0$, $g(S\alpha) = g(\alpha)$. Thus each strong Banach limit is a Banach limit, but not conversely.

Remark 5: Strong Banach limits g can be constructed in the following way. Take a Banach limit f according to Definition 2, assign to $\alpha = \{a_n\}$ a sequence $\beta = B\alpha = \{b_n\}$ according to (6) and put $g(\alpha) = f(\beta)$. Then (i) is clear, since for $a_n = 1$ we obtain $b_n = 1 + \varepsilon_n$, $\varepsilon_n \to 0$ and $f(\{b_n\}) = f(\{1, 1, \ldots\}) + f(\{\varepsilon_n\}) = 1$. The condition (ii) is trivially satisfied by condition (ii) of Definition 2. In order to show (iii) we insert $T\alpha$ in (6) and obtain $BT\alpha = \{b_n^*\}$ with

$$b_n^* = \frac{1}{\log 2} \sum_{j=2^{n-1}}^{2^n-1} (a_{2j-1} + a_{2j})/2j = b_{n+1} + \varepsilon_n,$$

$$g(T\alpha) = f(\{b_n^*\}) = f(\{b_{n+1}\}) + f(\{\varepsilon_n\}) = f(\{b_n\}) = g(\alpha).$$

If conversely a strong Banach limit g with (i)–(iii) is given, we can construct a Banach limit f by $f(\beta) = g(\alpha)$. This definition is independent of α, since $\beta = B\alpha_1 = B\alpha_2$ implies $g(\alpha_1 - \alpha_2) \geqq 0$ and $g(\alpha_1 - \alpha_2) \leqq 0$ by (ii) and therefore $g(\alpha_1) = g(\alpha_2)$. The conditions (i)–(ii) of Definition 2 are clearly satisfied. Since $S\beta = \{b_{n+1}\} = BT\alpha + B\gamma$ where γ converges to 0, the condition (iii) of Definition 2 is also satisfied in view of Remark 3.

Theorem 1: *The sequence $\alpha \in M$ is H_∞-limitable to the value a if and only if $g(\alpha) = a$ holds for every strong Banach limit g in M.*

Proof: By Corollary 1 the H_∞-limitability of the sequence α to the value a is equivalent to the almost convergence of the sequence $\beta = B\alpha$ to the value a. By Theorem A this is equivalent to $f(\beta) = a$ for every Banach limit f, and by the construction in Remark 5 this is equivalent to $g(\alpha) = a$ for every strong Banach limit g ∎

Example 1: We consider

$$\alpha = \{1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ldots\}$$

and obtain

$$TS\alpha = \{0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, \ldots\}.$$

By Remark 4 and (iii) we have $g(TS\alpha) = g(\alpha)$. On the other hand, $g(\alpha) + g(TS\alpha) = 1$ by (i). Consequently $g(\alpha) = 1/2$ for every strong Banach limit g, and the sequence α is H_∞-limitable to the value 1/2. This follows also immediately from Theorem B or Corollary 1 since $\beta = B\alpha = \{1, 0, 1, 0, 1, \ldots\} + \{\varepsilon_n\}$.

Remark 6: The condition (ii) is equivalent to the two conditions

(ii*) $\quad g(\alpha) \geqq 0 \quad$ for $\quad \alpha \geqq 0$,

(iv) $\quad g(\alpha) = 0 \quad$ for $\quad \beta = B\alpha = 0$.

Obviously (ii) implies (ii*) and (iv). Let conversely $B\alpha \geqq 0$. Then there is an α^* such that $\alpha^* \geqq 0$ and $B\alpha = B\alpha^*$. It follows $g(\alpha) = g(\alpha^*)$ from (iv) and $g(\alpha) \geqq 0$ from (ii*). The condition (ii) cannot be relaxed to the condition (ii*) alone. We give an example.

Example 2: Let $\alpha = \{a_n\}$ be given. We introduce

$$x_n = 2^{-n+1} \sum_{j=3\cdot 2^{n-1}+1}^{2^{n+1}} a_j$$

and put $g(\alpha) = f(\{x_n\})$ for some Banach limit f. Then g is a linear functional with (i), (ii*), and (iii). Now we can choose a sequence α^* such that $x_n = 1$ and therefore $g(\alpha^*) = 1$ but $B\alpha^* = 0$.

Remark 7: Replace in Definition 4 the operators B and T by B^* and T^*, respectively, which are given by

$$B^*\{a_n\} = \{b_n^*\}, \qquad b_n^* = \frac{1}{\log x} \sum_{j=[x^n]+1}^{[x^{n+1}]} \frac{a_j}{j}$$

and

$$T^*\{a_n\} = \{c_n^*\}, \qquad c_n^* = \frac{1}{x} \sum_{j=[xn]+1}^{[x(n+1)]} a_j,$$

where $x > 1$ is an arbitrary fixed real number. Then we obtain modified strong Banach limits g^*. As it is easy to see, all preceding assertions can also be proved for modified strong Banach limits g^*. Even *each strong Banach limit g is also a modified strong Banach limit g^* and conversely.* Namely, let $B^*\alpha = 0$. This means that the sequence α is H_∞-limitable to the value 0 which can be shown analogously as Corollary 1'. Therefore $g(\alpha) = 0$ for every strong Banach limit g by Theorem 1. Moreover we consider $\gamma = \alpha - T^*\alpha$. Then $B^*\gamma$ is almost convergent to 0 and consequently γ is H_∞-limitable to 0. It follows $g(\gamma) = 0$, $g(\alpha) = g(T^*\alpha)$ for every strong Banach limit g. Therefore looking to Remark 6 we see that every strong Banach limit is also a modified strong Banach limit. The converse can be shown in the same way.

Remark 8: We see furthermore that the condition (iii) in Definition 4 can be replaced by

(iii*) $\qquad g(\{a_1, a_2, \ldots\}) = g(\{a_0, a_1, a_1, a_2, a_2, \ldots\}),$

where a_0 is an arbitrary parameter. Namely, assume that γ is almost convergent to 0 and put

$$\beta_k = \{b_k, b_{k+1}, b_{k+1}, b_{k+2}, b_{k+2}, b_{k+2}, b_{k+2}, b_{k+3}, \ldots\},$$

where β_1 is defined by $B\beta_1 = B\gamma$. Then $g(\gamma) = g(\beta_1)$ by (ii) and $g(\beta_1) = g(\beta_k)$ by (iii*). But $|g(\beta_k)| \leqq \varepsilon$ for sufficiently large k, and therefore $g(\gamma) = 0$. Now $g(T\alpha) = g(\eta)$, where $2\eta = \{2a_0, a_1 + a_2, a_1 + a_2, a_3 + a_4, a_3 + a_4, \ldots\}$ according to (iii*). Since $\gamma = \eta - \alpha$ is almost convergent to 0, we have $g(\eta) = g(\alpha)$.

4. Uniform n^{-d}-limitability

In this section we wish to fill the gap between almost convergence and H_∞-limitability by a class of limitation methods.

Definition 5: A sequence $\{a_n\}$ of real numbers is called *uniformly n^{-d}-limitable to the value a*, $0 < d < 1$, if

$$\lim_{N\to\infty} \limsup_{k\to\infty} \left| N^{d-1} \sum_{n=k+1}^{L} a_n n^{-d} - a \right| = 0, \tag{8}$$

where $L = L(k, N)$ is the largest integer such that $\sum_{n=k+1}^{L} n^{-d} \leqq N^{1-d}$.

Remark 9: As in Definition 1 and in Theorem B, respectively, in Definition 5 for $\alpha \in M$ the $\limsup_{k\to\infty}$ can be replaced by the $\sup_{k=0,1,\ldots}$.

Remark 10: Let $0 \leqq c < d \leqq 1$ and let $\alpha \in M$ be uniformly n^{-c}-limitable. Then α is also uniformly n^{-d}-limitable, cf. SCHATTE ([9: Theorem 5] or [11: Lemma 1]). On the other hand, there are sequences α being uniformly n^{-d}-limitable but not uniformly n^{-c}-limitable, cf. the considerations following equality (29) in SCHATTE [9]. In this connection, uniformly n^{-0}-limitable means almost convergent and uniformly n^{-1}-limitable means H_∞-limitable.

We again intend to reduce uniform n^{-d}-limitability to almost convergence. To this end we introduce $d_n = [n^{1/(1-d)}]$ and $\beta_d = B_d\alpha = \{b_n\}$ by

$$b_n = (1 - d) \sum_{j=d_n}^{d_{n+1}-1} a_j j^{-d}. \tag{9}$$

Corollary 2: *The sequence α is uniformly n^{-d}-limitable to the value a if and only if the sequence $\beta_d = B_d\alpha$ is almost convergent to the value a.*

Proof: In (8) we can restrict ourselves to $k = d_K$, $N = d_r$ with integers K, r. Then we have

$$\left| N^{d-1} \sum_{n=k+1}^{L} a_n n^{-d} - a \right| = \left| \frac{1}{r} \sum_{n=K+1}^{K+r} b_n - a \right| + \gamma_{K,r}$$

where $\lim\limits_{r\to\infty} \limsup\limits_{K\to\infty} |\gamma_{K,r}| = 0$ ∎

Corollary 3: *The uniform n^{-d}-limitation is no matrix method.*

We introduce the transformation $T_d\alpha = \gamma_d' = \{c_n\}$. For $d_n \leqq j < d_{n+1}$ we set $m(j) = j - d_n + d_{n+1}$ and $c_j = a_{m(j)}$. For instance

$$T_{1/2}\alpha = \{a_4, a_5, a_6, a_9, \ldots, a_{13}, a_{16}, \ldots, a_{22}, a_{25}, \ldots, a_{33}, a_{36}, \ldots\}.$$

Definition 6: A linear functional g_d on M is called an *n^{-d}-Banach limit* if

(i) $g_d(\{1, 1, \ldots\}) = 1$,

(ii) $g_d(\alpha) \geqq 0$ for $B_d\alpha \geqq 0$,

(iii) $g_d(T_d\alpha) = g_d(\alpha)$.

Remark 11: If the sequence α is convergent to 0, then $\|T_d^k\alpha\| \leqq \varepsilon$ for sufficiently large k and consequently $g_d(\alpha) = 0$ for each n^{-d}-Banach limit.

Remark 12: To each Banach limit f an n^{-d}-Banach limit g_d can be assigned by $g_d(\alpha) = f(B_d\alpha)$ and conversely, analogously as in Remark 5. To this end, Remark 3 must be replaced by Remark 11.

Theorem 2: *The sequence $\alpha \in M$ is uniformly n^{-d}-limitable to the value a if and only if $g_d(\alpha) = a$ holds for every n^{-d}-Banach limit g_d in M* ∎

Example 3: Let $d = 1/2$, $\alpha = \{a_n\}$, where

$$a_n = \begin{cases} 1 \text{ if } (3k+1)^2 \leqq n < (3k+3)^2 \\ 0 \text{ if } (3k+3)^2 \leqq n < (3k+4)^2 \end{cases} \quad (k = 0, 1, 2, \ldots).$$

Then $\alpha + T_{1/2}\alpha + T_{1/2}^2\alpha = \{2, 2, \ldots\}$, and α is uniformly $n^{-1/2}$-limitable to the value 2/3. This can also be seen from Corollary 2 since

$$\beta_{1/2} = B_{1/2}\alpha = \{1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots\} + \{\varepsilon_n\}.$$

Remark 13: Let $0 \leqq c < d \leqq 1$, $\alpha \in M$, and g_d any n^{-d}-Banach limit. Then $B_c\alpha = 0$ means that the sequence α is uniformly n^{-c}-limitable to the value 0. But then α is also uniformly n^{-d}-limitable to 0 according to Remark 10. It follows $g_d(\alpha) = 0$. Moreover we consider $\gamma = \alpha - T_c\alpha$. Then $B_c\gamma$ is almost convergent to 0 and consequently γ is uniformly n^{-c}-limitable to 0 according to Corollary 2. But then γ

is also n^{-d}-limitable to 0 according to Remark 10, and it follows $g_d(\gamma) = 0$, $g_d(\alpha) = g_d(T_c\alpha)$. Thus we have seen that *each n^{-d}-Banach limit g_d is also an n^{-c}-Banach limit*. The converse is not true on account of Theorem 2 and since there are sequences α being n^{-d}-limitable but not n^{-c}-limitable, cf. Remark 10.

Clearly, an n^{-0}-Banach limit means a Banach limit and an n^{-1}-Banach limit means a strong Banach limit. Note that $T_0 = S$ and B_0 is the identity, but T_1 and B_1 are not defined. Thus it causes some difficulties to introduce more general p_n-Banach limits.

5. More on strong Banach limits

Finally we give yet another characterization of strong Banach limits for which a counterpart for n^{-d}-Banach limits is not known. Let

$$H\alpha = \{H_1(a_n)\} = \left\{\frac{1}{n}\sum_{j=1}^{n} a_j\right\}.$$

Theorem 3: *The linear functional g on M is a strong Banach limit if and only if*

(i) $g(\{1, 1, \ldots\}) = 1$,

(ii) $g(\alpha) \geqq 0$ *for* $\alpha \geqq 0$,

(iii) $g(S\alpha) = g(\alpha)$,

(iv) $g(H\alpha) = g(\alpha)$.

Proof: 1. Let the conditions of Definition 4 be satisfied. Then, according to Remark 4, the condition (iii) of Theorem 3 is also satisfied. In order to prove (iv) we write

$$\sum_{j=1}^{n} H_1(a_j)/j = \sum_{j=1}^{n} a_j \sum_{i=j}^{n} i^{-2} = \sum_{j=1}^{n} a_j/j + O(1)$$

as $n \to \infty$. It follows that $B(H\alpha - \alpha)$ is almost convergent to 0 and consequently $H\alpha - \alpha$ is H_∞-limitable to 0 according to Corollary 1, hence $g(H\alpha) = g(\alpha)$ on account of Theorem 1.

2. Let the conditions of Theorem 3 be satisfied. Assume further that α is H_∞-limitable to 0. Then $\|S^jH^k\alpha\| \leqq \varepsilon$ for sufficiently large k and j by Definition 3. Hence $|g(\alpha)| = |g(S^jH^k\alpha)| \leqq \varepsilon$ according to (iii), (iv) and therefore $g(\alpha) = 0$. Put $\gamma = \alpha - T\alpha$. Then γ is H_∞-limitable to 0 on account of Theorem B and $g(\alpha) = g(T\alpha)$. Let further $\beta = B\alpha = 0$. Then α is H_∞-limitable to 0 and $g(\alpha) = 0$. Now we can apply Remark 6 ∎

Corollary 4: *The sequence $\alpha \in M$ is H_∞-limitable to the value a if and only if $g(\alpha) = a$ holds for every linear functional g on M which satisfies the conditions* (i)–(iv) *of Theorem* 3.

Corollary 4 can also be concluded easily from results by Duran [4: Section 6]. But these results rely on a theorem due to Eberlein [5] a proof of which was not found in the literature.

Remark 14: The example $g(\{a_n\}) = a_1$ shows that the condition (iii) in Theorem 3 is indispensable. But the condition (iii) can be replaced by the weaker requirement that $g(\alpha) = 0$ if α converges to 0. This is possible because $H(\alpha - S\alpha)$ converges to 0 for $\alpha \in M$ and therefore $g(\alpha - S\alpha) = g\big(H(\alpha - S\alpha)\big) = 0$ by (iv). The condition (ii) in Theorem 3 is also indispensable, choose $g(\{a_n\}) = 3f(\{b_{2n-1}\}) - 2f(\{b_{2n}\})$, where $\{b_n\} = B\{a_n\}$, and where f is a Banach limit.

REFERENCES

[1] BERG, L.: Asymptotische Darstellungen und Entwicklungen. Berlin: Dt. Verlag Wiss. 1968.
[2] COHEN, D. I. A.: An explanation of the first digit phenomenon. J. Comb. Theory 20 (1976), 367–370.
[3] DRMOTA, M., and R. F. TICHY: A note on Riesz means and well-distribution. Math. Nachr. 139 (1988), 267–279.
[4] DURAN, J. P.: Almost convergence, summability and ergodicity. Can. J. Math. 26 (1974), 372–385.
[5] EBERLEIN, W. F.: On Hölder summability of infinite order. Notices Amer. Math. Soc. 19 (1972), A 164, Abstract # 691-46-21.
[6] KUIPERS, L., and H. NIEDERREITER: Uniform distribution of sequences. New York: J. Wiley & Sons 1974.
[7] LORENTZ, G. G.: A contribution to the theory of divergent sequences. Acta Math. 80 (1948), 167–190.
[8] SCHATTE, P.: Ein Kriterium für die H_∞-Limitierbarkeit. Math. Nachr. 64 (1974), 63–70.
[9] SCHATTE, P.: On the discrepancy of well-distributed sequences. Math. Nachr. 136 (1988), 267–280.
[10] SCHATTE, P.: On well-distribution with respect to weighted means. Math. Nachr. 138 (1988), 55–60.
[11] SCHATTE, P.: On uniform versions of the strong law of large numbers. Math. Nachr. 142 (1989), 149–166.
[12] SCHATTE, P.: On mantissa distributions in computing and Benford's law. J. Inf. Process. Cybern. EIK 24 (1988), 443–455.
[13] STIEGLITZ, M.: Eine Verallgemeinerung des Begriffes der Fastkonvergenz. Math. Japon. 18 (1973), 53–70.
[14] ZELLER, K., und W. BEEKMANN: Theorie der Limitierungsverfahren. Berlin: Springer-Verlag 1970.

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