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Iterated Dirac Operators in \mathbb{C}^n

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Wir geben eine Kiassifizierung der linearen konform-invarianten' Differentialoperatoren iber C", die ihre Wete in einer Cliffordschen Algebra annehmen. Solche Operatoren.schlielienden Diracschen Operator und seine Potenzen ein. Wir zeigen, daß die mit dem Diracschen Operator Wir geben eine Klassifizierung der linearen konform-invarianten Differentialoperatoren über (\mathbb{C}^n , die ihre Weste in einer Cliffordschen Algebra annehmen. Solche Operatoren schließen der Diracschen Operator und sein **J.** Ryan

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Diracschen Operator und seine Potenzen ein. Wir zeigen, daß • Wir geben eine Klassifizierung der linearen konform-invarianten Differentialoperatoren über \mathbb{C}^n , die ihre Werte in einer Cliffordschen Algebra annehmen. Solche Operatoren schließen den Diracschen Operator und se • Wir geben eine Klassifizierung der linearen konform-invarianten Differentialoperatoren über
 CP, die ihre Werte in einer Cliffordschen Algebra annehmen. Solche Operatoren schließen den

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Cⁿ, die ihre Weste in einer (Diracschen Operator und sein

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Mы даем классификацию линейных конформно-инвариантных дифференциальных in \mathbb{R}^n und \mathbb{C}^n verbundene Funktionentheorie sich auf alle diese Operator

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Mu даем классификацию линейных конформно-инвариантных ди

операторов над \mathbb{C}^n со значениями в алгебре Клиффорда. Эти оп

We give a classification of linear, conformally invariant, Clifford algebra valued differential operators over \mathbb{C}^n . Such operators comprise of the Dirac operator and its iterates. We show that the function theory, associated to the Dirac operator in \mathbb{R}^n and \mathbb{C}^n can be generalized to all **CHE CONSTRESS OF CCHA DIPARTS CONSTRESS OF CP. Such operators of the Dirac operator and it** these operators.
Introduction

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By introducing complex Clifford algebras'it has been possible to introduce a first order differential operator, over \mathbb{R}^n , whose square is the Laplacian, and to study the properties of analytic continuations to \mathbb{C}^n of functions which are annihilated by this operator $[6-8, 13, 14, 21-24]$. This operator gives a natural generalization of both the classical Cauchy-Riemann equations and the massless Dirac equation. The study of properties of functions which are annihilated by this generalized Cauchy-Riemann-Dirac operator is referred to as Clifford analysis $[6, 19, 20 - 24]$. In the 1930s Clifford analysis had been deseloped by FUETER [11, 12], and his collaborators; as a function theory over the quaternions, and by Moisin and Troporescu [18]. Also, earlier work on this analysis had been developed by DIXON [9]. More recently this analysis has been extended to higher dimensions by a number of authors (eg $[6-8, 13, 14, 19,$ $21-24$, 28]). **Example 10**

By introducing comporter differential operator [6 - 8, 13, 1

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In recent work $[1-3]$ AHLFORS, building on results' of VAHLEN $[27]$ and MAASS $[17]$, describes properties of Möbius transformations in \mathbb{R}^n by means of a group of matrices with entries in a Clifford algebra. Within mathematical physics the study of confor-Dirac operator is referred to as Clifford analysis [6, 19, 20 – 24]. In the 1930s Clifford
analysis had been developed by FuETER [11, 12], and his collaborators, as a function
theory over the quaternions, and by Morsuland operators over curved spaces, has been extensively pursued (see for example [10, 15]. In recent work $[1-3]$ AHLEORS, building on results of VAHLEN $[27]$ and MAASS $[17]$ describes properties of MÖBIUS transformations in \mathbb{R}^n by means of a group of matrices with entries in a Clifford algebra. Within ordinary d'Alemhertion acting on functions in Minkbwski space are conformally mally invariant differential operators on Minkowski space, and analogues of these
operators over curved spaces, has been extensively pursued (see for example [10, 15]
and references therein). In [15] JAKOBSEN and VERGNE s the matrices appearing in $[1-3]$ to show that the class of linear, conformally invariant, • holomorphic differential operators defined over \mathbb{C}^n comprises of a semigroup of iterated Dirac operators. In [10] Verma modules are used to describe conformally invariant In recent work $[1 - 3]$ AHLEOKS, buttum of results of VAHLEN [27] and
describes properties of MöBIUS transformations in \mathbb{R}^n by means of a group
with entries in a Clifford algebra. Within mathematical physics the st

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differential operators on Minkowski space.. However, the methods used here are function theoretic in nature. We obtain our result by first deducing a generalized Cauchy integral formula, or /generalized Green's formula, for solutions to each such operator. We then use these formulae to give a characterization of solutions to these equations over the Lie ball and apply arguments given in [24] and [8] to deduce the **1. Brash Controllering Solution**
 Results in the Solution of the methods used here are

function theoretic in nature. We obtain our result by first deducing a generalized

Cauchy integral formula, or *i*generalized Gree

everi dimensions to odd dimensions. This leads us to study particular types.of domains in \mathbb{C}^n , and to study conformal transformations over twofold covering spaces of some domains in \mathbb{C}^n , when *n* is odd.

Preliminaries

Let $A_n(\mathbb{C})$ be the complex, 2^n dimensional Clifford algebra described in [4, Part 1], [20, Chapter 13], and elsewhere. This algebra has an identity $l (=e_0)$, and basis elements $l, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_{j_1}, \ldots, e_{j_r}, \ldots, e_1 \ldots e_n$, where $j_1 < \cdots < j_r$ and $1 \leq r \leq n$. The elements e_1, \ldots, e_n satisfy the anticommutation relationship $e_j e_k + e_k e_j$ $= -2\delta_{jk}$, where δ_{jk} is the Kronecker delta. We call \mathbb{C}^n the complex space spanned. by e_1, \ldots, e_n . We call \mathbb{R}^n the real space spanned by e_1, \ldots, e_n . The isotropic, or null by $e_1, ..., e_n$. We call \mathbb{R}^n the real space spanned by $e_1, ..., e_n$. The isotropic, or null
cone $(z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0)$ is denoted by $N(0)$, and a general
vector $z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n$ is deno vector $z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n$ is denoted by z. It may be noted that each vector $z \in \mathbb{C}^n$ vector $z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n$ is denoted by z. It may be noted that each vector $z \in \mathbb{C}^n$.
 $\mathcal{N}(0)$ has a multiplicative inverse $z(z^2)^{-1} \in \mathbb{C}^n \setminus \mathcal{N}(0)$. Consequently, we have the Clif-

ford group Γ_n ford group $\Gamma_n(\mathbb{C}) \doteq \{ Z \in A_n(\mathbb{C}) : Z = z_1...z_k, z_j \in \mathbb{C}^n \setminus N(0) \text{ for } 1 \leq j \leq k \text{ and } k \text{ is an arbitrary positive integer} \}.$ On restricting the elements $Z \in \Gamma_n(\mathbb{C})$ so that for each z_j we have that $z_j^2 = \pm 1$ we obtain a subgroup of $\Gamma_n(\math$ cone $(z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0)$ is denoted by *N*(0), and a general vector $z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n$ is denoted by *z*. It may be noted that each vector $z \in \mathbb{C}^n$ $\setminus N(0)$ has a multiplicative inv and in the case where *k* is even we obtain a subgroup of $Pin(\mathbb{C}^n)$ which we denote by Let $A_n(\mathbb{C})$ be the complex, 2^n dimensional [20, Chapter 13], and elsewhere. This algebr

ments $l, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_{j_1} \ldots, e_{j_k}$
 $1 \leq r \leq n$. The elements e_1, \ldots, e_n satisfy the
 $= -2\delta_{ik}$, w bitrary positive integer). On restricting the elements $Z \in \Gamma_n(\mathbb{C})$ so
have that $z_j^2 = \pm 1$ we obtain a subgroup of $\Gamma_n(\mathbb{C})$ which we den
d in the case where k is even we obtain a subgroup of $Pin(\mathbb{C}^n)$ which
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Furthermore, we have [4, 20] an antiautomorphism $\sim: A_n(\mathbb{C}) \to A_n(\mathbb{C})$: $e_i \dots e_i$. $e_i, \ldots e_i$. For a general element $Z \in A_n(\mathbb{C})$ we denote $\sim (Z)$ by Z. Using the previously described anticommutation relationship it. may be observed that for each $a \in Pin(\mathbb{C}^n)$ we have that $a\mathbb{C}^n\tilde{a} = \mathbb{C}^n$ and for each $z \in \mathbb{C}^n$, $(az\tilde{a})^2 = z^2$. It follows that *Pin(***C**^{*n*}) is closely related to the complex orthogonal group $O(\mathbb{C}^n) = \{(a_i)\colon i\in I\}$ $1 \leq i, j \leq n, a_{ij} \in \mathbb{C}$ and (a_{ij}) $(a_{ij})^T = l$. In fact we have

Lemma 1: The group $Pin(\mathbb{C}^n)$ **is a four-fold covering of the group** $O(\mathbb{C}^n)$ **(i.e. there is a** *short exact sequence* $0 \to Z_4 \to Pin(\mathbb{C}^n) \to O(\mathbb{C}^n) \to 0$.

Outline, proof: On considering the group homomorphism $\theta\colon Pin(\mathbb{C}^n) \to O(\mathbb{C}^n)$ canonically induced by the map λ : $Pin(\mathbb{C}^n) \times \mathbb{C}^n$: $(a, z) \mapsto az\tilde{a}$ it may be observed that the elements 1, -1 , $\sqrt{-1}e_1...e_n$, $-\sqrt{-1}e_1...e_n \in Pin(\mathbb{C}^n)$ belong to the kernel of θ . It now follows from similar arguments to those detailed for the Euclidean case in [4, Part 1] that the homomorphism θ is surjective with kernel $\{1, -1, \sqrt{1-1}e_1 ... e_n\}$ $-\sqrt{-1}e_1...e_n$ | | -.

Using the Clifford algebra's anticommutation relationship we may from \mathbb{R}^n generate a real, 2^n -dimensional subalgebra of $A_n(\mathbb{C})$. This algebra is an example of a real Clifford algebra, and its properties are described in [4, Part 11, [20, Chapter 13] and elsewhere. We denote it by A_n . The group $A_n \cap Spin(\mathbb{C}^n)$ is denoted by $Spin(\mathbb{R}^n)$ and, as shown in $[4, 20]$, it is a double covering of the special orthogonal group $SO(n)$, which-acts on \mathbb{R}^n . The group $A_n \cap Pin(\mathbb{C}^n)$ is denoted by $Pin(\mathbb{R}^n)$ and it is a covering group of $O(n)$ (see [4, 20]).

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- - - -iterated .Dirac Operators in C" 387 Following results obtained by AHLFORS $[1-3]$; MAASS $[17]$ and VAHLEN $[27]$, we

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Following results obtained by AHLFORS $[1-3]$,

introduce the following type of matrices:

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that $a, b, c, d \in A_n$ introduce the following type of matrices:
Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that a, b , be a matrix such that $a, b, c, d \in A_n(\mathbb{C})$, and $a = a_1 \ldots a_l, b = b_1 \ldots b_m$, *Collowing results obtained by Africand* $[1-3]$, *MAASS* [17] and *VAHLEN* [27], we introduce the following type of matrices:

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that *a*, *b*, *c*, *d* \in *A_n*(**C**), and *a matrix.* As illustrated in [24], for each Clifford matrix the transformation *(az'+)(cz.d) ¹ -*

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (az + b) (cz + d)^{-1}
$$

is well defined, and gives a Möbius transformation in \mathbb{C}^n .

Suppose now that U is a domain in \mathbb{R}^n , $n>2$, and that $f: U \to A_n(\mathbb{C})$ is a function for which all partial derivatives exist. Suppose that *P* is a Clifford algebra valued constant coefficient differential operator of order $p \in \mathbb{Z}^+$, which acts on *f* on the left hand side (i.e. the operator *P* acting on */* does not involve multiplying'*/* on the right-hand side by an element of the noncommutative algebra $A_n(\mathbb{C})$. Then we may introduce Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that $a, b, c, d \in A_n(\mathbb{C})$, and $a = a_1 \dots a_l, b = b_1 \dots b_n$
 $c = c_1 \dots c_p, d = d_1 \dots d_q$, where $a_1 \dots a_l, b_1, \dots, b_m, c_1, \dots, c_p, d_1, \dots, d_q \in \mathbb{C}^n, l, m, p$,
 $\in \mathbb{Z}^+$ and $a\tilde{c}, \tilde{c}d, d\tilde{$

Definition 1 [8]: Suppose that for each $a \in Spin(\mathbb{R}^n)$ and for each */* with the pro-
perty Pf(axa) = 0 with respect to the variable *axa*, where $x \in \mathbb{R}^n$, we have that
Paf(axa) = 0 with respect to the variable perty $Pf(ax\tilde{a}) = 0$ with respect to the variable $ax\tilde{a}$, where $x \in \mathbb{R}^n$, we have that $P\bar{a}f(ax\bar{a})=0$ with respect to the variable x. Then P is called a *spin-Euclidean diffe-*

In [28], and elsewhere, it is observed that the operator $D = \sum_i e_i \partial/\partial x_i$ is a spin-Euclidean differential operator. It is also well-known (e.g. [28, Chapter 9] that the perty $I/(u\omega t) = 0$ with respect to the variable $x\alpha$, where $x \in \mathbb{R}^n$, we have that $P\bar{a}/(ax\bar{a}) = 0$ with respect to the variable x . Then P is called a *spin-Euclidean differential operator*, and f is said to nal group. $SO(n)$. As $Spin(\mathbb{R}^n)$ is a double covering of the group $SO(n)$, and \bar{a} is a constant, it follows that the Laplacian is a spin-Euclidean differential operator. On placing $D^0 = 1$ it is deduced in [8] that *Pulace D* = 0 with respect to the variable *x*. I field *P* is calculated *Perator*, and *f* is said to be *spin invariant with rese* In [28], and elsewhere, it is observed that the operator Euclidean differential operat efinition 1 [8]: Suppose that for each ' $a \in Spin(\mathbb{R}^n)$ and for each f with the pro-
 ty P/(aza) = 0 with respect to the variable axa, where $x \in \mathbb{R}^n$, we have that
 k/(aza) = 0 with respect to the variable ax f

• Theorem 1: *Every spin?-Euclidean differential operator of order p is of the form*

$$
\sum_{k=1}^{p} A_k D^k, \ with \ A_k = a_{0,k} + a_{1...n,k} e_1...e_n, \ where \ a_{0,k}, a_{1...n,k} \in \mathbb{C}.
$$

Proposition 1: *The only solutions to a spin-Euclidean differential operator which are invariant under dilation are solutions to an iterate of. the"operator' D.*

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Definition 2 [21]: A solution to the spin-Euclidean differential operator D is called a left regular function. A similar definition may be given for *right regular functions*.

An example of a function which is both left and right regular is the function $G(x) = x|x|^{-n}$, defined on $\mathbb{R}^n \setminus \{0\}$. Using this function we have [7] the following gene-
ralized Cauchy integral formula. placing $D^0 = 1$ it is deduced in [8] that

Theorem 1: Every spin-Euclidean differential operator of order p is of the form
 $\sum_{k=0}^{p} A_k D^k$, with $A_k = a_{0,k} + a_{1...n,k}e_1...e_n$, where $a_{0,k}, a_{1...n,k} \in \mathbb{C}$.

From the chain *f(x0)* = *ii,,f 0(X,- x) Wx/(x), _ (1)* invariant under dilation are solutions to an iterate of the operator D.

Definition 2 [21]: A solution to the spin-Euclidean differential operator

a left regular function. A similar definition may be given for right regu

Theorem 2: Suppose that $f: U \to A_n(\mathbb{C})$ is a left regular function, and $M \subseteq U$ is a *compact n-dimensional manifold. Then, for each point* $x_0 \in M$, the interior of M, we have

$$
f(x_0) = 1/\omega_n \int\limits_{\partial M} G(x_0-x) Wx f(x), \qquad \Box
$$

where ω_n *is the surface area of the unit sphere* S^{n-1} *, and,* $Wx = \sum c_i(-1)^i dx_i$ *.* = $1/\omega_n \int_{\partial M} G(x_0 - x) Wx f(x)$,
surface area of the unit sphere S^{n-1} , and $Wx = \sum_{j=1}^n e_j(-x_j)$

It may be observed from Theorem 2 that the operator D is a natural generalization of the Cauchy-Riemann operators. It may also be observed that this operator is also a Euclidean generalization of the Dirac operator described in [15] and elsewhere.

The operator \tilde{D} and the integral formula (1) have natural generalizations in \mathbb{C}^n . Before introducing these generalizations we require the following manifolds.

Definition₂ [21]: Suppose that *M* is a compact, smooth; connected, real *n*-dimensional manifold, with boundary, lying in, \mathbb{C}^n such that for each $z \in M$ we have (i) *• M* \cap *N(z)* = {z} and (ii) *TM*₂ \cap *N(z)* = {z}, where $N(z) = \{z' \in \mathbb{C}^n : (z - z')^2 = 0\},$ *then M* is called a *manifold of type one.* The operator D and the integral formula (1) have natural generalizations in \mathbb{C}^n .

The operator D and the integral formula (1) have natural generalizations in \mathbb{C}^n

Before introducing these generalizations we 388 *J.* RYAN:

The may be observed from Theorem 2 that the operator *D* is a natural generalization of the Cauchy-Riemann operators. It may also be observed that this operator is also a Euclidean generalization of the Di

 A_{TV} compact, *n*-dimensional manifold lying in \mathbb{R}^n is an example of a manifold of [23]. Any compact, *n*-dimensional manifold lying in \mathbb{R}^n is an example one. Further examples, and constructions of manifolds of
3].
In [21] we describe the following class of domains in \mathbb{C}^n . V

Definition 4: Suppose that *M* is a manifold of type one, then the component of \mathbb{C}^n ${N(z): z \in \partial M}$ containing M is called a *cell of harmonicity of type one*, and we denote it by M^+ .

When $M \subseteq \mathbb{R}^n$ these cells of harmonicity have previously been described in [5, 16, 26], and when *M* is the unit disc, *K*, in \mathbb{R}^n , the domain M^+ is the Lie ball K^+ $= \{z \in \mathbb{C} : (2^{-n}|z|)^2 + ((2^{-n}|z|)^4 - z^2(z)^2)^{1/2} < 1\}$ described in [26].

De finition'5 [23]: Suppose that U_C is a domain in \mathbb{C}^n and that $g\colon U_C\to A_n(\mathbb{C})$ is a When $M \subseteq \mathbb{R}^n$ these cells of harmonicity have previously been described in [5,

16, 26], and when M is the unit disc, K, in \mathbb{R}^n , the domain M^+ is the Lie ball K^+
 $= \{z \in \mathbb{C} : (2^{-n}|z|)^2 + ((2^{-n}|z|)^4 - z^2(z)^2$ When $M \equiv K^*$ these cens of harmometry have previously been described in 16, 26], and when M is the unit disc, K , in \mathbb{R}^n , the domain M^+ is the Lie bal $=\{z \in \mathbb{C} : (2^{-n}|z|)^2 + ((2^{-n}|z|)^4 - z^2(z)^2)^{1/2} \le 1\}$ des $\begin{align*}\n\text{note} \\
\mathbf{n} \quad \mathbf{5} \\
\mathbf{l} \quad K^+ \\
\mathbf{v} \quad \mathbf{is} \\
\mathbf{a} \quad U_{\mathbf{C}}, \\
\mathbf{t} \quad \mathbf{ion}, \\
\mathbf{v} \quad \mathbf{on} \quad \mathbf{io} \\
\mathbf{ro} \quad \mathbf{ro} \quad$ *A A A A E Z i*. Then *g* is called a *complex k-left regular fu* $k \in Z^+$. Then *g* is called a *complex k-left regular fu* and the *k-th order iterated Dirac operator* in \mathbb{C}^n .

Section $D_{\mathbb{C}}^k g(z) =$ where $D_C = \sum_{j=1}^{n} e_j \partial/\partial z_j$ and $k \in Z^+$. Then g is called a *complex k-lett* rand the operator D_C^k is called the *k-th order iterated Dirac operator* in

When *k* is even the equation $D_C^k g(z) = 0$ corresponds to the

functions described by AVANISSIAN in [5]. The operator $D_{\mathbb{C}}$ is a holomorphic generalization of the operator *D.*

In [23] we deduce the following Cauchy integral formula.

When *k* is even the equation $D_C^k g(z) = 0$ corresponds to the $k/2$ complex harmonic functions described by AVANISSIAN in [5]. The operator D_C is a holomorphic generalization of the operator *D*.
 In [23] we deduce th Theorem 3: *Suppose that f:* $U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex k-left regular function, with n exercibed by AVANISSIAN In* [9]. The operator D_C is a nononorphic generalization of the operator D .

In [23] we deduce the following Cauchy integral formula.

Theorem 3: Suppose that $f: U_C \rightarrow A_n(C)$ is a complex k-left $f(x) = \begin{cases} \n\frac{1}{2} & \text{if } x \leq n - 1. \text{ Suppose that } f: \n\frac{1}{2} & \text{if } x \in f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^n f(p) \cdot \frac{1}{2} & \text{if } p \leq 1. \n\end{cases}$

' (2)

V :'

$$
f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^n A_p G_p^+(z - z_0) Wz f(z),
$$

 $\mathbf{r} = \frac{1}{2} \sum_{i=1}^{N} \mathbf{r}_i$

 $where A_1 = (-1)^{n/2}, G_1^+(z) = z^{-n+1}, G_p^+(z) = z^{-n+p}, A_p$ is a constant with $D_{\mathbb{C}}A_pG_p^+(z)$ *one. Then for each point* $z_0 \in M^+ \cap U_C$ *we have*
 $f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^n A_p G_p^+(z - z_0) Wz$
 where $A_1 = (-1)^{n/2}$, $G_1^+(z) = z^{-n+1}$, $G_p^+(z) = z$
 $= A_{p-1} G_{p-1}^*(z)$, and $Wz = \sum_{i=1}^n (-1)^i e_i dz_i$.

Using the formula (2) it is straightforward to deduce the following holomorphic continuation.

v
Video Theorem 4 [23]: Suppose that, $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with n even and $k \leq n - 1$. Suppose also that $M \subseteq U_{\mathbb{C}}$ is an n-dimensional manifold *of type one lying in* $U_{\mathbb{C}}$. Then there is a holomorphic function $f^*: U_{\mathbb{C}} \cup M^+ \to A_n(\mathbb{C})$ *such that* $f^+|_{U_{\sigma} \cap M^+} = f$. \cdot

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Theorem 5 [24]: *Suppose that f:* $U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex 1-left regular function* with respect to the variable $(az + b)$ $(cz + d)^{-1}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix, and *n*_{is} even. Then the holomorphic function $J_1(cz + d) f((az + b) (cz + d)^{-1})$ is a complex Theorem 5 [24]: Suppose that $f: U_C \to A_n(C)$ is a complex 1-left regular function
with respect to the variable $(az + b)$ $(cz + d)^{-1}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix, and
n is even. Then the holomorphic functio $\times \{(cz + d)$ $(cz + d)\}^{-n/2}.$

Generalized Cauchy integral formulae

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This section is divided into two parts. In Part A we consider the cases where n is even and greater than two, and in Part B we consider the cases where n is odd, and greater than two. al formulae

b two parts. In Pa

in Part B we cons

icing the following
 ιt *f*: $U_{\mathbb{C}} \to A_n(\mathbb{C})$
 M a manifold of
 ι
 $\iint_{\mathbb{C}} A_p G_p(x_0 - x) dx$ *• f(x0)* = 1/a,, *f ^E. AG(x0 - x) WxD' I(x),* (3) -'

Part A. We begin by deducing the following extension to Theorem 3.

Theorem 6: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with $k \geq n$, $U_{\mathbb{C}} \cap \mathbb{R}^n + \emptyset$ and M a manifold of type one lying in $U_{\mathbb{C}} \cap \mathbb{R}^n$. Then, for each *point x0 € M we have*

$$
f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k A_p G_p(x_0 - x) \, \hat{W} x D^{p-1} f(x), \qquad (3)
$$

where $A_1 = 1$, A_p is a constant with $DA_pG_p(x) = -A_{p-1}G_{p-1}(x)$, $G_p(x) = x^{-n+p}$ for $f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{\kappa} A_p G_p(x_0 - x) Wx D^{p-1} f(x),$ (3)

where $A_1 = 1$, A_p is a constant with $DA_p G_p(x) = -A_{p-1} G_{p-1}(x)$, $G_p(x) = x^{-n+p}$ for
 $1 \leq p \leq n-1$, $G_n(x) = 1/2 \log (-x^2)$, and $G_p(x) = 1/2 x^{p-n} \log (-x^2) + n^{-1}(p - n)$
 $\times x^{$ *k* $\geq n$, $U_C \cap \mathbb{R}^n \neq \emptyset$ and *M*
 point $x_0 \in M$ we have
 $f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k A_p$
 i $\geq p \leq n-1$, A_p is a constant $1 \leq p \leq n-1$, $G_n(x) = 1/2$
 $\times x^{p-n}$ for $n+1 \leq p \leq k$.
 Outline proof: It follow $\begin{align*}\n & = 1, A_p \text{ is a} \\
 & n-1, G_n(x) = \\
 & r \cdot n + 1 \leq p \leq \\
 & \text{proof: It follows} \\
 & 1/\omega_n \int_{\partial K(x_n, r)} \sum_{p=1}^k \omega(p_n, r) \text{ is the real}\n \end{align*}$ *And* $f: U_C \to A_n(C)$ *is a complex k-left regular function, with

<i>AM a manifold of type one lying in* $U_C \cap \mathbb{R}^n$. Then, for each
 $\sum_{p=1}^k A_p G_p(x_0 - x) WxD^{p-1} f(x)$, (3)
 $\sum_{p=1}^k A_p G_p(x_0 - x) WxD^{p-1} f(x)$, (3)
 $\sum_{p=1}^k$

Outline proof: It follows from Stakes" heorcm that the integral (3) is equal to

$$
1/\omega_n \int\limits_{\partial K(x_0,r)} \sum_{p=1}^k A_p G_p(x_0-x) \cdot Wx D^{p-1} f(x), \qquad (4)
$$

where $K(x_0, r)$ is the real, *n*-dimensional disc lying in *M*, centred at x_0 , and with radius
r. As *r* log $r \rightarrow 0$ for $r \rightarrow 0$, it now follows from similar arguments to those used to
prove the generalized Cauchy i *r.* As *r* $\log r \rightarrow 0$ for $r \rightarrow 0$, it now follows from similar arguments to those used to prove the generalized Cauchy integral formula in [6] that the integral (4) is equal $f(x_0) = 1/\omega_r$

where $A_1 = 1$, A_p is
 $1 \leq p \leq n - 1$, $G_n(x)$
 $\times x^{p-n}$ for $n + 1 \leq$

Outline proof: It
 $1/\omega_n \int_{\partial K(x_0,r)} x \, dx$

where $K(x_0, r)$ is the i
 r . As $r \log r \to 0$ for

prove the generalize

to $f(x_0)$ \blacksquare

Before generalizing Theorem 6 to arbitrary manifolds of type one we require

 $\log (z - z_0)^2$ may be uniquely defined on the set $\partial M \times M$, where $z \in \partial M$ and $z_0 \in M$.

Proof: As *M* is a manifold of type one, then it follows from [26] that the boundary; ∂M , is homologous in $\mathbb{C}^n \setminus N(z_0)$ to the unit sphere $S^{n-1} + z_0$ lying in $\mathbb{R}^n + z_0$, for each $z_0 \in M$. As S^{n-1} is simply connected it follows that the expression log $(-(z - z_0)^2)$ may be uniquely defined for each $z \in S^{n-1} + z_0$. The result follows \blacksquare *k*_n. It is nomologous in $\mathbb{C}^n \setminus N(z_0)$ to the unit sphere $S^{n-1} + z_0$ tying in $\mathbb{R}^n + z_0 \in M$. As S^{n-1} is simply connected it follows that the expression log $(-\text{may be uniquely defined for each } z \in S^{n-1} + z_0$. The result follows *f* θ *e* i *e* i *e* i *e* i *e* i *de* i *a* i *s* θ *f* θ *each of* θ *f* orem 6 to arbitrary manifolds of ty
 Adimensional manifold of type one ly
 Adj defined on the set $\partial M \times M$, where
 A of type one, then it follows from [
 $N(z_0)$ to the unit sphere $S^{n-1} + z_0$ lyi

connected it f

Proposition 2: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex k-left regular function, with* may be uniquely defined for each $z \in S^{n-1} + z_0$. The result follows
 Proposition 2: Suppose that $f: U_C \to A_n(C)$ is a complex k-left regular function, with $k \ge n$. Suppose also that $z_0 \in U_C$ and $K(0, r) + z_0 \subset U_C$. Then t

may be uniquely defined for each
$$
z \in S^{n-1} + z_0
$$
. The result follows
\nProposition 2: Suppose that $f: U_C \to A_n(\mathbb{C})$ is a complex *k*-left regular function, with $k \ge n$. Suppose also that $z_0 \in U_C$ and $K(0, r) + z_0 \subset U_C$. Then the integral
\n
$$
1/\omega_n \int_{\partial K(0,r)+z_n} \sum_{p=1}^k A_p G_p^+(z_0 - z) Wz D_C^{p-1} f(z)
$$
\n
$$
= \omega_n
$$

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Outline proof: On choosing different covering values for $G_p^+(z_0 - z)$, for $n \leq p \leq k$ it follows from homogeneity arguments that the limit as *r* tends to zero for expression (5) is $f(z_0)$. The result now follows from Stokes' theorem \blacksquare

We now have

Theorem 7: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with $k \ge n$, and M is a manifold of type one lying in $U_{\mathbb{C}}$. Then, for each point $z_0 \in M$ and *for each choice of covering space values for* $G^+(z - z_0)$, with $n \leq p \leq k$, we have $f(x) = \frac{1}{2} \int_0^x f(x) \, dx$
 f(z₀) = $\frac{1}{\omega_n} \int_0^x f(x) \, dx$
 $f(x_0) = \frac{1}{\omega_n} \int_0^x f(x) \, dx$
 $f(x_0) = \frac{1}{\omega_n} \int_0^x f(x) \, dx$ *A*
 Ag different covering values for $G_p^+(z_0 - z)$, for $n \leq p \leq k$

arguments that the limit as r tends to zero for expression

ollows from Stokes' theorem
 Ag L C $\bigcup_{n=1}^{\infty} C_n \rightarrow A_n(\mathbb{C})$ *is a complex k-left*

$$
f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{\kappa} A_p G_p{}^+(z_0 - z) W_z D_{\mathbb{C}} p^{-1} f(z).
$$
 (6)

As is observed in [5] if *M* is a simply connected manifold of type one, it does not necessarily follow that the cell of harmonicity, M^+ , is simply connected. For example, given the annulus $A(1/2, 3/2) = \{x \in \mathbb{R}^n : 1/2 < -x^2 < 3/2\}$ the cell of harmonicity *A* $(1/2, 3/2)^+$ contains the path $e^{2\pi i\theta}e_1$, where $\theta \in [0, 1]$. By considering the continuous function q: $A(1/2, 3/2)^+ \rightarrow \mathbb{C} \setminus \{0\}$: $z \mapsto z^2$, it may be observed that this path is not We now have

We now have

Theorem 7: Suppose that $f: U_C \rightarrow A_n(C)$ is a complex k-left regular function, w
 $k \ge n$, and M is a manifold of type one lying in U_C . Then, for each point $z_0 \in M$ a

for each choice of covering Theorem 7: Suppose
 $k \ge n$, and M is a main

for each choice of cover
 $f(z_0) = 1/\omega_n \int_{\partial M}$

As is observed in [5]

necessarily follow that

given the annulus $A(1)$
 $A(1/2, 3/2)^+$ contains t

function q: $A(1/2, 3/2)$

As a consequence we have from the integral (6) that even for simply connected manifolds of type one the complex k-left regular functions , with $k \geq n$, do not necessarily, have unique holornorphic continuations to the cell of harmonicity. For example the holomorphic continuation of $log(-x^2)$ is a complex n-left regular function which is not uniquely defined on $A(1/2, 3/2)^+$. necessarily follow that the cell of harmonicity, M^* , is simply connected. For examply and $A(1/2, 3/2) + C₁$ and $e^{2\pi i\theta}e_1$, where $\theta \in \{0, 1\}$. By considering the continual $A(1/2, 3/2)^+ \to \mathbb{C} \setminus \{0\}$: z

In order to introduce suitable subdomains of $M⁺$ over which these functions are uniquely defined we begin by introducing the following definitions.

Definition 6: Suppose that M⁻is a manifold of type one and $z_0 \in M^+$ is a point such that for each $z \in N(z_0) \cap M$ the line segment joining z_0 to *z* lies in M^+ . Then the point z_0 is said to be *null connected to* M.

It is not in general the casethàt for an arbitrary manifold *M* of type one we have that each $z \in M^+$ is null connected to M. For example, for the point ie₁ $\in A(1/2, 3/2)^+$ the point $e_2 \in N(e_1) \cap A(1/2, 3/2)$, but the point $1/2ie_1 + 1/2e_2$ does not lie in $A(1/2, 3/2)^+$ even though it does lie on the line segment joining ie, to e_2 . the holomorphic continuation of $log(-x^2)$ is a complex *n*-1
is not uniquely defined on $A(1/2, 3/2)^+$.
In order to introduce suitable subdomains of M^+ over
uniquely defined we begin by introducing the following de
Defi In order to introduce suitable subdomains of M^+ over which
iquely defined we begin by introducing the following definitie
efinition 6: Suppose that M is a manifold of type one and z_0
at for each $z \in N(z_0) \cap M$ the

Definition 7: Suppose that M is a manifold of type one then the set of points ${z_0 \in M^+ : z_0 \text{ is null connected to } M}$ is called the *null connected subdomain of* M^+ , and it is denoted by NM^+ . •

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IL e mm a 3: *Suppose that M is a manifold of type one. 7'hen the null connected subdomain of* M^+ *is a domain.*

It is not in general the case that for an arbitrary manifold M of type one we have

that each $z \in M^+$ is null connected to M. For example, for the point $i \in \{A(1/2, 3/2)\}$

the point $i \in \{e\}$ $N(i_2) \cap A(1/2, 3/2)$, but Lemma 3: Suppose that M is a manifold of type one. Then the null connected subdomain
of M^+ is a domain.
Proof: Suppose that $z_0 \in NM^+$. Then, either $z_0 \in M$, or $z_0 \in M^+ \setminus M$. If $z_0 \in M$ then
we may choose a neighbo **we may choose a neighbourhood** $B(z_0) \subseteq M^+$ of z_0 such that for each pair of points $z_1, z_2 \in B(z_0)$ the line segment $\{2z_1 + (1 - \lambda) z_2 : \lambda \in [0, 1]\}$ lies $\text{in } B(z_0)$. In [23] we show that for each point $z \in M^+ \setminus M$ We now deduce
Lemma 3: Suppose that *M* is a manifold of type one. Then the null connected subdomain
of *M*⁺ is a domain.
Proof: Suppose that $z_0 \in NM^+$. Then, either $z_0 \in M$, or $z_0 \in M^+ \setminus M$. If $z_0 \in M$ then
we may the sphere S^{n-2} . It also follows from [23] that we may choose $B(z_0)$ so that $N(z) \cap M$ We now deduce

Lemma 3: Suppose that *M* is a manifold of type one. Then the null connected subdomain

of *M*⁺ is a domain.

Proof: Suppose that $z_0 \in NM^+$. Then, either $z_0 \in M$, or $z_0 \in M^+ \setminus M$. If $z_0 \in M$ then

we we may choose a neighbourhood $B(z_0) \subseteq M^+$ of z_0 such that for each pair of points $z_1, z_2 \in B(z_0)$ the line segment $\{2z_1 + (1 - \lambda) z_2 : \lambda \in [0, 1]\}$ lies in $B(z_0)$. In [23] we show that for each point $z \in M^+ \setminus M$ the $B(z_0)$ that each point $z \in B(z_0)$ is null connected to *M*. Suppose now that $z_0 \in M^+ \setminus M$ then as the set of line segments joining z_0 to the compact manifold $M \cap N(z_0)$ is compact, and contained in the open set M^+ , it follows that there is an open subset $U_{z_{\bullet}}$ of M^+ such that $z_0 \in U_{z_0}$ and for each $z \in U_{z_0}$ we have that $z \in NM^+$. Consequently, the set NM^+ is an open set. Lemma 3: Suppose that M is a manifold
of M^+ is a domain.
Proof: Suppose that $z_0 \in NM^+$. Then,
We may choose a neighbourhood $B(z_0) \subseteq z_1, z_2 \in B(z_0)$ the line segment $\{2z_1 + (1 \text{ show that for each point } z \in M^+ \setminus M \text{ the }\n$ the sphere S^{n- *I* that we may choose $D(z_0)$ so that s from the construction of the neighter of the neighter z_0 to the compact manifold $M \cap N$, it follows that there is an open $\in U_{z_0}$ we have that $z \in NM^+$. Con z_0) that each point $z \in B(z_0)$ is null connected to *M*. Suppose now that $z_0 \in M^+ \setminus M$ en as the set of line segments joining z_0 to the compact manifold $M \cap N(z_0)$ is comet, and contained in the open set M^+ , it

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To show that this open set is connected consider first a point $z_0 \in NM^+ \setminus M$. Then, from the open set U_i , we may construct the open subset of $M^*\setminus M$ ¹

To show that this open set is connected consider first a point $z_0 \in NM^+ \setminus M$. Then,

from the open set U_{z_0} we may construct the open subset of $M^+ \setminus M$
 $U'_{z_0} = \left(\bigcup_{z \in U_{z_0}} \bigcup_{z' \in N(z) \cap M} \bigcup_{i \in (0,1]} \bigcup_{z$

$$
U'_{z_0} = \bigcup_{z \in U_{z_0}} \bigcup_{z' \in N(z) \cap M} \bigcup_{\lambda \in (0,1]} \lambda z + (1 - \lambda) z' \big).
$$

 $z_1 \in (c_1'(\mathcal{U}_{z_0}^{\prime\prime}) \setminus U_{z_0}^{\prime\prime}) \cap (\mathcal{U}_{z_0}^{\prime\prime} \setminus U_{z_1}^{\prime\prime})$ such that a line segment joining z_1 to $M \cap N(z_1)$ is not entirely contained in M^+ . Suppose now that z_2 is a point on this line segment satisfying the condition $z_2 \in \text{cl}(M^+) \setminus M^+$. Then there is a point $z_3 \in \partial M$ such that $z_2 \in N(z_3)$. It may be observed that the complex hyperplane $\{z_3 + c(z_2 - z_3) : c \in \mathbb{C}\}\)$ is a subset of $N(z_3)$. Now consider the real three-dimensional hyperplane, H , containing this complex hyperplane, and the vector z_1 . As $z_1 \in (cl(U_{z_2}^{\prime\prime}) \setminus U_{z_2}^{\prime\prime})$ and $U_{z_2}^{\prime\prime}$ is an open set it follows that $H \cap U''_{z_0} \neq \emptyset$ and it is open. Therefore, we have that for each point z_4 \in *H* \cap U'' , the line passing through z_4 , and parallel to the line segment $\{2z_1 + (1 - \lambda)\}$ of $N(z_3)$. Now consider the real three-dimensional hyperplane, H , containing this complex hyperplane, and the vector z_1 . As $z_1 \in \{cl(U'_{2s}) \setminus U'_{2s}\}$ and U'_{2s} is an open set if collows that $H \cap U'_{2s} \neq \emptyset$ an plex hyperplane, and the vector z_1 . As $z_1 \in \{x_1 \cup z_2\}$ and Uz_2 is an open set $i \in \{0, 1\}$ follows that $H \cap U'_{z_2} \neq \emptyset$ and it is open. Therefore, we have that for each point $z_4 \in H \cap U''_{z_2}$ the line passi $U''_{z_0} + U'_{z_0}$. Consequently, each point $z_0 \in NM^+ \setminus M$ is path connected to the set M. As *M* is connected it follows that NM^+ is connected. Consequently, the set *NM* is a domain \blacksquare

In order to deduce hat *k1h* order complex left regular functions may be uniquely holomorphically extended to these domains we first require the following result.

Proposition 3: *For each closed path* $h: S^1 \to NM^+$ *there is a homotopy H*: $S^1 \times [0, 1]$ NM^+ such that for each closed pain $n: S^2 \to N/m$ increased nomology NM^+ such that for each $s \in S^1$ (i) $H(s, 0) = h(s)$ and (ii) $H(s, 1) \in M$.

Proof: As observed in the proof of Lemma 3 we prove in [23] that-for each point $z_0 \in M^+ \setminus M$ the set $N(z_0) \cap M$ is a manifold homeomorphic to the sphere S^{n-2} . It follows from the definition of a manifold of type one that for each point $z \in N(z_0) \cap M$ there does not exist any other point $z' \in N(z_0) \cap M$ such that $z' = z_0 + c(z-z_0)$ for, some $c \in \mathbb{C} \setminus \{0\}$. Consequently, we have that for each point $z_0 \in M^+ \setminus M$ there is a uni-.que non-zero complex number $c(z_0)$ such that $z_0 + c(z_0) (e_1 + ie_2) \in N(z_0) \cap M$. It now follows from the proof of Lemma 3 that for each path $h: S^1 \rightarrow NM^+$ we may produce a homotopy $H: S^1 \times [0, 1] \to NM^+,$ where $H(s, t) = h(s)$ for all s with $h(s)$ ϵ *M*, and $H(s, t) = h(s) + t c(z_0) (e_1 + ie_2)$, otherwise. This homotopy satisfies con-In order to deduce hat kth order complex left regular functions may be
holomorphically extended to these domains we first require the following r
Proposition 3: For each closed path $h: S^1 \to NM^+$ there is a homotopy $H: S \to$ some $c \in \mathbb{C} \setminus \{0\}$. Consequently, we have that for each point $z_0 \in$
que non-zero complex number $c(z_0)$ such that $z_0 + c(z_0)$ (e_1 -
now follows from the proof of Lemma 3 that for each path h
produce a homotopy

ditions (i) and (ii) \blacksquare
 Theorem 8: 'Suppose that f: $U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with

Theorem 8: 'Suppose that f: $U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with $k \geq n$, and *M* is a manifold of type one lying in $U_{\mathbb{C}}$. Then the function *f* has a unique *holomorphic continuation f*⁺ to the domain $U_{\mathbb{C}} \cup NM^+$.

Part B. We begin-by introducing the following extension to Theorem 6.

Theorem 9: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, U $n \mathbb{R}^n \neq \emptyset$ and M a manifold of type one lying in $U_\mathbb{C} \cap \mathbb{R}^n$. Then, for each point $x_0 \in M$, **Theorem 9:** Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular $\cap \mathbb{R}^n \neq \emptyset$, and M a manifold of type one lying in $U_{\mathbb{C}} \cap \mathbb{R}^n$. Then, for each we have
we have $f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k B_p G$

$$
f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k B_p G_p'(x_0 - x) W x D^{p-1} f(x),
$$

 $where \quad B_1 = 1, \quad B_p \quad is \quad a \quad constant$ $f(x_0) = 1 / \omega_n \int_{\partial M} \sum_{p=1}^k B_p G_p'$
where $B_1 = 1, B_p$ is a cons.
 $= (x^2)^{-1/2} x^{-n+1+p}$ for $1 \leq p \leq k$. *with* $DB_p G_p'(z) = B_{p-1}G'_{p-1}(x)$ *and* $G_p'(x)$ $\mathcal{L}_{\mathcal{L}}$

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The proof follows the same lines as the outline proof of Theorem 6 , so it is omitted.

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The proof follows the same lines as the outline proof of Theorem 6, so it

By equivalent arguments to those used to deduce Lemma 2 we also have

Lemma 4: For each real n-dimensional manifold of type one lyi **292** J. RYAN
 **The proof follows the same lines as the outline proof of Theorem 6, so it is omitted.
** *By* **equivalent arguments to those used to deduce Lemma 2 we also have

Lemma 4: For each real** *n***-dimensional manifol** Lemma 4: For each real *n*-dimensional manifold of type one lying in \mathbb{C}^n the expression $((z - z_0)(z - z_0))^{1/2}$ may be uniquely defined on the set $\partial M \times M$, where $z \in \partial M$ and $z_0 \in M$. *Z0 EM. . - .* 392 J. RYAN

The proof follows the same lines as the out
 By equivalent arguments to those used to
 Lemma 4: For each real n-dimensional man
 $((z - z_0) (z - z_0))^{1/2}$ may be *aniquely defin*
 $z_0 \in M$.

Consequently, we The proof follows the same lines as the outline proof of Theorem 6, so it is omitted.
By equivalent arguments to those used to deduce Lemma 2 we also have
Lemma 4: *For each real n-dimensional manifold of type one lying i* **f** follows the same lines as the outline proof of Theorem 6, s
alent arguments to those used to deduce Lemma 2 we also 1
4: For each real n-dimensional manifold of type one lying in $\mathbb{C}(z-z_0)^{1/2}$ may be uniquely def

Theorem 10: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, and *M* is a manifold of type one lying in $U_{\mathbb{C}}$. Then, for each point $z_0 \in M$ we have

$$
f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k B_p^l G_p^{'+}(z_0 - z) W_z D_C^{'p-1} f(z),
$$

where, $G_p{}^{\prime\,+}(z_o-z)$ is the holomorphic continuation to ∂M of $G_p{}^{\prime}(z_o-z)$, defined on $S^{n-1'} + z_0$, obtained via the homological equivalence in $\mathbb{C}^n \setminus N(z_0)$ of ∂M and $S^{n-1} + z_0$.

Observation 1: Both Definition 6 and 7 do not depend on the dimension of *M* being even. Also, the statement and proof of Lemma 3 and Proposition 2 are valid for odd-dimensional manifolds of type one. $f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^k B_p^1 G_p'^+(z_0 - z) WzD_{\mathbb{C}}^{p-1} f(z)$
where $G_p'^+(z_0 - z)$ is the holomorphic continuation to $\partial M^{n-1} + z_0$, obtained via the homological equivalence in \mathbb{C}^n .
Observation 1: Both Definition 6 *n* 6 and
d proof c
e one.
 $\lim_{z \to a} \exp(-iz)$
 $\lim_{z \to a} f^+, i$ $S^{n-1'} + z_0$, obtained via the homological equivalence in $\mathbb{C}^n \setminus N(z_0)$ of ∂M

Observation 1: Both Definition 6 and 7 do not depend on the d

being even. Also, the statement and proof of Lemma 3 and Proposition

o

Consequently, we have the following extension to odd dimensions of Theorem 8.

Theorem 11: Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, and $n = 1 \text{ mod } 2$. *Suppose also that* $M \subseteq U_{\mathbb{C}}$ *is a manifold of type one. Then the function f* has a unique holomorphic continuation, f^+ , to the domain $U_{\mathbb{C}} \cup NM^+$. *has a manifolds of type one lying in* U_C . Then, for each point $z_0 \in M$ we have $f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{k} B_p^1 G_p'^+(z_0 - z)$ $WzD_C^{p-1}f(z)$,

where $G_p'^+(z_0 - z)$ is the holomorphic continuation to ∂M of $G_p'(z_0 - z)$, **Theorem 11:** Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k -left regular $n = 1 \mod 2$. Suppose also that $M \subseteq U_{\mathbb{C}}$ is a manifold of type one. Then has a unique holomorphic continuation, f^* , to the domain U_{\math

Conformal invariance

. .

Proposition 4: *Suppose that f:* $U_{\mathbf{C}} \to A_n(\mathbf{C})$ is a complex 1-left regular function Then for each positive integer k the function $z^{k-1} f(z)$ is complex k-left regular. *anction.*
 $z^{k-1} f(z)$

-Proof: Suppose first that $k = 2p$. Then $D_{\mathbb{C}}^k = (-1)^p \left(\sum_{i=1}^n \frac{\partial^2}{\partial z_i^2} \right)$ and $z^{k-1} f(z)$ $= z(z_1^2 + \cdots + z_n^2)^{p-1} (-1)^{p-1} f(z)$. Now *? -*

and invariance

\nin this section by deducing the following result.

\nsition 4: Suppose that
$$
f: U_{\mathbb{C}} \to A_n(\mathbb{C})
$$
 is a complex 1-left regular function.

\nsoimpose first that $k = 2p$. Then $D_{\mathbb{C}}^k = (-1)^p \left(\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2} \right)$ and $z^{k-1} f(z)$.

\nSuppose first that $k = 2p$. Then $D_{\mathbb{C}}^k = (-1)^p \left(\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2} \right)$ and $z^{k-1} f(z)$.

\n
$$
\left(\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2} \right) \tilde{z}(z_1^2 + \cdots + z_n^2)^{p-1} f(z)
$$

\n
$$
= 2(p-1) z(z_1^2 + \cdots + z_n^2)^{p-2} f(z)
$$

\n
$$
+ 2(p-1) (2p-3) z(z_1^2 + \cdots + z_n^2)^{p-3} f(z)
$$

\n
$$
+ \sum_{j=1}^n z(z_1^2 + \cdots + z_n^2)^{p-2} 2(p-1) z_j \partial f(z) / \partial z_j
$$

\n
$$
= 3.
$$
 If $p = 1$, then it is straightforward to determine by direct calculation.

\n
$$
\sum_{j=1}^n \partial^2(z f(z) / \partial z_j^2 = 0.
$$

\n(8)

provided $p \geq 3$. If $p = 1$, then it is straightforward to determine by direct calculation that

a

$$
\sum_{j=1}^n \partial^2(z f(z)) / \partial z_j^2 = 0.
$$

 (8)

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If $p = 2$, then $\sum \partial^2 (z^3 f(z)) / \partial z_j^2 = 2z f(z) + 2z \sum z_j \partial f / \partial z_j$. It follows from expression (7) .that-

2, then
$$
\sum_{j=1}^{n} \frac{\partial^2 (z^3 f(z))}{\partial z_j^2} = 2z f(z) + 2z \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}.
$$
 It follows
\n
$$
\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}\right)^2 (z^3 f(z)) = 2 \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}\right) \left(z \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}\right).
$$

As *f(z)* is complex 1-left regular, $\left(\sum_{j=1}^{n} \frac{\partial^{2}}{\partial z_{j}}^{2}\right)\left(z \sum_{k=1}^{n} z_{k} \frac{\partial f}{\partial z_{k}}\right).$
 $\sum_{j=1}^{n} \frac{\partial^{2}}{\partial z_{j}}^{2} \left(z \sum_{k=1}^{n} z_{k} \frac{\partial f}{\partial z_{k}}\right) = 0.$ If, for $p \ge 3$, we, we, we have to see an the $z(z^{2})$ If $p = 2$, then $\sum_{j=1}^{n} \frac{\partial^2 (z^3 f(z))}{\partial z_j^2} = 2z f(z) + 2z \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}$. It follows from exp

(7) that
 $\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}\right)^2 (z^3 f(z)) = 2\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}\right) \left(z \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}\right)$.

As f allow the pth order complex Laplacian to act on the $z(z_1^2 + \cdots + z_n^2)^{p-1}$ part of $z(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, then it may be observed from expression (8) that this term is annihilated by this operator. As $f(z)$ is a complex 1-left regular function, we have that $f(z)$ is also annihilated by the complex Laplacian. Consequently $\left(\sum_{j=1}^n \partial^2/\partial z_j^2\right)^2 (z^3 f(z)) = 2$

As $f(z)$ is complex 1-left regular,

allow the pth order complex La
 $z(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, then it may

is annihilated by this operator.

have that $f(z)$ is also annihi
 $\left(\sum_{$ $z(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, then it may be observed is annihilated by this operator. As $f(z)$ is a have that $f(z)$ is also annihilated by the $\left(\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}\right)^p z \left(\sum_{j=1}^n z_j \frac{\partial}{\partial z_j}\right)^{p-1} f(z) = 0$. It ms *i* ion $\sum_{j=1}^{n} \frac{\partial^2(z^3 f(z))}{\partial z_j^2} = 2z f(z) + 2z \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}$. It follows from expression $\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} \left(z^3 f(z) \right) = 2 \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} \right) \left(z \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k} \right)$.

complex 1-left that, *'I n '* low the $z_1^2 + \cdots$ annihi

are the annihistic strategy of $\sum_{i=1}^n \frac{\partial^2}{\partial z_i}$

and Sunney 1-left regular, $\sum_{j=1}^{n} \frac{\partial^2}{\partial z}$
 I complex Laplacian t
 I (*z*), then it may be obs

his operator. As $f(z)$ is

also annihilated by
 $z_j \frac{\partial}{\partial z_j}$ $f(z) = 0$. It
 $\int^p z(-1)^{p-1} (z_1^2 + \cdots + z_n^2)^p f(z) = 2pz(z_1)$
 $\$

$$
\left(\sum_{j=1}^{n} \frac{\partial^{2}}{\partial z_{j}}^{2}\right)^{p} z(-1)^{p-1} (z_{1}^{2} + \cdots + z_{n}^{2})^{p-1} f(z) = 0.
$$

Suppose now that $k = 2p + 1$. Then, as $f(z)$ is a complex 1-left regular function we Suppose now that $k = 2p + 1$. Then, as $f(z)$ is a complex 1-left regular function we
have that $D_C(z_1^2 + \cdots + z_n^2)^p f(z) = 2pz(z_1^2 + \cdots + z_n^2)^p \zeta^{-1} f(z)$, and it follows from
the previous arguments that
 $\left(\sum_{j=1}^n \partial^2/\partial z_j^2\right)^$ the previous arguments that $\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}\right)^p z(-1)^{p-1} (z_1^2 + \cdots + z_n^2)^{p-1} f(z) = 0.$

is now that $k = 2p + 1$. Then, as $f(z)$ is a complex 1-left regular function we

t $D_C(z_1^2 + \cdots + z_n^2)^p f(z) = 2pz(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, and it follows fr

$$
\left(\sum_{j=1}^n \partial^2/\partial z_j^2\right)^p 2pz(z_1^2+\cdots+z_n^2)^{p-1} f(z)=0.
$$

Corollary: *Suppose that for* $0 \le l \le k - 1$ *the functions* $f_l: U_{\mathbb{C}} \to A_n(\mathbb{C})$ *are complex 1-left regular. Then the function*
 $\mathbb{F}: I \times \mathbb{R}^d \times \mathbb{C} \times \mathbb{R}^d \times \mathbb{R}^d$ (C) \mathbb{F}^{k-1} *plex 1-left regular. Then the function* $(k+2n^2)^{p-1} f(z) = 0.$
 $k \leq k$ is the state of the function \blacksquare
 $\sum_{k=0}^{k-1} z^k f_k(z)$

$$
F: U_{\mathbb{C}} \to A_n(\mathbb{C}): F(z) = \sum_{l=0}^{k-1} z^l f_l(z)
$$

is complex k-left regular.

Observation 2: It is not the case that every complex k -left regular function can be expressed in the form (9). From Theorem 4 we have that for *n* even each complex - 1-left regular function has a unique holomorphic continuation from a neighbourhood of a manifold of type one to its cell' of harmonicity. however, as observed earlier; or a manifold of type one to its cell of narmonicity. However, as observed earlier,
 $\log (-z^2)$ is a complex *n*-left regular function which is not uniquely defined on
 $A(1/2, 3/2)^+$.

As \mathbb{C}^n is contractible to a p $A(1/2,3/2)^+$.

As \mathbb{C}^n is contractible to a point we have from Theorems 7 and 10

Proposition 5: Suppose that $P_{(q)}$: $\mathbb{C}^n \to A_n(\mathbb{C})$ *is a complex, k-left regular polynomi'al, homogeneous of degree q with respect to the origin. Then*

$$
P_{(q)}(z) = \sum_{l=0}^{k-1} z^l P_l(z),
$$

where P_i is a complex 1-left regular function, homogeneous of degree $q - l$.

As the disc lying in \mathbb{R}^n , of radius $r \in \mathbb{R}^+$, is contractible within itself to a point, it follows from Theorems 8 and 11, Proposition 5 and the Taylor expansion given in [7, Theorem 10] that

S
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S

Theorem' 12: Suppose that $K(r)^+$ is the Lie ball of radius $r \in \mathbb{R}^+$, lying in \mathbb{C}^n and *f*: $K(r)^+ \rightarrow A_n(\mathbb{C})$ is a complex k-left regular function. Then
 $f(z) = \sum_{l=0}^{k-1} z^l f_l(z)$,

 $\frac{1}{\sqrt{2}}$

 $\mathcal{P}(\mathcal{S}) = \mathcal{P}(\mathcal{S})$, where $\mathcal{S}(\mathcal{S}) = \mathcal{S}(\mathcal{S})$

-

$$
f(z) = \sum_{l=0}^{k-1} z^l f_l(z),
$$

for each $z \in K(r)^+$, where each $f_i: K(r)^+ \to A_n(\mathbb{C})$ is a complex 1-left regular function. **•** Theorem' 12: Suppose that $K(r)^+$ is the Lie ball of radius $r \in \mathbb{R}^+$, lying in \mathbb{C}^n *a f*: $K(r)^+ \rightarrow A_n(\mathbb{C})$ is a complex k-left regular function. Then
 $f(z) = \sum_{l=0}^{k-1} z^l f_l(z)$,

for each $z \in K(r)^+$, where As $D_{\mathbb{C}}^k$ is a constant coefficient differential operator we have

Proposition 6: Suppose that f: $U_{\mathbf{C}} \to A_n(\mathbf{C})$ is a complex k-left regular function with *respect to the variable* $w = z + z_0$ *for some constant* $z_0 \in \mathbb{C}^n$. Then *f* is complex k-left *regular with respect to the variable z*. **794** *f <i>r regular 1. Ryand* $f: K(r)^+ \rightarrow A_n(\mathbb{C})$ *is a complex k-left regular function. Then***
** *f* $f: K(r)^+ \rightarrow A_n(\mathbb{C})$ **is a complex k-left regular function. Then
** $f(z) = \sum_{l=0}^{k-1} z^l f_l(z)$ **,
** *for each* $z \in K(r)^+$ **, wh** From 12: Suppose that $K(r)^+$ is the Lie ball of radius $K(r)^+ \rightarrow A_n(\mathbb{C})$ is a complex k-left regular function. Then
 $f(z) = \sum_{l=0}^{k-1} z^l f_l(z)$,

cach $z \in K(r)^+$, where each $f_l: K(r)^+ \rightarrow A_n(\mathbb{C})$ is a complex

As $D_{\mathbb{C}}^k$ Theorem 12: Suppose that $K(Y)^*$ is the Lie ball of radius $r \in \mathbb{R}^n$, tying in \mathbb{C}^n *a f* $K(Y)^+ \rightarrow A_n(\mathbb{C})$ is a complex k-left regular function. Then
 $f(z) = \sum_{i=0}^{k-1} z^i f_i(z)$,

for each $z \in K(Y)^*$, where each $\begin{array}{l} \text{constant}^i \text{co}\ \text{in} \ \text{$ $f(z) = \sum_{l=0}^{k-1} z^l f_l(z),$
 $f \in K(r)^+$, where each $f_l: K(r)^+ \rightarrow A_n(\mathbb{C})$ is a complex 1-left regular function.

^{*f*} is a constant coefficient differential operator we have

tion 6: Suppose that $f: U_c \rightarrow A_n(\mathbb{C})$ is a complex onstant coefficient differential operator we have

3: Suppose that $f: U'_\mathbf{C} \to A_n(\mathbf{C})$ is a complex k-left regular function

triable $w = z + z_0$ for some constant $z_0 \in \mathbf{C}^n$. Then f is comple-

pect to the variabl *tor* each $z \in K(r)$, where each f_1 , $K(r) \rightarrow A_n(\mathbb{C})$ is a constant

As $D_{\mathbb{C}}^k$ is a constant coefficient differential operato

Proposition 6: Suppose that $f: U_{\mathbb{C}} \rightarrow A_n(\mathbb{C})$ is a comprespect to the variable $w =$ on 6: Suppose that $f: U_C \rightarrow A_n(\mathbb{C})$ is a complex k-left regue

e variable $w = z + z_0$ for some constant $z_0 \in \mathbb{C}^n$. Then f

respect to the variable z.

position 6 and Theorem 12 we have
 $\begin{aligned} &13: Suppose \ that \ K(z_0, r)^+ \ is \ the \ Lie \$

Theorem 13: *Suppose that* $K(z_0, r)^+$ is the Lie ball of radius $r \in \mathbb{R}^+$ centred at $z_0 \in \mathbb{C}^n$, and $f: K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular function. Then

$$
f(\mathbf{k}) = \sum_{l=0}^{k-1} (z - z_0)^l f_l(z - z_0),
$$

for each z $\in K(z_0, r)^+$, where each $f:_{\mathfrak{l}} K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex 1-left regular function.

As $D_{\mathbb{C}}^k$ is a homogeneous differential operator we have

Proposition 7: *Suppose that f(w) is a complex k-left regular function with respect to' the variable w =* λz *, where* $\lambda \in \mathbb{C} \setminus \{0\}$ *. Then* $f(\lambda z)$ *is a complex k-left regular function with regular with respect to the variable z.*

From Proposition 6 and Theore

Theorem 13: Suppose that $K(z_0, r$

and $f: K(z_0, r)^+ \rightarrow A_n(\mathbb{C})$ is a comp
 $f(\mathbf{k}) = \sum_{i=0}^{k-1} (z - z_0)^i f_i(z - z_i)$

for each $z \in K(z_0, r)^+$, where each $f(\mathbf{k}) = \sum_{k=0}^{k-1} (z -$
for each $z \in K(z_0, r)^+$, with
 \leq tion.
As $D_{\mathbb{C}}^k$ is a homogen
Proposition 7: Supp-
the variable $w = \lambda z$, where
respect to the variable z.
We now deduce the P
begin with
Proposition 8: Supp

We now deduce the $Pin(\mathbb{C}^n)$ invariance of the complex k-left regular functions. We begin with

Proposition 8: *Suppose that* $a \in Pin(\mathbb{C}^n)$ and $K(\tilde{a}z_0a, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$ *and centred at az*₀ $\tilde{a} \in \mathbb{C}^n$, Suppose also that $f: K(a z_0 \tilde{a}, r)^+ \to A_n(\mathbb{C})$ is a complex k-left *reqular function with respect to the variable az* $\tilde{a} \in K(a z \tilde{a}, r)^+$ *. Then the function* $\tilde{a} f(a z \tilde{a})$ *i f*(x) = $\sum_{l=0}^{n} (z - z_0)^l f_l(z - z_0)$,
 for each z $\in K(z_0, r)^*$, where each *f*: $K(z_0, r)^* \rightarrow A_n(\mathbb{C})$ *is a complex 1-left regular function*

As D_C^k is a homogeneous differential operator we have

Proposition 7: deduce the $Pin(\mathbb{C}^n)$ invariance of the complex *k*-left regular functions. We
 h
 tion 8: Suppose that $a \in Pin(\mathbb{C}^n)$ *and* $K(\bar{a}z_0a, r)^+$ *is a Lie ball of radius* $r \in \mathbb{R}^+$ *

<i>ed at* $az_0\bar{a} \in \mathbb{C}^n$ *, Suppose x*, where $\lambda \in \mathbb{C} \setminus \{0\}$. Then $f(\lambda z)$ is
 ile z.
 ile z.
 ne Pin(\mathbb{C}^n) invariance of the c
 i \int
 1 neppose that $a \in Pin(\mathbb{C}^n)$ and $K(\in \mathbb{C}^n$, Suppose also that $f: K(\infty)$
 ight respect t begin with

Proposition 8: Suppose that $a \in Pin(\mathbb{C}^n)$ and $K(\tilde{a}z_0a, r)^+$ is a Lie ball of radius

and centred at $az_0\tilde{a} \in \mathbb{C}^n$, Suppose also that $f: K(az_0\tilde{a}, r)^+ \to A_n(\mathbb{C})$ is a comple

regular function wit

Proof: We have from Theorem 13 that $f(a\tilde{z}\tilde{a}) = \sum_{n=0}^{\infty} (az\tilde{a} - az_0\tilde{a})^t f_t(az\tilde{a})$, where Left regular with respect to the variable z .
have from Theorem 13 that $f(a\overline{z}\overline{a}) = \sum_{l=0}^{k-1}$
complex 1 left reqular function with respect each f_i is a complex 1-left regular function with respect to the variable $az\tilde{a}$. Now.

$$
\tilde{a} f(az \tilde{a}) = \sum_{l=0}^{k-1} \tilde{a} (az \tilde{a} - az_0 \tilde{a})^l f_l(az \tilde{a}) = \sum_{l=0}^{k-1} (a \tilde{a})^l (z - z_0)^l \tilde{a} f_l(az \tilde{a}). \qquad (10)
$$

It follows from Theorem 5 that each $\tilde{a}f_i(az\tilde{a})$ is a complex 1-left regular function with respect to the variable *z*. From expression (10) and the corollary to Proposition 4 we now have that the function $\tilde{a}/(az\tilde{a})$ is complex k -left regular \blacksquare

For each domain $U_{\mathbb{C}}$ and each $a \in Pin(\mathbb{C}^n)$ we may take sets of points $\{z_n\}_{n=0}^{\infty} \subset$ *• and* f_l *is a complex 1-left regular function with respect to the variable* $a z \bar{a}$ $\bar{a} f(a z \bar{a}) = \sum_{l=0}^{k-1} \bar{a} (a z \bar{a} - a z_0 \bar{a})^l f_l(a z \bar{a}) = \sum_{l=0}^{k-1} (a \bar{a})^l (z - z_0)^l \bar{a} f_l(a z \bar{a})$ *.***
** *•* **It follows from Th** Proof: We have from Theorem 13 that $f(az\tilde{a}) = \sum_{l=0}^{k-1} (az\tilde{a} - az_0\tilde{a})^l f_l(az\tilde{a})$, where
each f_l is a complex 1-left regular function with respect to the variable $az\tilde{a}$. Now.
 $\tilde{a}/(az\tilde{a}) = \sum_{l=0}^{k-1} \tilde$ $\tilde{a}_f(az\tilde{a}) = \sum_{l=0}^{k-1} \tilde{a}(az\tilde{a} - az_0\tilde{a})^l f_l(az\tilde{a}) = \sum_{l=0}^{k-1} (a\tilde{a})^l (z - z_0)^l \tilde{a} f_l(iz\tilde{a})$

1t follows from Theorem 5 that each $\tilde{a}f_l(az\tilde{a})$ is a complex 1-left regrespect to the variable z. Fr

Theorem 14: *Suppose that* $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function with *• respect to the variable azã. Then, the function f_a :* $\tilde{a}U_{\mathbb{C}}a \rightarrow A_{n}(\mathbb{C})$ *:* $f_{a}(z) = \tilde{a}/(az\tilde{a})$ *is complex k-left regular with respect to the variable z.*

Observation 3: Theorem 14 is also a consequence of Theorem 1. However, the methods used here to establish Theorem .14 differ from those used in [8] to establish Theorem 1. Later in this section we shall adapt the methods used here to establish Theorem 14 to deduce other results which are not consequences of Theorem 1.

In the cases where k is even the differentiable operator $D_{\mathbf{C}}{}^{\bm{k}}$ is an iterate of the complex Laplacian. As $a \in Pin(\mathbb{C}^n)$ is a constant, it follows from Theorem 14, or by direct calculation, that the function *f(azã)* is complex k-left regular with respect to the varia-Blead Dirac Iterated Dirac Iterated Dirac Iterated Dirac Iterated Dirac Iterated Dirac Iterated Dirac Iteration 3: Theorem 14 is also a consequence of The methods used here to establish Theorem 14 differ from those Theore

Observation 4: Proposition 8 may also be deduced by using Lemma 1 and directly. applying the iterated complex Laplacian to the function $f_{1,a}(z)$.

We now use our previous arguments to deduce the invariance of complex k-left regular functions under inversion. We begin with

Proposition 9: *Suppose that,* $K(z_0, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$, and centred at z_0 , and lying in $\mathbb{C}^n \setminus N(0)$. Suppose also that $f: K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular d *function in the variable* $w = z^{-1}$. Then the function ion 9: *Suppose that*, $K(z_0, r)^+$ *is a Lie ball of radius r*
 in $\mathbb{C}^n \setminus N(0)$. *Suppose also that* $f: K(z_0, r)^+ \to A_n(\mathbb{C})$ *is*
 in the variable $w = z^{-1}$. Then the function
 $inv(f): K^{-1}(z_0, r)^+ \to A_n(\mathbb{C})$: $inv(f)(z) = J_k(z$

inv
$$
(f): K^{-1}(z_0, r)^{\dagger} \to A_n(\mathbb{C}): \text{inv } (f) (z) = J_k(z) f(z^{-1}),
$$

• is complex k-left regular with respect to the variable z, where $K^{-1}(z_0, r)^{\dagger} = \{z \in \mathbb{C}^k \}$
 $z^{-1} \in K(z_0, r)^{\dagger} \}$ and $J_k(z) = G_1^{\dagger}(z)$ z^{k-1} for n even, $J_k(z) = G_1^{\dagger}(z)$ z^{k-1} for n odd.

Proof: From Theorem 13 we have that $f(z^{-1}) = \sum_{n=0}^{k-1} (z^{-1} - z_0)^l f_l(z^{-1})$, where each er inversion. We begin with

the mose that $K(z_0, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$, and centred at z_0 ,

Suppose also that $f: K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular
 $x \cdot w = z^{-1}$. Then the function
 $v,$ $f_i(z^{-1})$ is a complex k-left regular function in the variable z^{-1} . Now consider the function
 $\frac{k-1}{z} f(z) \frac{z^{-1}}{z^{-1}} f(z) \frac{z^{-1}}{z^{-1}} = z^{-1} f_i(z^{-1})$ We now use our previous arguments to deduce the invariance of complement equilar functions under inversion. We begin with

Proposition 9: Suppose that $K(z_0, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$, and central their an *Jk*(*f*): $K^{-1}(z_0, r)^{\dagger} \rightarrow A_n(\mathbb{C})$: $\text{inv}(f)(z) = J_k(z) f(z^{-1}),$
 xk-left regular with respect to the variable z, where $K^{-1}(z_0, r)^{\dagger} = \{z \in \mathbb{C}^k \setminus N(0) :$
 $\text{var}(z) = G_1^{-1}(z) z^{k-1} \text{ for } n \text{ even, } J_k(z) = G_1^{-1}(z) z^{k-1} \text{ for } n \text{$ Suppose also that $f: K(z_0, r)^+ \rightarrow A_n(\mathbb{C})$ is a complex k -left regular
 $w = z^{-1}$. Then the function
 $f, r)^{\dagger} \rightarrow A_n(\mathbb{C})$: inv $(f) (z) = J_k(z) f(z^{-1})$,
 x with respect to the variable z, where $K^{-1}(z_0, r)^{\dagger} = \{z \in \mathbb{C}^n \setminus$

$$
\begin{aligned} \n\chi(z) \, f(z^{-1}) &= \sum_{i=0}^{k-1} J_k(z) \, (z^{-1} - z_0)^i \, f_l(z^{-1}) \\ \n&= \sum_{i=0}^{k-1} J_k(z) \, \left(z / (z_1^2 + \dots + z_n^2) - z_0 \right)^i \, f_l(z^{-1}). \n\end{aligned}
$$

 $=\sum_{l=0}J_k(z)\left(\frac{z}{(z_1^2+\cdots+z_n^2)}-z_0\right)^l f_l(z^{-1}).$ (11)
For the case where $k=1$ and n is even the result is a direct consequence of Theorem 5. When $k = 1$ and *n* is odd, the result follows from direct analogues of the arguments given in [24]. $=\sum_{l=0}^{k-1} J_k(z) \left(z/(z_1^2) + \right.$

case where $k = 1$ and n is even
 $\frac{1}{k} = 1$ and n is odd, the result for $\left[\frac{24}{2}\right]$.

ose now that k is odd and greater

obtain, on placing $k = 2p + 1$,
 $(zz_0)^p J_1(z) f_{k-1}(z^{-1$ zase where $k = 1$ and n is even the result is a direct consequence of Theorem $k = 1$ and n is odd, the result follows from direct analogues of the arguments [24].

i.e now that k is odd and greater than one. Then on

Suppose now that k is odd and greater than one. Then on expanding the expression (11),'we obtain, on placing $k = 2p + 1$, the terms given ii
•, Supp
•, 11), we
•, and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **S** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

and

•

$$
z_0(zz_0)^{p-1} J_2(z) f_{k-1}(z^{-1}).
$$

(12)

As $J_1(z)$ $f_{k-1}(z^{-1})$ and $J_2(z)$ $f_{k-1}(z^{-1})$ are both complex harmonic functions with respect to the variable *z*, then it follows from the anticommutation relationship within the Clifford algebra, and the order of the operator. $D_{\mathbb{C}}^k$ that both expressions (12) and (13) are annihilated by this operator. The expansion of (11) also contains the term' $z^{p-1}z_0^pJ_2(z) f_{k-1}(z^{-1})$ and as $J_2(z) f_{k-1}(z^{-1})$ is a complex harmonic function in the variable *z*, it follows from the order of the operator $D_{\mathbb{C}}^*$ and the anticommutation relationship within the Clifford algebra that this expression is also annihilated by this operator. • . The expansion of (11) also contains the term $z^p J_1(z) f_{k-1}(z^{-1})$. As $J_1(z) f(z^{-1})$ is a complex 1-left regular function in the variable *z*, it follows from the corollary to Proposition 4 that this expression is also annihilated by the operator $D_{\mathbb{C}}^k$. As $z_0^2 \in \mathbb{C}$ and within the Clifford algebra that The expansion of (11) also contain
plex 1-left regular function in the
tion 4 that this expression is also **• that** this expression is also annihilated by this operator.
 • contains the term $z^p J_1(z) f_{k-1}(z^{-1})$ **. As** $J_1(z) f(z^{-1})$ **is a com-

in the variable z, it follows from the corollary to Proposition also annihilated by**

 $z^2 \in \mathbb{C}$, it may be observed that all other terms appearing in the expansion of (11) are of the form $\lambda_1(zz_0)^q J_1(z) f_1(z)$, $\lambda_2 z_0(zz_0)^{q_1} J_2(z) f_1(z)$, $\lambda_3 z^{q_1} z^{q_1} J_2(z) f_1(z^{-1})$ and $\lambda_4 z^{q_1} J_1(z)$ $x \times f_1(z^{-1})$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, q,q₁,q₂,q₃,q₄ $\in \mathbb{N}$, with $q \leq p$, q₁ $\leq p - 1$,
 $\lambda_1(z^{-1})$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, q,q₁,q₂,q₃,q₄ $\in \mathbb{N}$, with $q \leq p$, q₁ \leq $q_2 \leq 2p - 1$, $q_3 \leq p$ and $q_4 \leq p$. Consequently, we have that expression (11) is annihilated by the operator $D_{\mathbb{C}}^k$ when *k* is odd. '

When *k* is even, we have, on placing $k = 2p + 2$, that the expansion of expression (11) contains the terms $(zz_0)^p J_1(z) f_{k-1}(z^{-1}), z_0(zz_0)^{p-1} J_2(z) f_{k-1}(z^{-1}), z_0^2(zz_0)^p J_2(z) f_{k-1}(z)$ and $(z_0 z)^p z_0 J_2(z) f_{k-1}(z^{-1})$. By similar considerations to those used for the case where *k* is odd it may be deduced that all of these terms are annihilated by the operator $D_{\mathbb{C}}^k$. By similar arguments to those used in the first part of this proof it now follows that the expression (11) is annihilated by the operator $D_{\mathbb{C}}^k$ when *k* is even **U** be of the form $\lambda_1(2s_0)^q J_1(z) f_1(z)$, $\lambda_2 z_0(z_2)^{q_1} J_2(z) f_1(z)$, $\lambda_2 z^{q_1} z^{q_1} J_2(z)$, $f_1(z^{-1})$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, $q_1, q_2, q_3, q_4 \in \mathbb{N}$, with $\leq 2p - 1$, $q_3 \leq p$ and $q_4 \leq p$. Conseque

Note: As the map inv: $K^{-1}(z_0, r) t \rightarrow K(z_0, r) t : z \mapsto z^{-1}$ is a diffeomorphism and the domain $K(z_0, r)$ is contractible within itself to a point it follows that in the cases where *n* is odd the function $J_k(z) f(z^{-1})$ is uniquely defined on $K^{-1}(z_0, r)$ [†].

Theorem 15: Suppose that *n* is even, and that $U_{\mathbb{C}}$ is a domain lying in $\mathbb{C}^{n}\setminus N(0)$. *Suppose also that* $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex k-left regular function. Then* m 15: Suppose that *n* is even, and that U_C is
nlso that $f: U_C \rightarrow A_n(\mathbb{C})$ is a complex k-left regu
inv (f): $U_C^{-1} \rightarrow A_n(\mathbb{C})$: inv (f) (z) = $J_k(z)$ f(z⁻¹) *n* is odd the function $J_k(z) f(z^{-1})$ is uniquely defined on $K^{-1}(z_0, r)^{\dagger}$.

Using Proposition 9 we may now deduce

Theorem 15: Suppose *that n is even, and that* U_C is a domain lying in $\mathbb{C}^n \setminus N(0)$.

Suppose als

$$
\operatorname{inv}_1(f): U_{\mathbb{C}}^{-1} \to A_n(\mathbb{C}) : \operatorname{inv}(f)(z) = J_k(z) f(z^{-1})
$$

Then, on restricting *f* to $K(z_q, r_q)$ [†] for each q we have from Proposition 9 that inv(f): $K^{-1}(z_q, r_q)$ ^t $\rightarrow A_n(\mathbb{C})$ is a complex k-left regular function. Moreover, $U_q K^{-1} (z_q, r_q)$ [†] = $U_{\mathbb{C}}^{-1}$. As *n* is even $I_k(z) = G_k$ [†](z) z^{k-1} is uniquely defined on $U_{\mathbb{C}}^{-1}$. Consequently, $inv(f): U_{\mathbb{C}}^{-1} \to A_n(\mathbb{C})$ is a uniquely defined conplex k-left regular function \blacksquare $i s \ a \ comj \ \text{Proof} \ \text{Then,} \ \text{o} \ \text{inv}(f): \ H \cup_{q} K^{-1}(\text{Consequ}) \ \text{function} \ \text{function} \ \text{for} \ \text{coneq} \ \text{Conequ}$ Theorem 15: Suppose that *n* is even, and that U_C is a domain lying in $\mathbb{C}^n \setminus N(0)$.

Suppose also that $f: U_C \to A_n(\mathbb{C})$ is a complex *k*-left regular function. Then
 $\text{inv.}(f): U_C^{-1} \to A_n(\mathbb{C})$: $\text{inv.}(f)(z) = J_k(z) f(z^{-1$

On combining Propositions 6, 7 and 9, and Theorems 14 and 15 and [24] it is ..

also that the function $f((az + b) (cz + d)^{-1})$ is complex k-left regular in the variable **•** Consequently, \cdot **iii** \cdot *C_C* \cdot → $A_n(\mathbf{c})$ is a uniquely defined conplex *k*-left regular function **i**
 b Con combining Propositions 6, 7 and 9, and Theorems 14 and 15 and [24] it is straightforward to *k-left regular in the variable z, where* $f(xz + b)$ $(cz + d)^{-1}$. Then the function $J_k(cz + d) f((az + b) (cz + d)^{-1})$ is complex or each q we have from Proposition 9 that

omplex *k*-left regular function. Moreover,
 $I_k(z) = G_l^{\dagger}(z) z^{k-1}$ is uniquely defined on U_C^{-1} .

is a uniquely defined conplex *k*-left regular

and 9, and Theorems 14 and 15 $inv(f)$: K
 $\bigcup_q K^{-1}(z_q)$
Conseque
function
function
on constraight
 f
 $\bigcap_{q \geq 0}$
 $\bigcap_{r \geq$ *J*_{*k*}(α _{*i*})^{*t*} $= U_0^{-1}$. As *n* is even $I_k(z) = G_1!(z) z^{k-1}$ is uniquely defined on U_0^{-1} .
 *A*_{*k*}(*f*) \cdot *J*_{*k*}(*f*) \cdot *C_{<i>k*}⁻¹ \cdot *A_n*(**C**) is a uniquely defined conplex *k*-left regular On combining Propositions 6, 7 and 9, and Theorems 14 and 15 and [24] it is

straightforward to deduce

Theorem: 16 *Suppose that n is even, and that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a Clifford matrix. Suppose*

also that the f $f(x+e^2h + b)(cz + d)^{-1}$ is
 $f(x+e^2h + d)^{-1}$ i

$$
J_k(cz + d) = ((cz + d) (cz + d))^{(k-n)/2}, \quad k \text{ even}, \tag{14}
$$

$$
J_k(cz + d) = (cz \stackrel{\sim}{+} d) ((cz + d) (cz + d))^{(k-1-n)/2}, \quad k \text{ odd}. \tag{15}
$$

As *h* is even, it may be observed from expressions (14) and (15) that the functions $J_k(cz+d)$ are uniquely determined over each domain in \mathbb{C}^n for each *k*. $J_k(cz + d) = ((cz + d) (cz \nightharpoonup d))^{(k-n)/2}, \quad k$
 $J_k(cz + d) = (cz \nightharpoonup d) ((cz + d) (cz \nightharpoonup d))^{(k)}$

As *n* is even, it may be observed from expression
 $J_k(cz + d)$ are uniquely determined over each dom

Definition 8: For each domain $U_C \subseteq \mathbb{C}^$

$$
\{f\colon U^\mathbf{C}\to A_n(\mathbf{C})\colon f\text{ is a complex }k\text{-}left\text{ regular function}\}
$$

is denoted by $\Gamma_{l,k}(U_{\mathbb{C}})$.

It may be noted that this set is a right module over the algebra $A_n(\mathbb{C})$.

- Theorem 17: *Suppose that n, is even and that* $U_{1,\mathbb{C}}$ *and* $U_{2,\mathbb{C}}$ *are domains lying in* \mathbb{C}^n *such that for some Clifford matrix* $\binom{a}{c}$ *the transformation* $T_{[a\ b]}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}$: *Lierated Dirac Operator (i)*
 and that $U_{1,\mathbb{C}}$ *and* $U_{2,\mathbb{C}}$ *are dom*
 (a b) the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$
 phism. Then the two modules $\Gamma_{l,\mathbf{r}}$
 x) $\vec{T}_{l,\mathbf{r}}(f)$ (*z*) $(z \mapsto (az + b)$ $(cz + d)^{-1}$ *is a homeomorphism. Then the two modules* $\Gamma_{l,k}(U_{1,\mathbb{C}}), \Gamma_{l,k}(U_{2,\mathbb{C}})$ **are in the interest in the linear map**
 a a c c *cd cd c*

7

$$
\begin{aligned} \vec{T} \left(\begin{matrix} a & b \\ c & d \end{matrix} \right) &\colon \Gamma_{l,k}(U_{2,\mathbb{C}}) \to \Gamma_{l,k}(U_{1,\mathbb{C}}) &\colon \ \vec{T} \left(\begin{matrix} a & b \\ c & d \end{matrix} \right) (f) \ (z) \\ & = J_k(cz+d) \ f \big((az+b) \ (cz+d)^{-1} \big) .\end{aligned}
$$

We now return to consider the cases where *n is* odd. We begin with

Theorem 18: Suppose that $U_{\mathbf{C}}$ **is a domain lying in** $\mathbb{C}^n\setminus N(0)$ **and** $U_{\mathbf{C}}$ **is contractible** *within itself to a point. Then' for each conplex k-left regular function f:* $U_{\mathbb{C}} \to A_n(\mathbb{C})$ the *function* inv (*f*): $U_{\mathbb{C}}^{-1} \to A_n(\mathbb{C})$: inv (*f*) (z) = $J_k(z)$ *f*(z⁻¹) is complex k-left regular.

Proof: As $U_{\mathbb{C}}$ is contractible within itself to a point it follows from the homeomorphism inv: $U_{\mathbf{C}}^{-1} \to U_{\mathbf{C}}$: $z \mapsto z^{-1}$ that $U_{\mathbf{C}}^{-1}$ is also contractible within itself to a point: Consequently, $J_k^c(z)$ is a well-defined function on U_c^{-1} . The result now follows from the proof of Theorem 15 I

More generally we have

Theorem 19: Suppose that n is odd and that, $U_C = \sum_{i=1}^{n} I_i$ is a well-defined function on U_C^{-1} . The result now follows from the proof of Theorem 15 \blacksquare

More generally we have

Theorem 19: Suppose that n is odd *the transformation* $T_{\begin{pmatrix} a & b \ c & d \end{pmatrix}}: J(U_{\mathbb{C}}) \to U_{\mathbb{C}}: z \mapsto (az + b)$ $(cz + d)^{-1}$ is, a diffeomorphism. *Hible within itself to*
the transformation
Then, for f: $U_{\mathbb{C}} \rightarrow$ *f*: $U_{\mathbb{C}} \to A_n(\mathbb{C})$ a complex k-left regular function the function
 $J(f): J(U_{\mathbb{C}}) \to A_n(\mathbb{C}): J(f)(z) = J_k(cz + d) f((az + b) (cz + d))^{-1}$ **Proof:** As U_C is contractione within itself to a point it is

phism inv: $U_C^{-1} \rightarrow U_C$: $z \mapsto z^{-1}$ that U_C^{-1} is also contrac

Consequently, $J_k^C(z)$ is a well-defined function on U_C^{-1} . The

proof of Theorem 15 \blacks **(d) (d) (d)**

$$
J(f): J(U_{\mathbb{C}}) \to A_n(\mathbb{C}): J(f)(z) = J_k(cz+d) f((az+b) (cz+d))^{-1}
$$

$$
J_k(cz+d) = \begin{cases} ((cz+d) (cz \widetilde{+} d))^{(k-n)/2} & \text{for } k \text{ even,} \\ (cz \widetilde{+} d) ((cz+d) (cz \widetilde{+} d)^{(k-1-n)/2} & \text{for } k \text{ odd.} \end{cases}
$$

Outline proof: As $U_{\mathbb{C}}$ is contractible within itself to a point, then it follows that $J(U_{\mathbb{C}})$ is also contractible within itself to a point. Consequently, the function $J(U_{\mathbb{C}})$ is also contractible within itself to a point. Consequently, the function $J_k(cz + d)$ is well defined on the domain $J(U_{\mathbb{C}})$. The result now follows from the same reasonings as used to establish Theorem 16 \blacksquare

Theorem 19 also, gives us

Theorem^{'20}: *Suppose that n is 'odd and that* $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such *codd and that* $U_{1,\mathbb{C}}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation that the control of the transformation of the control of t **theorem 19 also gives us

Theorem 20:** Suppose that *n* is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such

that for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation $T_{\begin{pmatrix} a & b \\ c & d \$ *COMBIN 10* α *COMBIN*. Consequently
 COMBIN 10 α *COMBIN*
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 COMBIN 10 \mapsto $(az + b)$ $(cz + d)^{-1}$ is a homeomorphism. Suppose also that the domain $U_{1,C}$ is con*tractible within itself to a point. Then the two modules* $\Gamma_{l,k}(U_{1,C})$ *and* $\Gamma_{l,k}(U_{2,C})$ *are isomorphic via the linear map I_k*($cz + d$) = $\begin{cases} ((cz + d) (cz + d))^{(k-1)/2} & for k even, \\ (cz + d) ((cz + d) (cz + d)^{(k-1-n)/2}) & for k odd. \end{cases}$

Outline proof: As $U_{\mathbb{C}}$ is contractible within itself to a point, then if $J(U_{\mathbb{C}})$ is also contractible within itself to a point. Outline proof: As $U_{\mathbb{C}}$ is contractible within itself
 $J(U_{\mathbb{C}})$ is also contractible within itself to a poi
 $J_k(cz+d)$ is well defined on the domain $J(U_{\mathbb{C}})$. The

reasonings as used to establish Theorem 16

T also contractible within itself to a point. Co
) is well defined on the domain $J(U_{\mathbb{C}})$. The result is
as used to establish Theorem 16 \blacksquare
m 19 also gives us
n'20: *Suppose that n is 'odd'and that* $U_{1,\mathbb{C}}$ and

$$
\begin{aligned}\n\vec{T}_{\begin{pmatrix}a & b\\ c & d\end{pmatrix}}&: \Gamma_{l,k}(U_{1,\mathbb{C}}) \to \Gamma_{l,k}(U_{2,\mathbb{C}}): \vec{T}_{\begin{pmatrix}a & b\\ c & d\end{pmatrix}}(f) \ (z) \\
&\geq J_k(cz + d) \ f\big((az + b) \ (cz + d)\big).\n\end{aligned}
$$

 $\frac{1}{39}$ In the cases where $c = 0$ the function $J_k(cz + d)$ is a constant. Consequently, we have

Proposition 10: Suppose that *n* is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such **Proposition 10:** Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such that for some Clifford matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ the transformation $T_{\{a\,b\}}$: $U_{1,\mathbb{C}} \rightarrow U_{2,\mathbb{C}}$: $z \$ 398 J. RYAN

In the cases where $c = 0$ the function $J_k(cz + d)$ is a constant

have

Proposition 10: Suppose that n is odd and that $U_{1,C}$ and $U_{2,C}$ are

that for some Clifford matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ the transform $(az + b) d^{-1}$ *is a homeomorphism. Then the two modules* $\Gamma_{l,k}(U_{1,C})$ *and* $\Gamma_{l,k}(U_{2,C})$ *are isomorphic.* 18 J. RYAN

In the cases where $c = 0$ the function $J_k(cz + d)$ is

ve

roposition 10: Suppose that n is odd and that $U_{1,\mathbb{C}}$ a

t for some Clifford matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ the transform
 $\begin{pmatrix} z + b & d^{-1} & i s \ a & homeomorphism.$

Theorem 21: Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such that *for somorphic.*

In more general circumstar

Theorem 21: Suppose that n

for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
 $z \mapsto (az + b)$ $(cz + d)^{-1}$ is a h $= 0$ the function $J_k(cz)$
 se that n is odd and that
 natrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ the tran
 orphism. Then the two
 sstances we have
 at n is odd and that $U_{1,0}$
 a b, with $c\tilde{c} \neq 0$, the table *irst* 0, the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$: $U_{1,\mathbb{C}} \rightarrow U_{2,\mathbb{C}}$: $\int \mathbf{c} \, \mathbf{u} \, d\mathbf{v} \, d\mathbf{v}$ are domainting $\int_{\mathbf{c}} \mathbf{a} \, \mathbf{b}$
 $\int \mathbf{c} \, \mathbf{v} \, d\mathbf{v}$
 $\int \mathbf{c} \, \mathbf{v} \, d\mathbf{v}$ $z \mapsto (az + b)$ _{(cz} + d)⁻¹ is a homeomorphism. Suppose also that for the inclusion map *i*: $U_{2,\mathbb{C}} \to \mathbb{C}^n \setminus N(-c^{-1}d)$ the first singular homology group homomorphism $H_1(i)$: $H_1(U_{2,\mathbb{C}},\mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(-c^{-1}d) \mathbb{Z})$ is trivial. Then the function $J_k(cz + d)$ is well *defined on* $U_{2,\mathbb{C}}$ *, and the modules* $\Gamma_{l,k}(U_{1,\mathbb{C}})$ *and* $\Gamma_{l,k}(U_{2,\mathbb{C}})$ *are isomorphic.*

Proof: As $c\bar{c} = 0$, then we have that $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = ac^{-1} + \lambda(cz\tilde{c} + d\tilde{c})^{-1}$, where $\lambda = a\tilde{d} - b\tilde{c}$. As the group homomorphism $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(cz\tilde{c} + d\tilde{c}), \mathbb{Z})$ is trivial it follows that for each closed loop $p: S' \to U_{2,\mathbb{C}}$ there is a continuous extension $\hat{p}: D \to \mathbb{C}^n \setminus N(-c^{-1}d)$ to the disc D. As the function $J_k(cz + d)$ may be uniquely defined on $p(D) \subseteq \mathbb{C}^n \setminus N(-c^{-1}d)$, it follows that this function is well defined on each closed loop lying in $U_{2,C}$. Consequently, we have that for each complex k-left regular function $f((az + b) (cz + d)^{-1})$ defined on $U_{2,\mathbb{C}}$, the function $J_k(cz + d) f((az + b))$ \times (cz + d)⁻¹) is a well-defined complex k-left regular function on $U_{1,C}$, and the mo-
dules $T_{l,k}(U_{1,C})$ and $T_{l,k}(U_{2,C})$ are isomorphic 398 J. Rvax

In the cases where $c = 0$ the function $J_4(cx + d)$ is a constant. Conseq

have
 Proposition 10: Suppose that n is odd and that U_{1C} and U_{2C} are domains

that for some Clifford matrix $\begin{pmatrix} a & b \\ 0 & d \end{$ *t.* $0.2 \mathbb{C} \to \mathbb{C}^n$ $N_1(-c^{-2}d)$ *ine prst singular homotog*
 $H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(-c^{-1}d) \mathbb{Z})$ *is trivial. Then*

defined on $U_{2,\mathbb{C}}$, and the modules $\Gamma_{1,k}(U_{1,\mathbb{C}})$ and $\Gamma_{1,k}(U_{2,\mathbb{C}})$

 \cdot In-order to describe what happens in more general circumstances we require the dules $I_{1,k}(U_{1,\mathbb{C}})$ and $I_{1,k}(U_{2,\mathbb{C}})$ are isomorphic \blacksquare

In order to describe what happens in more general circumstances we require

following result.

Proposition 11: *For each integer n* > 2, *we have that*

Proof: As shown in [26] the space $\mathbb{C}^n \setminus N(0)$ can be homotopically deformed within itself to the set $B' = \{z \in \mathbb{C}^n : z = xe^{i\theta} : x \in S^{n-1} \text{ and } \theta \in [0, 2\pi]\}\$. From the set *B'* we obtain the set $B'' = \{ [z] : z \in B' \text{ and } [z] = \{z, -z \} \}$. Via the projection $p: B' \to B''$: \mapsto [z] we have that *B'* is a double covering of *B''*. Moreover, it may be observed that the set *B*" is a fibre bundle with base space RP^{n-1} , real projective $(n-1)$ -dimensional In order to describe what happens in more general circumstances we require
following result.
Proposition 11: For each integer $n > 2$, we have that $H_1(\mathbb{C}^n \setminus N(0), \mathbb{Z}) \cong \mathbb{Z}$.
Proof: As shown in [26] the space \math \rightarrow [z] we have that B' is a double covering of B''. Moreover, it may be observed that

the set B'' is a fibre bundle with base space RP^{n-1} , real projective $(n-1)$ -dimensional

space, and fibre the circle, S^1 .

Sup

space, and fibre the circle, S^1 .
Suppose now that $h': S^1 \to B'$ is a closed loop. Then on composing with the pro-
jection p we obtain a closed loop $h'': S' \to B''$. On composing this map with the fibre bundle projection $p_1: B'' \to RP^{n-1}$ we obtain a closed loop $h'' : S^1 \to RP^{n-1}$. On identifying RP^{n-1} with the $(n-1)$ dimensional disc K_{n-1} , with antipodal points on the boundary identified, it may be deduced, by elementary homotopy deformation arguments, that the loop $h'''(S¹)$ is either homotopic to a point or to a line segment bundle projection $p_1: B'' \to RP^{n-1}$ we obtain a closed loop $h'' : S^1 \to RP^{n-1}$. On identifying RP^{n-1} with the $(n-1)$ dimensional disc K_{n-1} , with antipodal points on the boundary identified, it may be deduced, by eleme Proof: As shown in [2]
itself to the set $B' = \{$
we obtain the set $B' = \{$
 $\mapsto [z]$ we have that B' is
the set B'' is a fibre bund
space, and fibre the circ
suppose now that h' :
jection p we obtain a cloundle proje passing through the centre of K_{n-1} and extending from a point on the boundary to its antipodal point. If this homotopy-deforms $h'''(S')$ to a point $[k] \in RP^{n-1}$, then there is an induced homotopic deformation of $h'''(S¹)$, within B'' , to a closed loop lying in the fibre, *81 ,* covering *k.* Furthermore, this homotopy deformation induces a'homotopy passing through the centre of K_{n-1} and extending from a point on the boundary to its antipodal point. If this homotopy deforms $h'''(S^1)$ to a point $[k] \in RP^{n-1}$, then there is an induced homotopic deformation of $h'''(S$ deformation of $h'(S^1)$, within B' , to a loop lying in the circle $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\} \subseteq B'$.
It may also be observed that the winding number of this loop is even. If, on the other hand, $h'''(S^1)$ is homotopic to a is an induced homotopic deformation of $h'''(S^1)$, within B'' , to a closed loop lying in
the fibre, S^1 , covering k. Furthermore, this homotopy deformation induces a homotopy
deformation of $h'(S^1)$, within B' , to a l

 $\frac{1}{2}$

arguments to those used in the previous paragraph, it may be deduced that this deformation induces a deformation, of $h'(\hat{S}^1)$ within B' , to a loop which is homotopic
to a loop which winds around $\{ke^{i\theta} : 0 \le \theta \le 2\pi\}$ an even number of times and is
joined to two semicircles. One of these semi to a loop which winds around $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ an even number of times and is joined to two semicircles. One of these semicircles is the set ${ke^{i\theta}}: 0 \le \theta \le \pi}$ and the other is a semicircle lying in the sphere $S^{n-1} \subset \mathbb{R}^n$, and joining k to $-k$. ph, it may be deduced that the
in B', to a loop which is home
 τ an even number of times a
s is the set $\{ke^{i\theta}: 0 \leq \theta \leq \pi\}$ and
 \mathbb{R}^n , and joining k to $-k$.
 $(z) = z^2$, it may be observed the *x* induces a deformation, of $h'(\hat{S}^1)$ within B' , to a loop which is homotopic
which winds around $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ an even number of times and is
two semicircles. One of these semicircles is the set $\{ke^{i$ formation induces a deformation, of $h'(\mathcal{S}^1)$ within B' , to a loop which is ho
to a loop which winds around $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ an even number of time
other is a semicircle. One of these semicircles is the se

On considering the map $\pi' : B' \to \mathbb{C}^n \setminus \{0\} : \pi'(z) = z^2$, it may be observed that of these loops only the ones which lie entirely in $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ and having zero winding number are contractible to a point. On restricting the map π' to the set:

$$
ke^{i\theta} \colon 0 \le \theta \le \pi
$$

we obtain a loop in $\mathbb{C}\setminus\{0\}$ which winds once around the origin. As the sphere S^{n-1} is simply connected the result follows **I**

Observation 5: For the case where $n = 2$ it is straightforward to adapt the proof of Proposition 11 to show that $H_1(\mathbb{C}^2 \setminus N(0), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 22: Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such *that for some Clifford matrix* $\begin{pmatrix} a & b \ & J \end{pmatrix}$, with $c\tilde{c} = 0$, the transformation T_{a} ϕ \: onsidering the map $\pi' : B' \to \mathbb{C}^n \setminus \{0\} : \pi'(z) = z^2$, it may be observed as which the entirely in $\{ke^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and here in the interest which the entirely in $\{ke^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and here in the i

 $\rightarrow U_{2,\mathbb{C}}:z\mapsto (az+b)(cz+d)^{-1}$ -is a *homeomorphism. Suppose also that for, the inclusion map i:* $U_{2,\mathbb{C}} \to \mathbb{C}^n \setminus N(-c^{-1}d)$ *the group homomorphism* $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z})$ \rightarrow *H₁*($\mathbb{C}^n\setminus N(-c^{-1}d), \mathbb{Z}$) is such that the image set $H(i)$ $(H_1(U_2, c, \mathbb{Z}))$ comprises solely *of even cycles (i.e. each closed loop 'has an even winding number). Then 'the function* $J_k(cz + d)$ is well defined on $U_{2,\mathbb{C}}$, and the modules $\Gamma_{1,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are iso-
morphic. is simply connected the result follows \blacksquare
 \blacksquare \blacksquare $\begin{align} e\ in \ \partial_t \partial_t \mathcal{L} \ \partial_t \mathcal{L} \ \partial_t \mathcal{L} \ \partial_t \mathcal{L} \end{align}$

Proof: Suppose that *l* is a closed loop in $U_{2,\mathbf{C}}$. Then for the map $p: U_{2,\mathbf{C}} \to$ *of even cycles (i.e. each closed loop has an even winding number). Then the function* $J_k(cz + d)$ *is well defined on* $U_{2,\mathbb{C}}$ *, and the modules* $\Gamma_{l,k}(U_{1,\mathbb{C}})$ *and* $\Gamma_{l,k}(U_{2,\mathbb{C}})$ *are isomorphic.*
Proof: Sup number of times. Consequently, the function $z \mapsto (z + c^{-1}d)^2$ the set $p(l)$ is a closed loop which winds around the origin an even

$$
c_1(z^2 + u)
$$
 is useful defined on $z_2 \in \mathbb{C}$, and the modules $Y_k(z_1, e)$ and $Y_k(z_2, u)$ are a
complete.
\n
$$
c_1(z + c^{-1}d)^2
$$
 the set $p(l)$ is a closed loop which winds around the origin an ev-
\number of times'. Consequently, the function
\n
$$
((z + c^{-1}d)^{-1} (z + c^{-1}d)^{-1})^{1/2} = ((cz + c^{-1}d)^{-1} (z + d\zeta^{-1})^{-1})^{1/2}
$$

\n
$$
= ((z + d)^{-1} (c\tilde{z} + d))^{-1/2} (c\tilde{c})^{1/2}
$$

\n
$$
= ((z + d)^{-1} (c\tilde{z} + d))^{-1/2} (c\tilde{c})^{1/2}
$$

is well defined on each such loop. It follows that J_k ($cz + d$) is a well defined function on $U_{2,\mathbb{C}}$. By similar arguments to those given in the proof of Theorem 2.1 it now. follows that the modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic \blacksquare

An example of a domain lying in \mathbb{C}^n which satisfy the properties described in Theorem 22 and contains closed loops which arc not homologous to zero in $\mathbb{C}^n \setminus N(-c^{-1}d)$

is $\mathbb{C}^{n,*} = \{z \in \mathbb{C}^n \setminus N(0) : z + \lambda x + i\mu y : x, y \in S^{n-1}, \text{ with } xy + yx = 0, \lambda, \mu \in \mathbb{R} \text{ and } |\lambda| > |\mu| \}$. is $\mathbb{C}^{n,*} = \{z \in \mathbb{C}^n \setminus N(0) : z + \lambda x + i\mu y : x, y \in S^{n-1}, \text{ with } xy + yx = 0, \lambda, \mu \in \mathbb{R} \text{ and }$ $|\lambda| > |\mu|$... $\begin{align*} (z+c^{-1}d)^{-1}(z+c^{-1}d)^{-1}\end{align*} \begin{align*} (z+c^{-1}d)^{-1}(z+c^{-1}d)^{-1}(z+d\delta^{-1})^{-1}\end{align*} \begin{align*} \mathcal{E}(\tilde{z}+c^{-1}d)^{-1}(z^2+d\delta^{-1})^{-1}\end{align*} \begin{align*} \mathcal{E}(\tilde{z}+c^{-1}d)^{-1}(z^2+d\delta^{-1})^{-1}\end{align*} \begin{align*} \mathcal{E}(\tilde{z}+c^{-1}d)^{-1}(z^2+d\delta^{-1})^{-1}\end{align*} \begin{align*} \mathcal{E$

As shown in the proof of Proposition 11 the domain $\mathbb{C}^n\setminus N(0)$ possesses, closed cycles whose winding number is odd. In order to deal with domains like $\mathbb{C}^n\setminus N(0)$ we require

Definition 9: Suppose that U is a domain in \mathbb{C}^n such that with respect to some point $z_1 \in \mathbb{C}^n$ we have $U \subseteq \mathbb{C}^n \setminus N(z_1)$, and that for the inclusion map $i: U \to \mathbb{C}^n \setminus N(z_1)$ the group homomorphism $H_1(i): H_1(U, \mathbb{Z}) \to H_1(\mathbb{C}^n N(z_1), \mathbb{Z})$ is surjective. Then we denote the Riemann surface, which is a two-fold covering of U , by U^2 , and we denote the right $A_n(\mathbb{C})$ module of complex'k-left regular functions defined on U^2 by $\Gamma_{l,k}^2(U, A_n(\mathbb{C}))$.

It is now a straightforward consequence of the constructions given in the proof of 400 J. RYAN_j

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It is now a straightforward

Proposition 11 to deduce

400 **J.** RYAN₁
 for same Clifford matrix (a b), with cc^z + 0, the transformation $T_{\begin{pmatrix}a&b\\c&d\end{pmatrix}}$. Theorem 23: Suppose that n is odd and that U_1 _C and U_2 _C are domains in \mathbb{C}^n surfor same Cliffo Theorem 23: Suppose that *n* is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in \mathbb{C}^n such that

 $z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Suppose also that for the inclusion map i: $U_{2,\mathbf{C}} \to \mathbb{C}^n \setminus N(-d\tilde{c})$ the group homomorphism $H_1(i): H_1(U_{2,\mathbf{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(d\tilde{c}), \mathbb{Z})$ *is surjective. Then the function* $J_k(cz + d)$ is well defined on the Riemann surface $U_{1,\mathbb{C}}^2$ *but not on* $U_{1,\mathbb{C}}$ *, and the modules* $\Gamma^2_{1,k}(U_{1,\mathbb{C}}, A_n(\mathbb{C}))$ and $\Gamma_{1,k}(U_{2,\mathbb{C}}, A_n(\mathbb{C}))$ are isomorphic. *Moreover, the modules* $\Gamma_{1,k}(U_{1,\mathbb{C}}, A_n(\mathbb{C}))$ and $\Gamma_{1,k}^2(U_{2,\mathbb{C}}, A_n(\mathbb{C}))$ are also isomorphic. It is now a straightforward consequently

Proposition 11 to deduce

Theorem 23: Suppose that *n* is odd a:

for same Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with
 $z \mapsto (az + b)(cz + d)^{-1}$ is a homeomoro
 $U_{2,\mathbf{C}} \rightarrow \mathbf{C}^n \setminus N$ For same Cuppoin matrix $\binom{d}{c}$, with $cc = 0$, the transformation $T_{\binom{a}{c}}$; $C_{1,C} \rightarrow C_{2,C}$.
 $z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism $H_1(i)$: $H_1(U_{2,C}, \mathbb{Z}) \rightarrow H_1(\mathbb{C}^n \setminus N(d\tilde{c}), \mathbb{Z})$

is surjective. Then the fun

On combining the results obtained in this section with Theorem 1 and Proposition 1

Theorem 24: The set of linear differential operators whose solution spaces are invariant under conformal transformations in \mathbb{C}^n is the set $C = {\lambda D^k : \lambda \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}^+ \}$.

For $\lambda_1 D_{k_1}, \lambda_2 D_{k_2} \in C$ we may define the product $\lambda_1 D^{k_1} \times \lambda_2 D^{k_2} = \lambda_1 \lambda_2 D^{k_1+k_2} \in C$. Under this product it may be observed that C is a semigroup which is canonically isomorphic to the semigroup $(\mathbb{C}\setminus\{0\})\times\mathbb{Z}^+$.

Concluding remark: In this paper we have used Clifford analysis to classify' linear, conformally invariant differential equations, and we have shown that each such equation possesses a homotopy invariant Cauehy integral formula. It follows that a large class of the results already known within Clifford analysis, and within potential theory, for the iterates of the Laplacian (e.g. $[5, 6]$), carry through to this On combining the results obtained in this section with Theorem 1 and Proposition 1
we have
Theorem 24: The set of tinear differential operators whose solution spaces are invariant
under conformal transformations in $\mathbb{C$ ' context. Co. 1986, p. 1676, p. 167-176. The three states are shown in the specified equation possesses a homotopy invariant Cauchy integral formula. It follows that a large cla

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