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Convergence Statements for Projection Type Linear Iterative Methods with Relaxations

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Um Lösungen oder verallgemeinerte Lösungen der linearen Operatorgleichung $Ax = b$ zu bestimmen, werden lineare Iterationsverfahren der Gestalt $x_{n+1} = (I - D_n A)\, x_n + D_n b$ betrachtet, bei denen die Operatoren $T_n = I - D_n A$ außerhalb ihrer Fixpunktmenge normreduzierend sind. Es werden Bedingungen angegeben, die die Konvergenz der Verfahren und bestimmte Eigenschaften der Grenzelemente sichern.

Чтобы находить решения или обобщённые решения линейного операторного уравнения $Ax = b$, рассматриваются итерационные методы $x_{n+1} = (I - D_n A)\, x_n + D_n b$, у которых операторы $T_n = I - D_n A$ сокращают норму своих подвижных элементов. Задаются условия, которые обеспечивают сходимость методов и определённые свойства предельных элементов.

To determine solutions or generalized solutions of the linear operator equation $Ax = b$ linear iterative methods of the form $x_{n+1} = (I - D_n)\, x_n + D_n b$ are considered for which the operators $T_n = I - D_n A$ are norm-reducing outside their fixed point set. Conditions are given which guarantee the convergence of the methods and certain properties of the limit elements.

1. Generalized solutions of linear operator equations

In this section we state and compare three concepts for generalized solutions of linear operator equations which are well known from the literature. We modify and complete the concepts in such a way that the close relations between them are especially accentuated.

Let X and Y be Hilbert spaces. We consider linear continuous operators $A \in \mathbf{L}(X, Y)$ from X into Y with the null space $\mathfrak{N}(A)$ and the range $\mathfrak{R}(A)$. The corresponding linear operator equation reads

$$Ax = b, \qquad x \in X, \quad b \in Y. \tag{1.1}$$

The first notion is of topological nature. It establishes a connection to approximation and to extremal problems.

Definition 1.1 (cf. [8, p. 40], [3, p. 221]): The element $\hat{v}$ in X is called an *extremal solution* (a virtual solution, a least-squares solution) of (1.1) if $\|A\hat{v} - b\| \leq \|Ax - b\|$ holds for all $x \in X$. An extremal solution v of minimal norm is said to be a *normal extremal* (least extremal, best approximate) *solution* of (1.1).

Extremal solutions of (1.1) exist iff b belongs to the algebraic direct sum $\mathfrak{R}(A) + \mathfrak{R}(A^*)$, where A^* denotes the adjoint of A. Under this condition there is one and only one normal extremal solution v of (1.1). It is the uniquely determined extremal solution $\hat{v}$ of (1.1) in $\overline{\mathfrak{R}(A^*)}$, which can be written in the form $v = A^+ b$. Thereby A^+ is the orthogonal generalized inverse of A.

The second notion is derived from the theory of generalized inverses.

Definition 1.2 (cf. [20], [12, p. 21]): Let the topological direct sum decompositions

$$X = \mathfrak{N}(A) \oplus \mathfrak{M}, \quad Y = \overline{\mathfrak{R}(A)} \oplus \mathfrak{N} \tag{1.2}$$

of X and Y be given. Let P and Q be the corresponding projectors determined by $\mathfrak{N}(P) = \mathfrak{N}(A)$, $\mathfrak{R}(P) = \mathfrak{M}$, $\mathfrak{R}(Q) = \overline{\mathfrak{R}(A)}$, $\mathfrak{N}(Q) = \mathfrak{N}$. A solution v of the (projectional) equation

$$Ax = Qb \tag{1.3}$$

is called a *Q-generalized solution* of (1.1). If v additionally fulfils the equation

$$Px = 0, \tag{1.4}$$

then v is said to be a *P-normal Q-generalized solution* of (1.1).

Q-generalized solutions of (1.1) exist iff b belongs to $\mathfrak{R}(A) \dotplus \mathfrak{N}$. Under this condition there is one and only one P-normal Q-generalized solution of (1.1). It is the uniquely determined Q-generalized solution of (1.1) in $\mathfrak{M}$. It can be written as $v = A^-_{P,Q} b$, where $A^- = A^-_{P,Q}$ is the generalized inverse of A with respect to the pair (P, Q) of projectors P, Q satisfying the relations $A^-A = I - P$, $AA^- \mid \mathfrak{D}(A^-) = Q \mid \mathfrak{D}(A^-)$. Here $\mathfrak{D}(A^-) = \mathfrak{R}(A) \dotplus \mathfrak{N}$ denotes the domain of A^- (see, e.g., [8, pp. 33–34]).

The third notion results from the theory of iterative methods. It is properly speaking of algebraic nature.

Definition 1.3 (cf. [6, p. 32], [11, p. 104], [13, p. 209]): Let $\mathfrak{Z}$ be a set of Hilbert spaces Z and $\mathfrak{S}$ be a set of operators $G \in \mathbf{L}(Y, Z)$ with $Z = Z(G) \in \mathfrak{Z}$. A solution v of the system

$$GAx = Gb, \quad G \in \mathfrak{S}, \tag{1.5}$$

of linear operator equations is called a *$\mathfrak{S}$-generalized solution* of (1.1). Let $\mathfrak{Z}'$ be a further set of Hilbert spaces Z' and $\mathfrak{H}$ be a corresponding set of operators $H \in \mathbf{L}(X, Z')$ with $Z' = Z'(H) \in \mathfrak{Z}'$. If v additionally to (1.5) fulfils the system

$$Hx = 0, \quad H \in \mathfrak{H}, \tag{1.6}$$

of homogenuous linear operator equations, then v is said to be a *$\mathfrak{H}$-normal $\mathfrak{S}$-generalized solution* of (1.1). Besides, we shall briefly speak of *normal (generalized) solutions* if the (generalized) solutions have minimal norm (see also Definition 1.1).

If $X(A, b)$ denotes the set of solutions and $X'(A, b)$ the set of generalized solutions of (1.1) (in the sense of one of the mentioned definitions), then the relation $X'(A, b) \supseteq X(A, b)$ is satisfied. For the first two definitions there holds, moreover, $X'(A, b) = X(A, b)$ if $X(A, b) \neq \emptyset$. Obviously $X(A, b)$ and $X'(A, b)$ are closed convex sets. Thus the normal (generalized) solutions are uniquely determined.

If we choose $\mathfrak{M} = \overline{\mathfrak{R}(A^*)} = \mathfrak{N}(A)^\perp$, $\mathfrak{N} = \mathfrak{N}(A^*) = \overline{\mathfrak{R}(A)}^\perp$ in (1.2), then the corresponding projectors P, Q become orthogonal. Therefore each (normal) extremal solution of (1.1) with the above P, Q is also a (P-normal) Q-generalized solution of (1.1). Reversely, each (P-normal) Q-generalized solution of (1.1) with arbitrary P, Q can be interpreted as a (normal) extremal solution of (1.1) with respect to suitable modified scalar products in X and Y (see, e.g., [6, p. 56]).

Evidently the equations $Ax = Qb$ and $QAx = Qb$ are equivalent. Thus each (P-normal) Q-generalized solution of (1.1) in the sense of Definition 1.2 is also a ($\{P\}$-normal) $\{Q\}$-generalized solution of (1.1) in the sense of Definition 1.3.

The discussion shows that the generality of the given concepts for generalized solutions increases from definition to definition.

2. Relaxations

This section contains some notions and results from [15, 16] which are used in the following sections.

Let H be a Hilbert space. We consider operators $T \in \mathrm{L}(H)$ and introduce a special kind of nonexpansive linear operators.

Definition 2.1: T is called a *relaxation* if the condition $\|Tx\| < \|x\|$ holds for all $x \notin \mathfrak{N}(I - T)$.

Theorem 2.2: *$T \in \mathrm{L}(H)$ is a relaxation iff the relations*

$$T\mathfrak{N}(I - T)^{\perp} \subseteqq \mathfrak{N}(I - T)^{\perp}, \qquad \|Tx\| < \|x\| \ \forall\, x \in \mathfrak{N}(I - T)^{\perp} \setminus \{0\}$$

are satisfied.

Definition 2.3: T is called a *strong relaxation* if the relations $T\mathfrak{N}(I - T)^{\perp} \subseteqq \mathfrak{N}(I - T)^{\perp}$, $\|T \mid \mathfrak{N}(I - T)^{\perp}\| < 1$ *are fulfilled.*

By Theorem 2.2 each strong relaxation is also a relaxation. But there are relaxations which are not strong (see [16]).

Definition 2.4: Let T be a relaxation. Then the number $\nu = \nu(T) = \|T \mid \mathfrak{N}(I - T)^{\perp}\|$ is called the *relaxation degree* of T.

The relaxation degree $\nu(T)$ lies between 0 and 1. It is less than 1 iff T is a strong relaxation.

Definition 2.5 (see [5]): T is called *norm-attaining* if there exists an element u in $H \setminus \{0\}$ satisfying $\|Tu\| = \|T\|\,\|u\|$.

Lemma 2.6: *If the restriction $T \mid \mathfrak{N}(I - T)^{\perp}$ of a relaxation T is norm-attaining, then T is a strong relaxation.*

Compact operators T are norm-attaining (see [12, p. 15]). Therefore compact and especially finite-dimensional relaxations T are always strong relaxations.

Since a relaxation T is completely reduced by the pair $(\mathfrak{N}(I - T), \mathfrak{N}(I - T)^{\perp})$ of orthogonal subspaces (for the notion see [18, p. 268]), we can connect T with the orthoprojector T' determined by $\mathfrak{N}(T') = \mathfrak{N}(I - T)$. In this sense T is said to be a *relaxation of the* corresponding *orthoprojector* T'. Therefore a relaxation T of T' can be characterized by the relations

$$Tx = x \quad \text{for} \quad x \in \mathfrak{N}(T'), \qquad \|Tx\| < \|x\| \quad \text{for} \quad x \notin \mathfrak{N}(T').$$

If T is moreover a strong relaxation, the relations read

$$T \mid \mathfrak{N}(T') = I \mid \mathfrak{N}(T'), \qquad \|T \mid \mathfrak{N}(T')\| < 1.$$

Theorem 2.7: *T is a relaxation of T' iff*

a) $T' = T'T = TT'$, $\|Tx - T'x\| < \|x\|$ *for* $x \neq 0$

holds. T is a strong relaxation of T' iff

b) $T' = T'T = TT'$, $\|T - T'\| < 1$

holds. Condition a) *supplies* $\Re(T') = \Re(I - T)$, $\Re(T') = \overline{\Re(I - T)}$. *Under condition* b) *it results* $\Re(T') = \Re(I - T)$.

If T is a relaxation of T', then the relaxation degree $\nu(T)$ can be expressed by $\nu(T) = \|T \mid \Re(T')\| = \|T - T'\|$ (see Definition 2.4). Contractive operators T (i.e. $\|T\| < 1$) are strong relaxations of 0 with $\nu(T) = \|T\|$. Orthoprojectors $T \neq I$ are (degenerate) strong relaxations of themselves with $\nu(T) = 0$.

Definition 2.8: Let $T' \neq I$ be an orthoprojector. Then $T = (1 - \lambda) I + \lambda T'$, $|1 - \lambda| < 1$, is called a *scalar relaxation* of T' with the (relaxation) *parameter* λ.

A scalar relaxation T of T' is a strong relaxation of T' with $\nu(T) = |1 - \lambda|$.

Theorem 2.9: *Let T_i be relaxations of the orthoprojectors T_i' for $i = 1, \ldots, k$. Then the product $T = T_k \ldots T_1$ is a relaxation of the orthoprojector T' determined by* $\Re(T') = \bigcap \{\Re(T_i') : i = 1, \ldots, k\}$.

3. Product sequences of relaxations

In this section we list some notions and an important result from [17].

Definition 3.1: The strictly monotone increasing mapping $(k(n))$ from the set $\mathbf{N}$ of natural numbers into itself is called *partition sequence* if the conditions $k(0) = 0$, $\sup \{k(n + 1) - k(n) : n \in \mathbf{N}\} < \infty$ are satisfied.

Definition 3.2 (cf. [1, p. 448], [14, pp. 92–93]): A sequence (t_n) is called *almost cyclic* if there are a finite selection set $\mathcal{T} = \{t_{n(1)}, \ldots, t_{n(l)}\}$ of (t_n) and an integer $m > 0$ such that

$$\mathcal{T} = \{t_n, \ldots, t_{n+m-1}\} \quad \text{for all } n. \tag{3.1}$$

Let (t_n) be almost cyclic. The selection set $\mathcal{T}$ of (t_n) with the property (3.1) for some m is said to be the *basic set* of (t_n). There exist partition sequences $(k(n))$ for (t_n) fulfilling the condition $\mathcal{T} = \{t_{k(n)}, \ldots, t_{k(n+1)-1}\}$ for all n, where $\mathcal{T}$ denotes the basic set of (t_n). If the numbers $k(n)$ are chosen minimal, we call the corresponding partition sequence $(k(n))$ the *characteristic sequence* of (t_n). Evidently, cyclic sequences (t_n) are almost cyclic.

Now let H be a Hilbert space. We consider sequences (T_n) of operators $T_n \in \mathbf{L}(H)$ and the derived product sequences (P_n) with

$$P_n = T_n \ldots T_1 T_0. \tag{3.2}$$

At first we formulate conditions ensuring the convergence of (P_n).

Assumption 3.3:

a) (T_n') is an almost cyclic sequence of orthoprojectors with the basic set $\{T'_{n(1)}, \ldots, T'_{n(l)}\}$ and the characteristic sequence $(k(n))$.

b) $\dim \operatorname{span} \bigcup_{j=1}^{l} \Re(T'_{n(j)}) < \infty$ (span: closed linear hull).

c) T_n are relaxations of T_n' for all n.

d) $\sum_{n=0}^{\infty} \mu^{(n)}$ is divergent, where

$$\mu^{(n)} = \min \{\mu_i : i = k(n), \ldots, k(n + 1) - 1\},$$

$$\mu_n = 1 - \nu(T_n) = 1 - \|T_n - T_n'\|.$$

Remark 3.3′: **1.** The conditions a), c) and d′) $\overline{\lim}\,\|T_n - T_n'\| < 1$ supply the condition d). **2.** For scalar relaxations $T_n' = (1 - \lambda_n) I + \lambda_n T_n'$ we have $\mu_n = 1 - |1 - \lambda_n|$ (see Definition 2.8 and the statement after it). If the parameters λ_n are supposed to be real, this means $\mu_n = \min\{\lambda_n, 2 - \lambda_n\}$. In this case $\sum \mu_n$ is divergent iff $\sum \lambda_n(2 - \lambda_n)$ is divergent. The latter condition plays an important role in the convergence theory of SOR methods (see e.g. [21]). **3.** Because of condition d) infinitely many T_n must be strong relaxations.

Assumption 3.4:

a′) (T_n) is an almost cyclic sequence of relaxations with the basic set $\{T_{n(1)}, \ldots, T_{n(l)}\}$ and the characteristic sequence $\big(k(n)\big)$,

b′) $T^{(n)} = T_{k(n+1)-1} \cdots T_{k(n)}$ is norm-attaining on $\mathfrak{N} = \operatorname{span}\{\cup\, \mathfrak{N}(I - T_{n(j)}) : j = 1, \ldots, l\}$ for all n.

Remark 3.4′: **1.** If, as usual, we denote the corresponding orthoprojectors to T_n by T_n', then in view of Theorem 2.7 (cf. Assumption 3.3/b)) the set $\mathfrak{N}$ in b′) can be written in the form $\mathfrak{N} = \operatorname{span}\{\cup\, \mathfrak{N}(T'_{n(j)}) : j = 1, \ldots, l\}$. **2.** The above conditions are partly more special and partly more general than those of Assumption 3.3. On the one side the use of only finitely many different relaxations is a strong restriction. On the other side the strong condition b) can be weakend to b′). The latter is for instance satisfied if one of the relaxations $T_{n(j)}$ $(j = 1, \ldots, l)$ is compact on $\mathfrak{N}$. Since $T^{(n)}$ contains this relaxation as a factor, $T^{(n)}$ is compact on $\mathfrak{N}$, too, and therefore norm-attaining on $\mathfrak{N}$ (see [12, p. 15]).

Theorem 3.5: *Let Assumption* 3.3 *or Assumption* 3.4 *be fulfilled. Then the product sequence* (P_n) *from* (3.2) *converges uniformly to the orthoprojector* P *determined by*

$$\mathfrak{N}(P) = \bigcap_{j=1}^{l} \mathfrak{N}(T'_{n(j)}) = \bigcap_{n} \mathfrak{N}(I - T_n'), \qquad \mathfrak{N}(P) = \operatorname{span} \bigcup_{j=1}^{l} \mathfrak{N}(T'_{n(j)}) = \operatorname{span} \bigcup_{n} \mathfrak{N}(I - T_n'). \tag{3.3}$$

Remark 3.5′: **1.** If Assumption 3.3 holds with exception of b), then (P_n) still accumulates to P pointwise in the weak topology. **2.** In view of Theorem 2.7 the infinite intersections and unions in (3.3) can be reduced to finite intersections and unions containing the operators $T_{n(j)}$ $(j = 1, \ldots, l)$. **3.** Let $T^{(n)} = T_{k(n+1)-1} \cdots T_{k(n)}$ be the partial products induced by the corresponding characteristic sequence $\big(k(n)\big)$ occuring in Assumption 3.3 and Assumption 3.4, respectively. The proof of Theorem 3.5 shows that there are certain constants $\nu^{(n)} < 1$ with

$$\|T^{(n)} \mid \mathfrak{N}\| \leqq \nu^{(n)}. \tag{3.4}$$

If we define

$$m = m(n) = \max\{i \in \mathbf{N} : k(i) - 1 \leqq n\}, \tag{3.5}$$

then we get the error estimate

$$\|P_n - P\| \leqq \nu^{(m-1)} \cdots \nu^{(1)} \nu^{(0)}. \tag{3.6}$$

4. Iterative methods with relaxations as iteration operators

Let X and Y be Hilbert spaces. We consider a linear operator equation

$$Ax = b \qquad \big(A \in \mathbf{L}(X, Y), b \in Y\big). \tag{4.1}$$

Using a sequence (D_n) of operators $D_n \in \mathbf{L}(Y, X)$ we can generate a linear instationary iterative one-step process

$$x_{n+1} = T_n x_n + D_n b = P_n x_0 + B_n b, \tag{4.2}$$

where

$$T_n = I - D_n A, \qquad P_n = T_n \dots T_1 T_0, \qquad B_n = \sum_{i=0}^{n} T_n \dots T_{i+1} D_i.$$

We are interested in conditions ensuring that the limit elements $x_\infty = \lim x_n$ of the iterative process (4.2) exist and represent solutions or generalized solutions of the equation (4.1) (see Section 1). For global convergence results we need the existence of the limit operators $P_\infty = \lim P_n$ and $B_\infty = \lim B_n$ (see (4.2)).

Let (D_n') be such a sequence of operators $D_n' \in \mathbf{L}(Y, X)$ that the corresponding operators $T_n' = I - D_n' A$ are orthoprojectors. If we choose $D_n = D_n'$ and $T_n = T_n'$ in (4.2), respectively, well-known *iterative methods of the projection type* arise (for special cases see, e.g., [6, 7, 10, 11]).

By including a sequence (λ_n) of relaxation parameters λ_n we can influence the speed of convergence or the precision of the results. Thus the choice $D_n = \lambda_n D_n'$ supplies so-called (scalar) *relaxation methods*. The corresponding iteration operators

$$T_n = I - \lambda_n D_n' A = (1 - \lambda_n) I + \lambda_n T_n'$$

are scalar relaxations in the sense of Definition 2.8. Such relaxation methods are investigated, e.g., in [12–14]. A further generalization is obtained if the scalar relaxations T_n are replaced by (operator) relaxations in the sense of Definition 2.1. The basis of a convergence theory for such general relaxation methods is in parts already contained in [12]. First explicit results are given in [4] for cyclic methods in finite-dimensional spaces. The results of [2] can also be arranged in our framework.

Definition 4.1: The iterative method (4.2) is said to be *consistent* if it possesses fixed points, that are elements x' in X satisfying $x' = T_n x' + D_n b$ for all n.

Obviously the method (4.2) is consistent iff the corresponding equation (4.1) has $\{D_n\}$-generalized solutions (see Definition 1.3). Namely, each fixed point of (4.2) is a $\{D_n\}$-generalized solution of (4.1) and vice versa. The method (4.2) is automatically consistent if the equation (4.1) is solvable. A consistent iterative method (4.2) can be written in the form

$$x_{n+1} = P_n x_0 + (I - P_n) x' = x' + P_n(x_0 - x'), \tag{4.3}$$

where x' is any $\{D_n\}$-generalized solution of (4.1).

5. About the convergence of consistent iterative relaxation methods and the properties of the limits elements

We consider an operator equation (4.1) and an iterative relaxation method (4.2), where the iteration operators T_n or at least the corresponding orthoprojectors T_n' constitute an almost cyclic sequence (see Assumption 3.4 and Assumption 3.3, respectively).

Theorem 5.1: *Let Assumption 3.3 or Assumption 3.4 be fulfilled. Besides we additionally integrate some of the following assumptions:*

a) (4.2) *is consistent for the given* $b \in Y$.

b) (4.2) *is consistent for all* $b \in Y$.

c) (4.1) *is solvable.*

d) *For* $b = 0$ *each* $\{D_n\}$*-generalized solution of* (4.1) *is also a solution of* (4.1).

Then the following statements hold:

1. *(P_n) converges uniformly to the orthoprojector $P_\infty = P$ determined by the relations*

$$\mathfrak{N}(P) = \bigcap_{j=1}^{l} \mathfrak{N}(T'_{n(j)}) = \bigcap_{n} \mathfrak{N}(D_n A);$$

$$\mathfrak{R}(P) = \operatorname{span} \bigcup_{j=1}^{l} \mathfrak{R}(T'_{n(j)}) = \operatorname{span} \bigcup_{n} \mathfrak{R}(D_n A).$$

2. *The additional assumption* a) *supplies*: *(x_n) converges to*

$$x_\infty = Px_0 + (I - P)\, x' = x' + P(x_0 - x'), \tag{5.1}$$

where x' is any $\{D_n\}$-generalized solution of (4.1). *Furthermore, x_∞ itself is a $\{D_n\}$-generalized solution of* (4.1) *with $Px_\infty = Px_0$ and the normal $\{D_n\}$-generalized solution of* (4.1) *for $x_0 \in \mathfrak{N}(P)$.*

3. *The additional assumption* b) *supplies: The further relation $\mathfrak{N}(P) = \operatorname{span}\{\bigcup \mathfrak{N}(D_n) : n \in \mathbf{N}\}$ is fulfilled. Moreover, (B_n) converges strongly to a left-orthogonal outer inverse $B_\infty = B$ of A. The limit element x_∞ has the representation*

$$x_\infty = Px_0 + Bb. \tag{5.2}$$

4. *The additional assumption* d) *supplies: P is the orthogonal projection onto $\mathfrak{N}(A)$ and can be expressed by*

$$P = I - A^+A, \tag{5.3}$$

where A^+ is the orthogonal generalized inverse of A. This implies $\mathfrak{N}(P) = \mathfrak{N}(A)$, $\mathfrak{R}(P) = \overline{\mathfrak{N}(A^*)}$.

5. *The additional assumptions* b) *and* d) *supply: B is a left-orthogonal generalized inverse of A. Furthermore, x_∞ is an AB-generalized solution of* (4.1) *with $(I - A^+A)\, x_\infty = (I - A^+A)\, x_0$ and the normal AB-generalized solution Bb of* (4.1) *for $x_0 \in \overline{\mathfrak{N}(A^*)}$.*

6. *The additional assumptions* c) *and* d) *supply: x_∞ is a solution of* (4.1) *with $(I - A^+A)\, x_\infty = (I - A^+A)\, x_0$ and the normal solution A^+b of* (4.1) *for $x_0 \in \overline{\mathfrak{N}(A^*)}$.*

Proof: **1.** The first statement is an immediate consequence of Theorem 3.5 if the relation $T_n = I - D_nA$ is observed.

2. Assumption a) guarantees that the method (4.2) can be written in the form (4.3). Then statement 1 implies the convergence of (x_n) and the representation (5.1) of x_∞. This representation leads to

$$D_nAx_\infty = D_nAPx_0 + D_nA(I - P)\, x' = D_nAx' = D_nb$$

for all n since $\mathfrak{N}(P) = \bigcap \{\mathfrak{N}(D_nA) : n \in \mathbf{N}\}$ by statement 1. Starting again with (5.1), it follows that

$$Px_\infty = P^2x_0 + P(I - P)\, x' = Px_0. \tag{5.4}$$

For $x_0 \in \mathfrak{N}(P)$ we obtain $x_\infty = (I - P)\, x'$ from (5.1). Observing the orthogonality of Px_0 and $(I - P)\, x'$, the Pythagorean relation implies

$$\|Px_0 + (I - P)\, x'\|^2 = \|Px_0\|^2 + \|(I - P)\, x'\|^2 \geqq \|(I - P)\, x'\|^2.$$

Since all $\{D_n\}$-generalized solutions x' of (4.1) are fixed points and therefore limit elements of (4.2), all assertions of statement 2 are shown.

3. By assumption b) the relation $D_n b \in \mathfrak{R}(D_n A)$ is satisfied for all n and all $b \in Y$. This means $\mathfrak{R}(D_n) \subseteq \mathfrak{R}(D_n A)$ for all n. As $\mathfrak{R}(D_n A) \subseteq \mathfrak{R}(D_n)$ holds automatically for all n, we get the equation $\mathfrak{R}(D_n A) = \mathfrak{R}(D_n)$ for all n. Therefore, it follows that $\mathfrak{R}(P) = \operatorname{span} \bigcup \{\mathfrak{R}(D_n A) : n \in \mathbf{N}\} = \operatorname{span} \bigcup \{\mathfrak{R}(D_n) : n \in \mathbf{N}\}$ if statement 1 is taken into account. Comparing (4.2) and (4.3) we obtain $B_n b = (I - P_n) x'$, $x' = x'(b)$. Thus $(B_n b)$ converges to $(I - P) x'(b)$ for all $b \in Y$. Let B be the limit operator of (B_n) in the sense of the strong operator topology. Using the relation $P_n = I - B_n A$, we get $BA = I - P = (I - P)^* = (BA)^*$. It is easy to show the identity $B_m = \sum_{i=0}^{m} D_i (I - A D_{i-1}) \dots (I - A D_0)$. Therefore, $B_m b \in \operatorname{span} \bigcup \{\mathfrak{R}(D_n) : n \in \mathbf{N}\} = \mathfrak{R}(P)$ for all m and all $b \in Y$. Thus we find $P B_m = 0$ for all m and $PB = 0$. Finally, $BAB = (I - P) B = B - PB = B$. Hence B is a left-orthogonal outer inverse of A (for the notion see e.g. [8]). If we pass in (4.2) to the limit, we obtain (5.2).

4. Assumption d) means $\bigcap \{\mathfrak{R}(D_n A) : n \in \mathbf{N}\} \subseteq \mathfrak{R}(A)$. Because of $\mathfrak{R}(A) \subseteq \bigcap \{\mathfrak{R}(D_n A) : n \in \mathbf{N}\} = \mathfrak{R}(P)$ it is $\mathfrak{R}(P) = \mathfrak{R}(A)$. Therefore P is the orthoprojector onto $\mathfrak{R}(A)$ along $\overline{\mathfrak{R}(A^*)}$ which can be represented in the form (5.3). □

5. By statement 3 the operator B is a left-orthogonal outer inverse of A. Statement 4 shows $ABA = A(I - P) = A - AP = A$, so that B is an inner inverse and consequently even a generalized inverse of A. In view of $AP = 0$ we get from (5.2) $Ax_\infty = APx_0 + ABb = ABb$, where AB is a projector onto $\mathfrak{R}(A)$. Thus x_∞ is an AB-generalized solution of (4.1). The relations (5.3) and (5.4) can be combined to $(I - A^+A) x_\infty = (I - A^+A) x_0$. The general solution set of $Ax = ABb$ has the form $X(A, ABb) = \mathfrak{R}(A) + x_\infty$. For $x_0 \in \overline{\mathfrak{R}(A^*)}$ it is also $x_\infty \in \overline{\mathfrak{R}(A^*)}$ if we take (5.1) and $\overline{\mathfrak{R}(A^*)} = \mathfrak{R}(P) = \mathfrak{R}(I - P)$ into consideration. Therefore x_∞ is the normal AB-generalized solution of (4.1) in this case. The representation $x_\infty = Bb$ results from (5.2).

6. Assumption c) implies assumption a). Under assumption c) therefore (5.1) is satisfied with any solution x' of (4.1). Besides, (5.4) holds. In view of d) we get (5.3) and $AP = 0$. Hence we find $Ax_\infty = APx_0 + A(I - P) x' = Ax' = b$ and $(I - A^+A) \times x_\infty = (I - A^+A) x_0$. The general solution set of $Ax = b$ has the form $X(A, b) = \mathfrak{R}(A) + x_\infty$. Analogously as in the proof of statement 5 we can show that x_∞ is the normal solution of (4.1) for $x_0 \in \overline{\mathfrak{R}(A^*)}$. Therefore x_∞ can be written as $A^+ b$ ■

Remark 5.2: **1.** The assumptions b) and d) imply that AB is a continuous projector onto $\mathfrak{R}(A)$. Hence $\mathfrak{R}(A)$ and $\mathfrak{R}(A^*)$ are closed. Thus we can replace $\overline{\mathfrak{R}(A^*)}$ by $\mathfrak{R}(A^*)$ in the statements 5 and 6 of Theorem 5.1. **2.** Under the assumptions for statement 2, statement 5 and statement 6 the set of all limit elements $x_\infty = x_\infty(x_0)$ coincides with the set of all $\{D_n\}$-generalized solutions, AB-generalized solutions and solutions of (4.1), respectively. Thus all solutions or generalized solutions of (4.1) in the above-mentioned sense can be reached by the selection of suitable starting elements x_0. **3.** Under the assumptions of Theorem 5.1 the limit elements x_∞ of (4.2) are in general no extremal solutions. But they can be interpreted as extremal solutions with respect to a suitable scalar product in Y if the assumptions of statement 5 are fulfilled (see Section 1). **4.** Using (4.3), (5.1) and (3.6) we get the error estimate

$$\|x_\infty - x_{n+1}\| = \|(P - P_n)(x_0 - x')\| \leqq \nu^{(m-1)} \dots \nu^{(0)} \|x_0 - x'\|,$$

where $\nu^{(n)}$ and m are determined by (3.4) and (3.5). **5.** If Assumption 3.3 is fulfilled with exception of condition b), then the statements of Theorem 5.1 hold in some respect still in the sense of weak topology (see Remark 3.5′).

6. About the convergence of cyclic iterative relaxation methods and the properties of the limit elements

At first we consider an equation (4.1) and a cyclic iterative method (4.2) belonging to it. We suppose that the generating sequence (D_n) has the cyclic length L. Then there are at most L different operators $D_0, D_1, \ldots, D_{L-1}$ in the sequence (D_n). A linear stationary iterative one-step process arises if in each case L single iteration steps are united to one step. The resulting method has the form

$$x^{n+1} = Tx^n + Db = T^{n+1}x^0 + B^{(n)}b \tag{6.1}$$

with

$$x^0 = x_0, \qquad D = \sum_{j=0}^{L-1} T_{L-1} \ldots T_{j+1} D_j,$$
$$T = I - DA = T_{L-1} \ldots T_1 T_0, \qquad B^{(n)} = \sum_{i=0}^{n} T^i D. \tag{6.2}$$

Now we turn to relaxation methods. Thus (T_n) is a cyclic sequence of relaxations T_n containing at most L different relaxations $T_0, T_1, \ldots, T_{L-1}$ of corresponding orthoprojectors $T_0', T_1', \ldots, T_{L-1}'$.

Theorem 6.1: *Let T be norm-attaining on $\mathfrak{N} = \operatorname{span} \bigcup \{\mathfrak{N}(T_j') : j = 0, \ldots, L-1\}$. Besides, some of the following assumptions are additionally integrated:*

a) (6.1) *is consistent for the given $b \in Y$.*

b) (6.1) *is consistent for all $b \in Y$.*

c) (4.1) *is solvable.*

d) *For $b = 0$ each $\{D\}$-generalized solution of* (4.1) *is also a solution of* (4.1).

Then the listed statements hold:

1. *Both (P_n) and (T^n) converge uniformly to the orthoprojector P determined by the relations*

$$\mathfrak{N}(P) = \bigcap_{j=0}^{L-1} \mathfrak{N}(T_j') = \bigcap_{j=0}^{L-1} \mathfrak{N}(D_j A) = \mathfrak{N}(DA),$$
$$\mathfrak{N}(P) = \operatorname{span} \bigcup_{j=0}^{L-1} \mathfrak{N}(T_j') = \operatorname{span} \bigcup_{j=0}^{L-1} \mathfrak{N}(D_j A) = \mathfrak{N}(DA). \tag{6.3}$$

2. *The additional assumption* a) *supplies: (x^n) converges to*

$$x^\infty = Px^0 + (I - P)\, x' = x' + P(x^0 - x'), \tag{6.4}$$

where x' is any $\{D\}$-generalized solution of (4.1). *Furthermore x^∞ itself is a $\{D\}$-generalized solution of* (4.1) *with $Px^\infty = Px^0$ and the normal $\{D\}$-generalized solution of* (4.1) *for $x^0 \in \mathfrak{N}(DA)$.*

3. *The additional assumption* b) *supplies: The further relation $\mathfrak{N}(P) = \mathfrak{N}(D)$ is fulfilled. Moreover $(B^{(n)})$ converges uniformly to a left-orthogonal outer inverse $B = (DA + P)^{-1} D$ of A. The limit element x^∞ has the representation*

$$x^\infty = Px^0 + Bb. \tag{6.5}$$

4. *The additional assumption* d) *supplies: P is the orthogonal projection onto $\mathfrak{N}(A)$ along $\overline{\mathfrak{N}(A^*)}$ and can be expressed in the form $P = I - A^+A$.*

5. *The additional assumptions* b) *and* d) *supply: B is a left-orthogonal generalized inverse of A and can be written as $B = (I - TA^+A)^{-1} D$. Furthermore x^∞ is an AB-generalized solution of* (4.1) *with* $(I - A^+A)\, x^\infty = (I - A^+A)\, x^0$ *and the normal AB-generalized solution Bb of* (4.1) *for* $x^0 \in \overline{\mathfrak{R}(A^*)}$.

6. *The additional assumptions* c) *and* d) *supply: x^∞ is a solution of* (4.1) *with* $(I - A^+A)\, x^\infty = (I - A^+A)\, x^0$ *and the normal solution A^+b of* (4.1) *for* $x^0 \in \overline{\mathfrak{R}(A^*)}$.

Proof: The sequence (T_n) of relaxations is almost cyclic with the basic set $\{T_0, T_1, \ldots, T_{L-1}\}$ and the characteristic sequence (nL): Thus $T^{(n)} = T_{(n+1)L-1} \ldots T_{nL} = T$ for all n. Therefore (T_n) satisfies the conditions of Assumption 3.4.

1. By Theorem 5.1 the sequence (P_n) converges uniformly to the orthoprojector P determined by the relations

$$\mathfrak{R}(P) = \bigcap_{j=0}^{L-1} \mathfrak{R}(T_j') = \bigcap_n \mathfrak{R}(D_nA) = \bigcap_{j=0}^{L-1} \mathfrak{R}(D_jA),$$

$$\mathfrak{R}(P) = \operatorname{span} \bigcup_{j=0}^{L-1} \mathfrak{R}(T_j') = \operatorname{span} \bigcup_n \mathfrak{R}(D_nA) = \operatorname{span} \bigcup_{j=0}^{L-1} \mathfrak{R}(D_jA).$$

(T^n) converges uniformly to P, too, since it is a partial sequence of (P_n). By Theorem 2.9 the product $T = T_{L-1} \ldots T_0$ is a relaxation of P. T is supposed to be norm-attaining on $\mathfrak{R} = \mathfrak{R}(P)$. Hence T is even a strong relaxation in view of Lemma 2.6. Thus the relations $\|T \mid \mathfrak{R}\| < 1$ and $\mathfrak{R}(P) = \mathfrak{R}(I - T) = \mathfrak{R}(DA)$, $\mathfrak{R}(P) = \mathfrak{R}(I - T) = \mathfrak{R}(DA)$ hold by virtue of Theorem 2.7 and (6.2).

The constant sequence (T) also fulfils the conditions of Assumption 3.4. Hence most of the remaining assertions are simple consequences of Theorem 5.1. We restrict ourselves to the few other assertions.

3. We have $\mathfrak{R}(DA) \subseteq \mathfrak{R}(D)$. Assumption b) leads to $\mathfrak{R}(D) \subseteq \mathfrak{R}(DA)$. That means $\mathfrak{R}(D) = \mathfrak{R}(DA) = \mathfrak{R}(P)$. Statement 3 of Theorem 5.1 shows the strong convergence of $(B^{(n)})$ to $B^{(\infty)} = B$. But if you take the relations

$$P = TP = PT, \qquad (T - P)^n = \big(T(I - P)\big)^n = T^n(I - P),$$

$$\|T - P\| = \|T(I - P)\| = \|T \mid \mathfrak{R}(P)\| < 1$$

into account,

$$B = \sum_{i=0}^{\infty} T^i D = \sum_{i=0}^{\infty} T^i (I - P)\, D = \sum_{i=0}^{\infty} (T - P)^i\, D$$

$$= (I - T + P)^{-1} D = (DA + P)^{-1} D$$

holds uniformly.

5. Under the assumptions b) and d) we get $I - T + P = I - T(I - P) = I - TA^+A$ in view of statement 4 and $B = (I - TA^+A)^{-1} D$ in view of statement 3. All remaining assertions are clear if Theorem 5.1 is observed ∎

Remark 6.2: $\overline{\mathfrak{R}(A^*)}$ can be substituted by $\mathfrak{R}(A^*)$ in the statements 5 and 6 of Theorem 6.1 (see Remark 5.2/1.).

Remark 6.3: Condition a) in Theorem 6.1 can be equivalently replaced by one of the conditions $\mathfrak{R}(DA) = \mathfrak{R}(D)$ or $\mathfrak{R}(D) \subseteq \mathfrak{R}$. Condition d) in Theorem 6.1 is equivalent to $\mathfrak{R}(DA) = \mathfrak{R}(A)$. Hence some of the statements follow also by results contained in [9].

Remark 6.4: It is possible to strengthen the assumptions of Theorem 6.1 in such a way that they do not depend on the order in which the operators D_j $(j = 0, \ldots, L-1)$ occur. The modified assumptions read:

i) T_j is compact on $\mathfrak{N}$ $(j = 0, \ldots, L-1)$.

a') $D_j b \in \mathfrak{N}$ $(j = 0, \ldots, L-1)$.

b') span $\bigcup \{\mathfrak{N}(D_j): j = 0, \ldots, L-1\} \subseteq \mathfrak{N}$.

c') (4.1) is solvable.

d') $\bigcap \{\mathfrak{N}(D_j A): j = 0, \ldots, L-1\} = \mathfrak{N}(A)$.

Under similar assumptions some results of Theorem 6.1 are obtained in [4] for finite-dimensional spaces. It is easy to see that the assumptions of Theorem 6.1 are replaceable by the corresponding modified assumptions without changing the statements.

Let $\pi = (\pi(0), \ldots, \pi(L-1))$ be a permutation of $(0, \ldots, L-1)$. We consider an iterative method

$$x_\pi^{n+1} = T_\pi x^n + D_\pi b = T_\pi^{n+1} x^0 + B_\pi^{(n)} b$$

which differs from (6.1) by a permuted order of the generating operators D_j $(j = 0, \ldots, L-1)$. Under the assumptions i) and b') the iterative sequence (x_π^n) converges for an arbitrary π to

$$x_\pi^\infty = P_\pi x^0 + (I - P_\pi) x_\pi' = P_\pi x^0 + B_\pi b.$$

An analysis shows that $P_\pi = P$ does not depend on π while in contrast x_π' and B_π in general depend on π. But if we add the assumptions c') and d') to i), the dependence vanishes totally.

Remark 6.5: Let the conditions i) and a') of Remark 6.4 be fulfilled. Then not only the sequence (x^n) with $x^n(x_0) = x_{nL}(x_0)$ but also the other cyclic partial sequences $(x_{nL+i}(x_0))$ $(i = 1, \ldots, L-1)$ converge. Let $(x^n(x_0, \pi))$ denote the cyclically united iterative method with the permuted order π of the generating operators D_n, that is

$$D_{\pi(0)}, D_{\pi(1)}, \ldots, D_{\pi(L-1)}, D_{\pi(0)}, D_{\pi(1)}, \ldots$$

Furthermore let π_i be the order arising from the original order by a right shift of i positions in the cyclic sequence, that is

$$D_i, D_{i+1}, \ldots, D_{L+i-1}, D_i, D_{i+1}, \ldots$$

Evidently we obtain for the cyclic partial sequences $x_{nL+i}(x_0) = x^n(x_i, \pi_i)$ $(i = 0, \ldots, L-1)$. The limits are $\bar{x}_i = \bar{x}_i(x_0) = x^\infty(x_i, \pi_i)$ $(i = 0, \ldots, L-1)$. The sequence $(x_{nL+i}(\bar{x}_0))$ is cyclic with the property $x_{nL+i}(\bar{x}_0) = \bar{x}_i(\bar{x}_0)$. Besides, $Px_n(x_0) = Px_0$ for all n in view of $Px_{n+1} = PT_n x_n + PD_n b = Px_n$. If $x'(x_0)$ is the limit of a convergent partial sequence belonging to $(x_n(x_0))$, there also holds $Px'(x_0) = Px_0$. That means $P\bar{x}_i(x_0) = Px_0 = P\bar{x}_0(x_0) = P\bar{x}_0 = P\bar{x}_i(\bar{x}_0)$ and $x_{nL+i}(\bar{x}_0) = \bar{x}_i(x_0) = \bar{x}_i$. Starting with the limit $\bar{x}_0$ of the first cyclic partial sequence $(x_{nL}(x_0))$, the limits $\bar{x}_i$ of the cyclic partial sequences $(x_{nL+i}(x_0))$ are successively mapped one upon the other during the iterative process (4.2), that is $T_{nL+i}\bar{x}_i + D_{nL+i} b = \bar{x}_{i+1}$ $(i = 0, \ldots, L-1)$ with $\bar{x}_L = \bar{x}_0$.

Remark 6.6: Under the assumptions for statement 2, statement 4 and statement 5 of Theorem 6.1, the set of all limit elements $x^\infty = x^\infty(x^0)$ coincides with the set of all $\{D\}$-generalized solutions, AB-generalized solutions and solutions of (4.1), respectively. Thus all solutions or generalized solutions of (4.1) in the above-mentioned sense can be reached by the selection of suitable starting elements x^0.

Remark 6.7: Under the assumptions of Theorem 6.1 the iteration operator T of (6.1) is contractive on $\mathfrak{N}(P)$, that is $\|T \mid \mathfrak{N}(P)\| \leqq \alpha$ for a certain $\alpha < 1$. Then it follows

$$\|T^n - P\| = \|T^n(I - P)\| \leqq \alpha^n. \tag{6.6}$$

Thus we find

$$\|x^\infty - x^n\| = \|(P - T^n)(x^0 - x')\| \leqq \alpha^n \|x^0 - x'\|,$$

using (4.3) with $n-1$ instead of n and $P_{n-1} = T^n$ as well as (6.4) and (6.6). On the other hand we get from (6.1) and (6.5)

$$\|x^\infty - x^n\| = \|(B - B^{(n-1)})(b - Ax^0)\| = \left\| \sum_{i=n}^{\infty} T^i D(b - Ax^0) \right\|$$
$$\leqq \sum_{i=n}^{\infty} \alpha^i \|D\| \|b - Ax^0\| = \alpha^n \|D\| \|b - Ax^0\|/(1 - \alpha).$$

Remark 6.8: If the basic assumption of Theorem 6.1 that T is norm-attaining on $\mathfrak{N}$ is omitted, then (P_n) and (T^n) still accumulate to the orthogonal projector P in the weak sense, where the relations (6.3) for $\mathfrak{R}(P)$ and $\mathfrak{N}(P)$ hold with the exception that $\mathfrak{N}(P) = \mathfrak{N}(DA)$ has to be replaced by $\mathfrak{N}(P) = \overline{\mathfrak{R}(DA)}$. The remaining statements are also still true if the convergence is interpreted as accumulation in the weak sense.

7. Special classes of iterative relaxation methods

We turn to special iterative methods (4.2). We additionally consider a sequence (Z_n) of Hilbert spaces Z_n and two sequences (G_n), (R_n) of operators G_n, $R_n \in \mathbf{L}(Y, Z_n)$. Furthermore we suppose the composite operators

$$A_n = G_n A \in \mathbf{L}(X, Z_n) \tag{7.1}$$

to be normally solvable, that is $\mathfrak{R}(A_n) = \overline{\mathfrak{R}(A_n)}$ for all n. Then they possess orthogonal generalized inverses $A_n^+ \in \mathbf{L}(Z_n, X)$. Besides the adjoints A_n^* of A_n are normally solvable, too (see, e.g., [8]).

Now we can choose the generating operators D_n of (4.2) in the form

$$D_n = A_n^+ R_n \tag{7.2}$$

for all n. It is easy to see that D_n can equivalently be expressed by

$$D_n = A_n^* R_n', \tag{7.2'}$$

where the relations

$$R_n' = (A_n A_n^*)^+ R_n, \qquad R_n = A_n A_n^* R_n'$$

hold. In the case $\mathfrak{R}(A_n) = Z_n$ we obtain

$$R_n' = (A_n A_n^*)^{-1} R_n, \qquad D_n = A_n^* (A_n A_n^*)^{-1} R_n.$$

Lemma 7.1: *The iteration operators $T_n = I - D_n A = I - A_n^+ R_n A$ of (4.2)/(7.2) are relaxations of the orthoprojectors $T_n' = I - A_n^+ A_n$ iff the following two conditions are satisfied*:

a) $A_n^+ R_n A (I - A_n^+ A_n) = 0$, b) $\|A_n^+ (A_n - R_n A) x\| < \|x\|$ *for* $x \neq 0$.

These conditions a), b) *imply*

$$\mathfrak{R}(T_n') = \mathfrak{R}(A_n) = \mathfrak{R}(D_n A),$$
$$\mathfrak{N}(T_n') = \mathfrak{R}(A_n^*) = \overline{\mathfrak{R}(D_n A)} = \overline{\mathfrak{R}(D_n)}. \tag{7.3}$$

Proof: At first we find

$$T_n' T_n = (I - A_n^+ A_n)(I - A_n^+ R_n A)$$
$$= I - A_n^+ A_n - A_n^+ R_n A + A_n^+ R_n A = T_n'.$$

Starting with a) we get the relations

$$A_n^+ R_n A = A_n^+ R_n A A_n^+ A_n,$$

$$T_n T_n' = (I - A_n^+ R_n A)(I - A_n^+ A_n)$$
$$= I - A_n^+ R_n A - A_n^+ A_n + A_n^+ R_n A A_n^+ A_n = I - A_n^+ A_n = T_n'.$$

Reversely the relations $T_n T_n' = T_n'$ lead again to a). Besides,

$$T_n - T_n' = I - A_n^+ R_n A - I + A_n^+ A_n = A_n^+ (A_n - R_n A)$$

and

$$\mathfrak{N}(D_n A) = \mathfrak{N}(I - T_n), \qquad \mathfrak{N}(T_n') = \mathfrak{N}(A_n^+ A_n) = \mathfrak{N}(A_n),$$

$$\overline{\mathfrak{N}(D_n A)} = \overline{\mathfrak{N}(I - T_n)}, \qquad \mathfrak{N}(T_n') = \mathfrak{N}(A_n^+ A_n) = \mathfrak{N}(A_n^+) = \mathfrak{N}(A_n^*),$$

$$\mathfrak{N}(D_n A) \subseteq \mathfrak{N}(D_n).$$

In view of (7.2′) it follows moreover that $\mathfrak{N}(D_n) \subseteq \mathfrak{N}(A_n^*)$. Now all assertions result from Theorem 2.7 ∎

Remark 7.2: *An analogous lemma can be stated including* strong relaxations T_n. For this b) has to be replaced by b′) $\|A_n^+(A_n - R_n A)\| < 1$. The conditions a), b′) supply $N(T_n') = R(A_n) = R(D_n A) = R(D_n)$ (see also Theorem 2.7).

Lemma 7.3: *If the iteration operators* $T_n = I - A_n^+ R_n A$ *of* (4.2)/(7.2) *are strong relaxations of* $T_n' = I - A_n^+ A_n$, *then* (4.2)/(7.2) *is consistent for all* $b \in Y$.

The assertion is evident if Remark 7.2 is observed. The statement of Lemma 7.3 contains just the additional assumption b) of Theorem 5.1.

Let (G_n) be almost cyclic. Then (T_n') is almost cyclic, too. The operators T_n are relaxations of T_n' provided that the conditions a), b) of Lemma 7.1 are fulfilled (cf. Assumption 3.3).

Let (G_n) and (R_n) be cyclic. Then (T_n) is cyclic, too. Thus a cyclic sequence of relaxations T_n arises if additionally the conditions a), b) of Lemma 7.1 hold (cf. Assumption 3.4 and Section 6). For such cyclic relaxation methods the relations

$$\mathfrak{N}(D) \subseteq \operatorname{span} \bigcup_{i=0}^{L-1} \mathfrak{N}(D_i) \subseteq \operatorname{span} \bigcup_{i=0}^{L-1} \mathfrak{N}(T_i') = \mathfrak{N}$$

are satisfied (see Remark 6.3, Remark 6.4). Using the preceding results of this section we can concretize Theorem 5.1 and Theorem 6.1 for relaxation methods with generating operators (D_n) of the kind (7.2).

Now we want to specialize (D_n) step by step. Let (A_n) be a sequence of operators $A_n \in \mathrm{L}(Z_n)$: The definition

$$R_n = A_n G_n \tag{7.4}$$

supplies according to (7.2) the generating operators

$$D_n = A_n^+ A_n G_n. \tag{7.5}$$

Then the condition a) of Lemma 7.1 is fulfilled automatically since we get

$$A_n^+ R_n A (I - A_n^+ A_n) = A_n^+ A_n A_n (I - A_n^+ A_n)$$
$$= A_n^+ A_n (A_n - A_n A_n^+ A_n) = 0.$$

Hence the following statement results from Lemma 7.1 and Remark 7.2.

Corollary 7.4: *The iteration operators* $T_n = I - D_n A = I - A_n^+ \Lambda_n A_n$ *of* (4.2)/(7.5) *are relaxations (strong relaxations) of* $T_n' = I - A_n^+ A_n$ *iff the relations* $\|A_n^+(I - \Lambda_n) A_n x\| < \|x\|$ *for* $x \neq 0$ $\left(\|A_n^+(I - \Lambda_n) A_n\| < 1\right)$ *are satisfied.*

For the special choice $\Lambda_n = \lambda_n I$ we get a class of iterative methods which are investigated in [11–14]. Here the operators T_n are strong relaxations of T_n' iff $|1 - \lambda_n| < 1$ holds (see Section 2). In the case $Y = \mathbf{R}^N$ we can define the operators G_n by $G_n y = y_n$, where the elements y_n consist of some components of $y \in \mathbf{R}^N$. Hence the linear equation (4.1) is separated by (G_n) into a system $(Ax)_n = b_n$ of linear equations. If each component of $y \in \mathbf{R}^N$ occurs in at least one y_n, then $\bigcap_n \mathfrak{N}(D_n A) = \bigcap_n \mathfrak{N}(A_n) = \mathfrak{N}(A)$. Therefore condition d) of Theorem 5.1 is fulfilled.

For $X = \mathbf{R}^M$, $Y = \mathbf{R}^N$ the linear operator A can be represented as a matrix and the linear operators G_n can be chosen to be row selection matrices of A consisting of some of the columns of the unit matrix $I_N \in \mathbf{L}(\mathbf{R}^N)$. Thus the so-called *PSH methods* arise. The simplest form of such methods was proposed by Kaczmarz in 1937. Later this *Kaczmarz algorithm* was rediscovered in the context of x-ray reconstruction techniques (see, e.g., [7, 12, 19]).

Remark 7.5: **1.** In the paper [2] convergence statements for (4.2)/(7.5) are obtained under the following assumptions: a) $X = \mathbf{R}^M$, $Y = \mathbf{R}^{LN}$, $Z_n = \mathbf{R}^L$ for all n. b) (G_n) is a cyclic sequence of matrices selecting in each case the following L rows of $A \in \mathbf{L}(\mathbf{R}^M, \mathbf{R}^{LN})$. c) (Λ_n) is a bounded sequence of matrices $\Lambda_n \in \mathbf{L}(\mathbf{R}^L)$ with $\overline{\lim_{n \to \infty}} \|A_n^+(I - \Lambda_n) A_n\| < 1$. These assumptions integrate into our general framework (see, e.g., condition d') in Remark 3.3' and Corollary 7.4).

2. The paper [19] contains a convergence statement for (4.2)/(7.5) assuming the following special conditions: a') $X = \mathbf{R}^M$, $Y = \mathbf{R}^N$, $Z_n = \mathbf{R}$ for all n, b') (G_n) is a cyclic sequence of matrices selecting step by step all rows of A. c') (Λ_n) is a sequence of matrices satisfying $\Lambda_n = \lambda_n I$, $0 \leq \lambda_n \leq 2$, for all n, $\sum \mu^{(k)}$ is divergent, where $\mu^{(k)} = \min\{\mu_{kN+j-1} : j = 1, \ldots, N\}$, $\mu_n = \min\{\lambda_n, 2 - \lambda_n\}$. d') (4.1) is solvable. Again our general assumptions are fulfilled (see, e.g., Assumption 3.3 and Remark 3.3').

3. The results in [2] and [19] are covered by our results. Moreover we can derive some further results for these special cases.

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