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Iterated Dirac Operators in $\mathbb{C}^n$

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Wir geben eine Klassifizierung der linearen konform-invarianten Differentialoperatoren über $\mathbb{C}^n$, die ihre Werte in einer Cliffordschen Algebra annehmen. Solche Operatoren schließen den Diracschen Operator und seine Potenzen ein. Wir zeigen, daß die mit dem Diracschen Operator in $\mathbb{R}^n$ und $\mathbb{C}^n$ verbundene Funktionentheorie sich auf alle diese Operatoren verallgemeinern läßt.

Мы даем классификацию линейных конформно-инвариантных дифференциальных операторов над $\mathbb{C}^n$ со значениями в алгебре Клиффорда. Эти операторы включают в себя оператор Дирака и его степени. Мы доказываем, что теория функций, связанная с оператором Дирака в $\mathbb{R}^n$ и $\mathbb{C}^n$, обобщается на все эти операторы.

We give a classification of linear, conformally invariant, Clifford algebra valued differential operators over $\mathbb{C}^n$. Such operators comprise of the Dirac operator and its iterates. We show that the function theory associated to the Dirac operator in $\mathbb{R}^n$ and $\mathbb{C}^n$ can be generalized to all these operators.

Introduction

By introducing complex Clifford algebras it has been possible to introduce a first order differential operator, over $\mathbb{R}^n$, whose square is the Laplacian, and to study the properties of analytic continuations to $\mathbb{C}^n$ of functions which are annihilated by this operator [6–8, 13, 14, 21–24]. This operator gives a natural generalization of both the classical Cauchy-Riemann equations and the massless Dirac equation. The study of properties of functions which are annihilated by this generalized Cauchy-Riemann-Dirac operator is referred to as Clifford analysis [6, 19, 20–24]. In the 1930s Clifford analysis had been developed by Fueter [11, 12], and his collaborators, as a function theory over the quaternions, and by Moisil and Téodorescu [18]. Also, earlier work on this analysis had been developed by Dixon [9]. More recently this analysis has been extended to higher dimensions by a number of authors (eg [6–8, 13, 14, 19, 21–24, 28]).

In recent work [1–3] Ahlfors, building on results of Vahlen [27] and Maass [17], describes properties of Möbius transformations in $\mathbb{R}^n$ by means of a group of matrices with entries in a Clifford algebra. Within mathematical physics the study of conformally invariant differential operators on Minkowski space, and analogues of these operators over curved spaces, has been extensively pursued (see for example [10, 15] and references therein). In [15] Jakobsen and Vergne show that powers $\Box^l$ of the ordinary d'Alembertion acting on functions in Minkowski space are conformally invariant in the sense described here. In this paper we use the complex extension of the matrices appearing in [1–3] to show that the class of linear, conformally invariant, holomorphic differential operators defined over $\mathbb{C}^n$ comprises of a semigroup of iterated Dirac operators. In [10] Verma modules are used to describe conformally invariant differential operators on Minkowski space. However, the methods used here are function theoretic in nature. We obtain our result by first deducing a generalized Cauchy integral formula, or generalized Green's formula, for solutions to each such operator. We then use these formulae to give a characterization of solutions to these equations over the Lie ball and apply arguments given in [24] and [8] to deduce the result.

In this paper we also extend a number of results in complex Clifford analysis from even dimensions to odd dimensions. This leads us to study particular types of domains in $\mathbb{C}^n$, and to study conformal transformations over twofold covering spaces of some domains in $\mathbb{C}^n$, when n is odd.

Preliminaries

Let $A_n(\mathbb{C})$ be the complex, 2^n dimensional Clifford algebra described in [4, Part 1], [20, Chapter 13], and elsewhere. This algebra has an identity l $(= e_0)$, and basis elements $l, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$, where $j_1 < \cdots < j_r$ and $1 \leqq r \leqq n$. The elements $e_1, \ldots, e_n$ satisfy the anticommutation relationship $e_je_k + e_ke_j = -2\delta_{jk}$, where δ_{jk} is the Kronecker delta. We call $\mathbb{C}^n$ the complex space spanned by $e_1, \ldots, e_n$. We call $\mathbb{R}^n$ the real space spanned by $e_1, \ldots, e_n$. The isotropic, or null cone $(z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n \colon z_1^2 + \cdots + z_n^2 = 0)$ is denoted by $N(0)$, and a general vector $z_1e_1 + \cdots + z_ne_n \in \mathbb{C}^n$ is denoted by z. It may be noted that each vector $z \in \mathbb{C}^n \setminus N(0)$ has a multiplicative inverse $z(z^2)^{-1} \in \mathbb{C}^n \setminus N(0)$. Consequently, we have the Clifford group $\Gamma_n(\mathbb{C}) \triangleq \{Z \in A_n(\mathbb{C}) \colon Z = z_1 \ldots z_k, z_j \in \mathbb{C}^n \setminus N(0)$ for $1 \leqq j \leqq k$ and k is an arbitrary positive integer$\}$. On restricting the elements $Z \in \Gamma_n(\mathbb{C})$ so that for each z_j we have that $z_j^2 = \pm 1$ we obtain a subgroup of $\Gamma_n(\mathbb{C})$ which we denote by $Pin(\mathbb{C}^n)$, and in the case where k is even we obtain a subgroup of $Pin(\mathbb{C}^n)$ which we denote by $Spin(\mathbb{C}^n)$.

Furthermore, we have [4, 20] an antiautomorphism $\sim \colon A_n(\mathbb{C}) \to A_n(\mathbb{C}) \colon e_{j_1} \ldots e_{j_r} \mapsto e_{j_r} \ldots e_{j_1}$. For a general element $Z \in A_n(\mathbb{C})$ we denote $\sim(Z)$ by $\tilde{Z}$. Using the previously described anticommutation relationship it may be observed that for each $a \in Pin(\mathbb{C}^n)$ we have that $a\mathbb{C}^n\tilde{a} = \mathbb{C}^n$ and for each $z \in \mathbb{C}^n$, $(az\tilde{a})^2 = z^2$. It follows that $Pin(\mathbb{C}^n)$ is closely related to the complex orthogonal group $O(\mathbb{C}^n) = \{(a_{ij}) \colon 1 \leqq i, j \leqq n, a_{ij} \in \mathbb{C}$ and $(a_{ij})(a_{ij})^{\top} = l\}$. In fact we have

Lemma 1: *The group $Pin(\mathbb{C}^n)$ is a four-fold covering of the group $O(\mathbb{C}^n)$ (i.e. there is a short exact sequence $0 \to Z_4 \to Pin(\mathbb{C}^n) \to O(\mathbb{C}^n) \to 0$).*

Outline proof: On considering the group homomorphism $\theta \colon Pin(\mathbb{C}^n) \to O(\mathbb{C}^n)$ canonically induced by the map $\lambda \colon Pin(\mathbb{C}^n) \times \mathbb{C}^n \colon (a, z) \mapsto az\tilde{a}$ it may be observed that the elements $1, -1, \sqrt{-1}e_1 \ldots e_n, -\sqrt{-1}e_1 \ldots e_n \in Pin(\mathbb{C}^n)$ belong to the kernel of θ. It now follows from similar arguments to those detailed for the Euclidean case in [4, Part 1] that the homomorphism θ is surjective with kernel $\left\{1, -1, \sqrt{-1}e_1 \ldots e_n, -\sqrt{-1}e_1 \ldots e_n\right\}$ ∎

Using the Clifford algebra's anticommutation relationship we may from $\mathbb{R}^n$ generate a real, 2^n-dimensional subalgebra of $A_n(\mathbb{C})$. This algebra is an example of a real Clifford algebra, and its properties are described in [4, Part 1], [20, Chapter 13] and elsewhere. We denote it by A_n. The group $A_n \cap Spin(\mathbb{C}^n)$ is denoted by $Spin(\mathbb{R}^n)$ and, as shown in [4, 20], it is a double covering of the special orthogonal group $SO(n)$, which acts on $\mathbb{R}^n$. The group $A_n \cap Pin(\mathbb{C}^n)$ is denoted by $Pin(\mathbb{R}^n)$ and it is a covering group of $O(n)$ (see [4, 20]).

Following results obtained by AHLFORS [1–3], MAASS [17] and VAHLEN [27], we introduce the following type of matrices:

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix such that $a, b, c, d \in A_n(\mathbb{C})$, and $a = a_1 \ldots a_l$, $b = b_1 \ldots b_m$, $c = c_1 \ldots c_p$, $d = d_1 \ldots d_q$, where $a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_p, d_1, \ldots, d_q \in \mathbb{C}^n$, $l, m, p, q \in \mathbb{Z}^+$ and $a\tilde{c}$, $\tilde{c}d$, $d\tilde{b}$, $\tilde{b}a \in \mathbb{C}^n$ and $a\tilde{d} - b\tilde{c} \in \mathbb{C}\setminus\{0\}$. Then this matrix is called a *Clifford matrix*. As illustrated in [24], for each Clifford matrix the transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (az + b)(cz + d)^{-1}$$

is well defined, and gives a Möbius transformation in $\mathbb{C}^n$.

Suppose now that U is a domain in $\mathbb{R}^n$, $n > 2$, and that $f\colon U \to A_n(\mathbb{C})$ is a function for which all partial derivatives exist. Suppose that P is a Clifford algebra valued constant coefficient differential operator of order $p \in \mathbb{Z}^+$, which acts on f on the left hand side (i.e. the operator P acting on f does not involve multiplying f on the right-hand side by an element of the noncommutative algebra $A_n(\mathbb{C})$). Then we may introduce the following definition.

Definition 1 [8]: Suppose that for each $a \in Spin(\mathbb{R}^n)$ and for each f with the property $Pf(ax\tilde{a}) = 0$ with respect to the variable $ax\tilde{a}$, where $x \in \mathbb{R}^n$, we have that $P\tilde{a}f(ax\tilde{a}) = 0$ with respect to the variable x. Then P is called a *spin-Euclidean differential operator*, and f is said to be *spin invariant with resepct to* P.

In [28], and elsewhere, it is observed that the operator $D = \sum_{j=1}^{n} e_j \partial/\partial x_j$ is a spin-Euclidean differential operator. It is also well-known (e.g. [28, Chapter 9] that the Laplacian $\Delta = -DD = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is invariant under actions of the special orthogonal group $SO(n)$. As $Spin(\mathbb{R}^n)$ is a double covering of the group $SO(n)$, and $\tilde{a}$ is a constant, it follows that the Laplacian is a spin-Euclidean differential operator. On placing $D^0 = 1$ it is deduced in [8] that

Theorem 1: *Every spin-Euclidean differential operator of order p is of the form*

$$\sum_{k=0}^{p} A_k D^k, \text{ with } A_k = a_{0,k} + a_{1\ldots n,k} e_1 \ldots e_n, \text{ where } a_{0,k}, a_{1\ldots n,k} \in \mathbb{C}.$$

From the chain rule we have

Proposition 1: *The only solutions to a spin-Euclidean differential operator which are invariant under dilation are solutions to an iterate of the operator D.*

Definition 2 [21]: A solution to the spin-Euclidean differential operator D is called a *left regular function*. A similar definition may be given for *right regular functions*.

An example of a function which is both left and right regular is the function $G(x) = x|x|^{-n}$, defined on $\mathbb{R}^n\setminus\{0\}$. Using this function we have [7] the following generalized Cauchy integral formula.

Theorem 2: *Suppose that $f\colon U \to A_n(\mathbb{C})$ is a left regular function, and $M \subseteq U$ is a compact n-dimensional manifold. Then, for each point $x_0 \in M$, the interior of M, we have*

$$f(x_0) = 1/\omega_n \int_{\partial M} G(x_0 - x)\, Wx\, f(x), \tag{1}$$

where ω_n is the surface area of the unit sphere S^{n-1}, and $Wx = \sum_{j=1}^{n} e_j(-1)^j \hat{d}x_j$.

It may be observed from Theorem 2 that the operator D is a natural generalization of the Cauchy-Riemann operators. It may also be observed that this operator is also a Euclidean generalization of the Dirac operator described in [15] and elsewhere.

The operator D and the integral formula (1) have natural generalizations in $\mathbb{C}^n$. Before introducing these generalizations we require the following manifolds.

Definition 3 [21]: Suppose that M is a compact, smooth, connected, real n-dimensional manifold, with boundary, lying in $\mathbb{C}^n$ such that for each $z \in M$ we have (i) $M \cap N(z) = \{z\}$ and (ii) $TM_z \cap N(z) = \{z\}$, where $N(z) = \{z' \in \mathbb{C}^n : (z - z')^2 = 0\}$, then M is called a *manifold of type one.*

Any compact, n-dimensional manifold lying in $\mathbb{R}^n$ is an example of a manifold of type one. Further examples, and constructions of manifolds of type one are given in [23].

In [21] we describe the following class of domains in $\mathbb{C}^n$.

Definition 4: Suppose that M is a manifold of type one, then the component of $\mathbb{C}^n \setminus \{N(z) : z \in \partial M\}$ containing M is called a *cell of harmonicity of type one*, and we denote it by M^+.

When $M \subseteqq \mathbb{R}^n$ these cells of harmonicity have previously been described in [5, 16, 26], and when M is the unit disc, K, in $\mathbb{R}^n$, the domain M^+ is the Lie ball $K^+ = \left\{z \in \mathbb{C}: (2^{-n}|z|)^2 + \left((2^{-n}|z|)^4 - z^2(z)^2\right)^{1/2} < 1\right\}$ described in [26].

Definition 5 [23]: Suppose that $U_{\mathbb{C}}$ is a domain in $\mathbb{C}^n$ and that $g: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a holomorphic function which satisfies the equation $D_{\mathbb{C}}{}^k g(z) = 0$ for each $z \in U_{\mathbb{C}}$, where $D_{\mathbb{C}} = \sum_{j=1}^{n} e_j\, \partial/\partial z_j$ and $k \in Z^+$. Then g is called a *complex k-left regular function*, and the operator $D_{\mathbb{C}}{}^k$ is called the *k-th order iterated Dirac operator* in $\mathbb{C}^n$.

When k is even the equation $D_{\mathbb{C}}{}^k g(z) = 0$ corresponds to the $k/2$ complex harmonic functions described by AVANISSIAN in [5]. The operator $D_{\mathbb{C}}$ is a holomorphic generalization of the operator D.

In [23] we deduce the following Cauchy integral formula.

Theorem 3: *Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with n even and $k \leqq n - 1$. Suppose also that $M_1 \subseteqq U_{\mathbb{C}}$ is an n-dimensional manifold of type one. Then for each point $z_0 \in M^+ \cap U_{\mathbb{C}}$ we have*

$$f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{n} A_p G_p{}^+(z - z_0)\, Wz\, f(z), \tag{2}$$

where $A_1 = (-1)^{n/2}$, $G_1{}^+(z) = z^{-n+1}$, $G_p{}^+(z) = z^{-n+p}$, A_p is a constant with $D_{\mathbb{C}} A_p G_p{}^+(z) = A_{p-1} G_{p-1}^+(z)$, and $Wz = \sum_{j=1}^{n} (-1)^j e_j \hat{d} z_j$.

Using the formula (2) it is straightforward to deduce the following holomorphic continuation.

Theorem 4 [23]: *Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with n even and $k \leqq n - 1$. Suppose also that $M \subseteqq U_{\mathbb{C}}$ is an n-dimensional manifold of type one lying in $U_{\mathbb{C}}$. Then there is a holomorphic function $f^+: U_{\mathbb{C}} \cup M^+ \to A_n(\mathbb{C})$ such that $f^+|_{U_{\mathbb{C}} \cap M^+} = f$.*

Theorem 5 [24]: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex* 1-*left regular function with respect to the variable* $(az + b)(cz + d)^{-1}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a Clifford matrix, and* n *is even. Then the holomorphic function* $J_1(cz + d)\, f\big((az + b)(cz + d)^{-1}\big)$ *is a complex* 1-*left regular function with respect to the variable* z, *where* $J_1(cz + d) = (cz \tilde{+} d) \times \{(cz + d)(cz \tilde{+} d)\}^{-n/2}$.

Generalized Cauchy integral formulae

This section is divided into two parts. In Part A we consider the cases where n is even and greater than two, and in Part B we consider the cases where n is odd, and greater than two.

Part A. We begin by deducing the following extension to Theorem 3.

Theorem 6: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex* k-*left regular function, with* $k \geqq n$, $U_{\mathbb{C}} \cap \mathbb{R}^n \neq \emptyset$ *and* M *a manifold of type one lying in* $U_{\mathbb{C}} \cap \mathbb{R}^n$. *Then, for each point* $x_0 \in M$ *we have*

$$f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{k} A_p G_p(x_0 - x)\, WxD^{p-1} f(x), \tag{3}$$

where $A_1 = 1$, A_p *is a constant with* $DA_pG_p(x) = -A_{p-1}G_{p-1}(x)$, $G_p(x) = x^{-n+p}$ *for* $1 \leqq p \leqq n-1$, $G_n(x) = 1/2 \log(-x^2)$, *and* $G_p(x) = 1/2\, x^{p-n} \log(-x^2) + n^{-1}(p-n) \times x^{p-n}$ *for* $n + 1 \leqq p \leqq k$.

Outline proof: It follows from Stokes' theorem that the integral (3) is equal to

$$1/\omega_n \int_{\partial K(x_0,r)} \sum_{p=1}^{k} A_p G_p(x_0 - x)\, WxD^{p-1} f(x), \tag{4}$$

where $K(x_0, r)$ is the real, n-dimensional disc lying in M, centred at x_0, and with radius r. As $r \log r \to 0$ for $r \to 0$, it now follows from similar arguments to those used to prove the generalized Cauchy integral formula in [6] that the integral (4) is equal to $f(x_0)$ ∎

Before generalizing Theorem 6 to arbitrary manifolds of type one we require

Lemma 2: *For each real* n-*dimensional manifold of type one lying in* $\mathbb{C}^n$ *the expression* $\log(z - z_0)^2$ *may be uniquely defined on the set* $\partial M \times M$, *where* $z \in \partial M$ and $z_0 \in M$.

Proof: As M is a manifold of type one, then it follows from [26] that the boundary, ∂M, is homologous in $\mathbb{C}^n \setminus N(z_0)$ to the unit sphere $S^{n-1} + z_0$ lying in $\mathbb{R}^n + z_0$, for each $z_0 \in M$. As S^{n-1} is simply connected it follows that the expression $\log\big(-(z - z_0)^2\big)$ may be uniquely defined for each $z \in S^{n-1} + z_0$. The result follows ∎

Proposition 2: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex* k-*left regular function, with* $k \geqq n$. *Suppose also that* $z_0 \in U_{\mathbb{C}}$ *and* $K(0, r) + z_0 \subset U_{\mathbb{C}}$. *Then the integral*

$$1/\omega_n \int_{\partial K(0,r)+z_0} \sum_{p=1}^{k} A_p G_p^{+}(z_0 - z)\, WzD_{\mathbb{C}}^{p-1} f(z) \tag{5}$$

evaluates to $f(z_0)$ *for each of the covering space values of* $G_p^{+}(z_0 - z)$, *with* $n \leqq p \leqq k$.

Outline proof: On choosing different covering values for $G_p^+(z_0 - z)$, for $n \leqq p \leqq k$ it follows from homogeneity arguments that the limit as r tends to zero for expression (5) is $f(z_0)$. The result now follows from Stokes' theorem ∎

We now have

Theorem 7: *Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with $k \geqq n$, and M is a manifold of type one lying in $U_{\mathbb{C}}$. Then, for each point $z_0 \in \dot{M}$ and for each choice of covering space values for $G^+(z - z_0)$, with $n \leqq p \leqq k$, we have*

$$f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{k} A_p G_p^+(z_0 - z)\, Wz D_{\mathbb{C}}^{p-1} f(z). \tag{6}$$

As is observed in [5] if M is a simply connected manifold of type one, it does not necessarily follow that the cell of harmonicity, M^+, is simply connected. For example, given the annulus $A(1/2, 3/2) = \{x \in \mathbb{R}^n : 1/2 < -x^2 < 3/2\}$ the cell of harmonicity $A(1/2, 3/2)^+$ contains the path $e^{2\pi i\theta}e_1$, where $\theta \in [0, 1]$. By considering the continuous function $q: A(1/2, 3/2)^+ \to \mathbb{C}\setminus\{0\}: z \mapsto z^2$, it may be observed that this path is not homotopic to any path in $\mathbb{R}^n\setminus\{0\}$.

As a consequence we have from the integral (6) that even for simply connected manifolds of type one the complex k-left regular functions, with $k \geqq n$, do not necessarily have unique holomorphic continuations to the cell of harmonicity. For example the holomorphic continuation of $\log(-x^2)$ is a complex n-left regular function which is not uniquely defined on $A(1/2, 3/2)^+$.

In order to introduce suitable subdomains of M^+ over which these functions are uniquely defined we begin by introducing the following definitions.

Definition 6: Suppose that M is a manifold of type one and $z_0 \in M^+$ is a point such that for each $z \in N(z_0) \cap M$ the line segment joining z_0 to z lies in M^+. Then the point z_0 is said to be *null connected to M*.

It is not in general the case that for an arbitrary manifold M of type one we have that each $z \in M^+$ is null connected to M. For example, for the point $ie_1 \in A(1/2, 3/2)^+$ the point $e_2 \in N(ie_1) \cap A(1/2, 3/2)$, but the point $1/2ie_1 + 1/2e_2$ does not lie in $A(1/2, 3/2)^+$ even though it does lie on the line segment joining ie_1 to e_2.

Definition 7: Suppose that M is a manifold of type one then the set of points $\{z_0 \in M^+ : z_0$ is null connected to $M\}$ is called the *null connected subdomain of M^+*, and it is denoted by NM^+.

We now deduce

Lemma 3: *Suppose that M is a manifold of type one. Then the null connected subdomain of M^+ is a domain.*

Proof: Suppose that $z_0 \in NM^+$. Then, either $z_0 \in \dot{M}$, or $z_0 \in M^+\setminus\dot{M}$. If $z_0 \in \dot{M}$ then we may choose a neighbourhood $B(z_0) \subseteqq M^+$ of z_0 such that for each pair of points $z_1, z_2 \in B(z_0)$ the line segment $\{\lambda z_1 + (1 - \lambda) z_2 : \lambda \in [0, 1]\}$ lies in $B(z_0)$. In [23] we show that for each point $z \in M^+\setminus\dot{M}$ the set $N(z) \cap \dot{M}$ is a manifold homeomorphic to the sphere S^{n-2}. It also follows from [23] that we may choose $B(z_0)$ so that $N(z) \cap M \subseteqq B(z_0)$ for each $z \in B(z_0)$. It now follows from the construction of the neighbourhood $B(z_0)$ that each point $z \in B(z_0)$ is null connected to M. Suppose now that $z_0 \in M^+\setminus\dot{M}$ then as the set of line segments joining z_0 to the compact manifold $\dot{M} \cap N(z_0)$ is compact, and contained in the open set M^+, it follows that there is an open subset U_{z_0} of M^+ such that $z_0 \in U_{z_0}$ and for each $z \in U_{z_0}$ we have that $z \in NM^+$. Consequently, the set NM^+ is an open set.

To show that this open set is connected consider first a point $z_0 \in NM^+\setminus M$. Then, from the open set U_{z_0} we may construct the open subset of $M^+\setminus M$

$$U'_{z_0} = \left(\bigcup_{z\in U_{z_0}} \bigcup_{z'\in N(z)\cap M} \bigcup_{\lambda\in(0,1]} \lambda z + (1-\lambda) z'\right).$$

Let $U''_{z_0} = NM^+ \cap U'_{z_0}$, and suppose that $U''_{z_0} \neq U'_{z_0}$. Then there exists a point $z_1 \in \left(\mathrm{cl}\,(U''_{z_0})\setminus U''_{z_0}\right) \cap (U'_{z_0}\setminus U''_{z_0})$ such that a line segment joining z_1 to $M \cap N(z_1)$ is not entirely contained in M^+. Suppose now that z_2 is a point on this line segment satisfying the condition $z_2 \in \mathrm{cl}\,(M^+)\setminus M^+$. Then there is a point $z_3 \in \partial M$ such that $z_2 \in N(z_3)$. It may be observed that the complex hyperplane $\{z_3 + c(z_2 - z_3): c \in \mathbb{C}\}$ is a subset of $N(z_3)$. Now consider the real three-dimensional hyperplane, H, containing this complex hyperplane, and the vector z_1. As $z_1 \in \left(\mathrm{cl}(U''_{z_0})\setminus U''_{z_0}\right)$ and U''_{z_0} is an open set it follows that $H \cap U''_{z_0} \neq \emptyset$ and it is open. Therefore, we have that for each point $z_4 \in H \cap U''_{z_0}$ the line passing through z_4, and parallel to the line segment $\{\lambda z_1 + (1-\lambda) \times z_2: \lambda \in [0, 1]\}$, intersects with the complex hyperplane $\{z_3 + c(z_2 - z_3): c \in \mathbb{C}\}$. Also, for z_4 sufficiently close to z_1 we have that these line segments also lie in U''_{z_0}. Thus $(H \cap U''_{z_0}) \cap N(z_3) \neq \emptyset$. As $z_3 \in \partial M$ this contradicts our assumption that $U''_{z_0} \neq U'_{z_0}$. Consequently, each point $z_0 \in NM^+\setminus M$ is path connected to the set M. As M is connected it follows that NM^+ is connected. Consequently, the set NM is a domain ∎

In order to deduce hat kth order complex left regular functions may be uniquely holomorphically extended to these domains we first require the following result.

Proposition 3: *For each closed path $h: S^1 \to NM^+$ there is a homotopy $H: S^1 \times [0, 1] \to NM^+$ such that for each $s \in S^1$* (i) $H(s, 0) = h(s)$ *and* (ii) $H(s, 1) \in M$.

Proof: As observed in the proof of Lemma 3 we prove in [23] that for each point $z_0 \in M^+\setminus M$ the set $N(z_0) \cap M$ is a manifold homeomorphic to the sphere S^{n-2}. It follows from the definition of a manifold of type one that for each point $z \in N(z_0) \cap M$ there does not exist any other point $z' \in N(z_0) \cap M$ such that $z' = z_0 + c(z - z_0)$ for some $c \in \mathbb{C}\setminus\{0\}$. Consequently, we have that for each point $z_0 \in M^+\setminus M$ there is a unique non-zero complex number $c(z_0)$ such that $z_0 + c(z_0)(e_1 + ie_2) \in N(z_0) \cap M$. It now follows from the proof of Lemma 3 that for each path $h: S^1 \to NM^+$ we may produce a homotopy $H: S^1 \times [0, 1] \to NM^+$, where $H(s, t) = h(s)$ for all s with $h(s) \in M$, and $H(s, t) = h(s) + tc(z_0)(e_1 + ie_2)$, otherwise. This homotopy satisfies conditions (i) and (ii) ∎

Using Proposition 3 and Theorem 7 it is now straightforward to deduce.

Theorem 8: *Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, with $k \geq n$, and M is a manifold of type one lying in $U_{\mathbb{C}}$. Then the function f has a unique holomorphic continuation f^+ to the domain $U_{\mathbb{C}} \cup NM^+$.*

Part B. We begin by introducing the following extension to Theorem 6.

Theorem 9: *Suppose that $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function, $U_{\mathbb{C}} \cap \mathbb{R}^n \neq \emptyset$, and M a manifold of type one lying in $U_{\mathbb{C}} \cap \mathbb{R}^n$. Then, for each point $x_0 \in M$, we have*

$$f(x_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{k} B_p G_p'(x_0 - x)\, WxD^{p-1} f(x),$$

where $B_1 = 1$, B_p is a constant with $DB_pG_p'(z) = B_{p-1}G'_{p-1}(x)$ and $G_p'(x) = (x^2)^{-1/2}\, x^{-n+1+p}$ for $1 \leq p \leq k$.

The proof follows the same lines as the outline proof of Theorem 6, so it is omitted. By equivalent arguments to those used to deduce Lemma 2 we also have

Lemma 4: *For each real n-dimensional manifold of type one lying in* $\mathbb{C}^n$ *the expression* $\left((z - z_0)(z - z_0)\right)^{1/2}$ *may be uniquely defined on the set* $\partial M \times M$, *where* $z \in \partial M$ and $z_0 \in M$.

Consequently, we now have

Theorem 10: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex k-left regular function, and M is a manifold of type one lying in* $U_{\mathbb{C}}$. *Then, for each point* $z_0 \in M$ *we have*

$$f(z_0) = 1/\omega_n \int_{\partial M} \sum_{p=1}^{k} B_p{}' G_p{}'^{+}(z_0 - z)\, WzD_{\mathbb{C}}{}^{p-1} f(z),$$

where $G_p{}'^{+}(z_0 - z)$ *is the holomorphic continuation to* ∂M *of* $G_p{}'(z_0 - z)$, *defined on* $S^{n-1} + z_0$, *obtained via the homological equivalence in* $\mathbb{C}^n \setminus N(z_0)$ *of* ∂M *and* $S^{n-1} + z_0$.

Observation 1: Both Definition 6 and 7 do not depend on the dimension of M being even. Also, the statement and proof of Lemma 3 and Proposition 2 are valid for odd-dimensional manifolds of type one.

Consequently, we have the following extension to odd dimensions of Theorem 8.

Theorem 11: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex k-left regular function, and* $n = 1 \bmod 2$. *Suppose also that* $M \subseteqq U_{\mathbb{C}}$ *is a manifold of type one. Then the function f has a unique holomorphic continuation,* f^+, *to the domain* $U_{\mathbb{C}}' \cup NM^+$.

Conformal invariance

We begin this section by deducing the following result.

Proposition 4: *Suppose that* $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ *is a complex 1-left regular function. Then for each positive integer k the function* $z^{k-1} f(z)$ *is complex k-left regular.*

Proof: Suppose first that $k = 2p$. Then $D_{\mathbb{C}}{}^k = (-1)^p \left(\sum_{j=1}^{n} \partial^2/\partial z_j{}^2\right)$ and $z^{k-1} f(z) = z(z_1{}^2 + \cdots + z_n{}^2)^{p-1} (-1)^{p-1} f(z)$. Now

$$\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j{}^2}\right) z(z_1{}^2 + \cdots + z_n{}^2)^{p-1} f(z)$$

$$= 2(p-1)\, z(z_1{}^2 + \cdots + z_n{}^2)^{p-2} f(z)$$

$$+ 2(p-1)(2p-3)\, z(z_1{}^2 + \cdots + z_n{}^2)^{p-3} f(z)$$

$$+ \sum_{j=1}^{n} z(z_1{}^2 + \cdots + z_n{}^2)^{p-2}\, 2(p-1)\, z_j\, \partial f(z)/\partial z_j \tag{7}$$

provided $p \geqq 3$. If $p = 1$, then it is straightforward to determine by direct calculation that

$$\sum_{j=1}^{n} \partial^2\left(z f(z)\right)/\partial z_j{}^2 = 0. \tag{8}$$

If $p = 2$, then $\sum_{j=1}^{n} \partial^2(z^3 f(z))/\partial z_j^2 = 2z f(z) + 2z \sum_{j=1}^{n} z_j \partial f/\partial z_j$. It follows from expression (7) that

$$\left(\sum_{j=1}^{n} \partial^2/\partial z_j^2\right)^2 (z^3 f(z)) = 2\left(\sum_{j=1}^{n} \partial^2/\partial z_j^2\right)\left(z \sum_{k=1}^{n} z_k \partial f/\partial z_k\right).$$

As $f(z)$ is complex 1-left regular, $\sum_{j=1}^{n} \partial^2/\partial z_j^2 \left(z \sum_{k=1}^{n} z_k\, \partial f(z)/\partial z_k\right) = 0$. If, for $p \geqq 3$, we allow the pth order complex Laplacian to act on the $z(z_1^2 + \cdots + z_n^2)^{p-1}$ part of $z(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, then it may be observed from expression (8) that this term is annihilated by this operator. As $f(z)$ is a complex 1-left regular function, we have that $f(z)$ is also annihilated by the complex Laplacian. Consequently $\left(\sum_{j=1}^{n} \partial^2/\partial z_j^2\right)^p z \left(\sum_{j=1}^{n} z_j\, \partial/\partial z_j\right)^{p-1} f(z) = 0$. It may now be observed from expression (8) that

$$\left(\sum_{j=1}^{n} \partial^2/\partial z_j^2\right)^p z(-1)^{p-1} (z_1^2 + \cdots + z_n^2)^{p-1} f(z) = 0.$$

Suppose now that $k = 2p + 1$. Then, as $f(z)$ is a complex 1-left regular function we have that $D_{\mathbb{C}}(z_1^2 + \cdots + z_n^2)^p f(z) = 2pz(z_1^2 + \cdots + z_n^2)^{p-1} f(z)$, and it follows from the previous arguments that

$$\left(\sum_{j=1}^{n} \partial^2/\partial z_j^2\right)^p 2pz(z_1^2 + \cdots + z_n^2)^{p-1} f(z) = 0.$$

Consequently, $z^{2p} f(z)$ is a complex k-left regular function ∎

Corollary: *Suppose that for $0 \leqq l \leqq k - 1$ the functions $f_l\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ are complex 1-left regular. Then the function*

$$F\colon U_{\mathbb{C}} \to A_n(\mathbb{C})\colon F(z) = \sum_{l=0}^{k-1} z^l f_l(z) \tag{9}$$

is complex k-left regular.

Observation 2: It is not the case that every complex k-left regular function can be expressed in the form (9). From Theorem 4 we have that for n even each complex 1-left regular function has a unique holomorphic continuation from a neighbourhood of a manifold of type one to its cell of harmonicity. However, as observed earlier, $\log(-z^2)$ is a complex n-left regular function which is not uniquely defined on $A(1/2, 3/2)^+$.

As $\mathbb{C}^n$ is contractible to a point we have from Theorems 7 and 10

Proposition 5: *Suppose that $P_{(q)}\colon \mathbb{C}^n \to A_n(\mathbb{C})$ is a complex, k-left regular polynomial, homogeneous of degree q with respect to the origin. Then*

$$P_{(q)}(z) = \sum_{l=0}^{k-1} z^l P_l(z),$$

where P_l is a complex 1-left regular function, homogeneous of degree $q - l$.

As the disc lying in $\mathbb{R}^n$, of radius $r \in \mathbb{R}^+$, is contractible within itself to a point, it follows from Theorems 8 and 11, Proposition 5 and the Taylor expansion given in [7, Theorem 10] that

Theorem 12: *Suppose that $K(r)^+$ is the Lie ball of radius $r \in \mathbb{R}^+$, lying in $\mathbb{C}^n$ and $f\colon K(r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular function. Then*

$$f(z) = \sum_{l=0}^{k-1} z^l f_l(z),$$

for each $z \in K(r)^+$, where each $f_l\colon K(r)^+ \to A_n(\mathbb{C})$ is a complex 1-left regular function.

As $D_{\mathbb{C}}^k$ is a constant coefficient differential operator we have

Proposition 6: *Suppose that $f\colon U_{\mathbb{C}}' \to A_n(\mathbb{C})$ is a complex k-left regular function with respect to the variable $w = z + z_0$ for some constant $z_0 \in \mathbb{C}^n$. Then f is complex k-left regular with respect to the variable z.*

From Proposition 6 and Theorem 12 we have

Theorem 13: *Suppose that $K(z_0, r)^+$ is the Lie ball of radius $r \in \mathbb{R}^+$ centred at $z_0 \in \mathbb{C}^n$, and $f\colon K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular function. Then*

$$f(z) = \sum_{l=0}^{k-1} (z - z_0)^l f_l(z - z_0),$$

for each $z \in K(z_0, r)^+$, where each $f_l\colon K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex 1-left regular function.

As $D_{\mathbb{C}}^k$ is a homogeneous differential operator we have

Proposition 7: *Suppose that $f(w)$ is a complex k-left regular function with respect to the variable $w = \lambda z$, where $\lambda \in \mathbb{C}\setminus\{0\}$. Then $f(\lambda z)$ is a complex k-left regular function with respect to the variable z.*

We now deduce the $Pin(\mathbb{C}^n)$ invariance of the complex k-left regular functions. We begin with

Proposition 8: *Suppose that $a \in Pin(\mathbb{C}^n)$ and $K(\tilde{a}z_0a, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$ and centred at $az_0\tilde{a} \in \mathbb{C}^n$. Suppose also that $f\colon K(az_0\tilde{a}, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular function with respect to the variable $az\tilde{a} \in K(az\tilde{a}, r)^+$. Then the function $\tilde{a}f(az\tilde{a})$ is complex k-left regular with respect to the variable z.*

Proof: We have from Theorem 13 that $f(az\tilde{a}) = \sum_{l=0}^{k-1} (az\tilde{a} - az_0\tilde{a})^l f_l(az\tilde{a})$, where each f_l is a complex 1-left regular function with respect to the variable $az\tilde{a}$. Now

$$\tilde{a}f(az\tilde{a}) = \sum_{l=0}^{k-1} \tilde{a}(az\tilde{a} - az_0\tilde{a})^l f_l(az\tilde{a}) = \sum_{l=0}^{k-1} (a\tilde{a})^l (z - z_0)^l \tilde{a}f_l(az\tilde{a}). \tag{10}$$

It follows from Theorem 5 that each $\tilde{a}f_l(az\tilde{a})$ is a complex 1-left regular function with respect to the variable z. From expression (10) and the corollary to Proposition 4 we now have that the function $\tilde{a}f(az\tilde{a})$ is complex k-left regular ∎

For each domain $U_{\mathbb{C}}$ and each $a \in Pin(\mathbb{C}^n)$ we may take sets of points $\{z_p\}_{p=0}^{\infty} \subset \mathbb{C}$ and $\{r(z_p)\} \subset \mathbb{R}^+$ such that $\{az_p\tilde{a}\}_{p=0}^{\infty} \subseteq U_{\mathbb{C}}$ and $\bigcup_{p=1}^{\infty} K(az_p\tilde{a}, r(z_p)) = U_{\mathbb{C}}$.

As a consequence the following result follows from Proposition 8.

Theorem 14: *Suppose that $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function with respect to the variable $az\tilde{a}$. Then, the function $f_a\colon \tilde{a}U_{\mathbb{C}}a \to A_n(\mathbb{C})\colon f_a(z) = \tilde{a}f(az\tilde{a})$ is complex k-left regular with respect to the variable z.*

Observation 3: Theorem 14 is also a consequence of Theorem 1. However, the methods used here to establish Theorem 14 differ from those used in [8] to establish Theorem 1. Later in this section we shall adapt the methods used here to establish Theorem 14 to deduce other results which are not consequences of Theorem 1.

In the cases where k is even the differentiable operator $D_{\mathbb{C}}{}^k$ is an iterate of the complex Laplacian. As $a \in Pin(\mathbb{C}^n)$ is a constant, it follows from Theorem 14, or by direct calculation, that the function $f(az\bar{a})$ is complex k-left regular with respect to the variable z.

Observation 4: Proposition 8 may also be deduced by using Lemma 1 and directly applying the iterated complex Laplacian to the function $f_{1,a}(z)$.

We now use our previous arguments to deduce the invariance of complex k-left regular functions under inversion. We begin with

Proposition 9: *Suppose that $K(z_0, r)^+$ is a Lie ball of radius $r \in \mathbb{R}^+$, and centred at z_0, and lying in $\mathbb{C}^n \setminus N(0)$. Suppose also that $f\colon K(z_0, r)^+ \to A_n(\mathbb{C})$ is a complex k-left regular function in the variable $w = z^{-1}$. Then the function*

$$\operatorname{inv}(f)\colon K^{-1}(z_0, r)^\dagger \to A_n(\mathbb{C})\colon \operatorname{inv}(f)(z) = J_k(z)\, f(z^{-1}),$$

is complex k-left regular with respect to the variable z, where $K^{-1}(z_0, r)^\dagger = \{z \in \mathbb{C}^n \setminus N(0)\colon z^{-1} \in K(z_0, r)^\dagger\}$ and $J_k(z) = G_1{}^\dagger(z)\, z^{k-1}$ for n even, $J_k(z) = G_1{}'^\dagger(z)\, z^{k-1}$ for n odd.

Proof: From Theorem 13 we have that $f(z^{-1}) = \sum_{l=0}^{k-1} (z^{-1} - z_0)^l f_l(z^{-1})$, where each $f_l(z^{-1})$ is a complex k-left regular function in the variable z^{-1}. Now consider the function

$$\begin{aligned} J_k(z)\, f(z^{-1}) &= \sum_{l=0}^{k-1} J_k(z)\, (z^{-1} - z_0)^l f_l(z^{-1}) \\ &= \sum_{l=0}^{k-1} J_k(z)\, \big(z/(z_1^2 + \dots + z_n^2) - z_0\big)^l f_l(z^{-1}). \end{aligned} \tag{11}$$

For the case where $k = 1$ and n is even the result is a direct consequence of Theorem 5. When $k = 1$ and n is odd, the result follows from direct analogues of the arguments given in [24].

Suppose now that k is odd and greater than one. Then on expanding the expression (11), we obtain, on placing $k = 2p + 1$, the terms

$$(zz_0)^p\, J_1(z)\, f_{k-1}(z^{-1}) \tag{12}$$

and

$$z_0(zz_0)^{p-1} J_2(z)\, f_{k-1}(z^{-1}). \tag{13}$$

As $J_1(z)\, f_{k-1}(z^{-1})$ and $J_2(z)\, f_{k-1}(z^{-1})$ are both complex harmonic functions with respect to the variable z, then it follows from the anticommutation relationship within the Clifford algebra, and the order of the operator $D_{\mathbb{C}}{}^k$ that both expressions (12) and (13) are annihilated by this operator. The expansion of (11) also contains the term $z^{p-1} z_0{}^p J_2(z)\, f_{k-1}(z^{-1})$ and as $J_2(z)\, f_{k-1}(z^{-1})$ is a complex harmonic function in the variable z, it follows from the order of the operator $D_{\mathbb{C}}{}^k$ and the anticommutation relationship within the Clifford algebra that this expression is also annihilated by this operator. The expansion of (11) also contains the term $z^p J_1(z)\, f_{k-1}(z^{-1})$. As $J_1(z)\, f(z^{-1})$ is a complex 1-left regular function in the variable z, it follows from the corollary to Proposition 4 that this expression is also annihilated by the operator $D_{\mathbb{C}}{}^k$. As $z_0{}^2 \in \mathbb{C}$ and

$z^2 \in \mathbb{C}$, it may be observed that all other terms appearing in the expansion of (11) are of the form $\lambda_1(zz_0)^q J_1(z) f_l(z)$, $\lambda_2 z_0(zz_0)^{q_1} J_2(z) f_l(z)$, $\lambda_3 z^{q_2} z^{q_3} J_2(z) f_l(z^{-1})$ and $\lambda_4 z^{q_4} J_1(z) \times f_l(z^{-1})$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, $q, q_1, q_2, q_3, q_4 \in \mathbb{N}$, with $q \leqq p$, $q_1 \leqq p-1$, $q_2 \leqq 2p-1$, $q_3 \leqq p$ and $q_4 \leqq p$. Consequently, we have that expression (11) is annihilated by the operator $D_{\mathbb{C}}{}^k$ when k is odd.

When k is even, we have, on placing $k = 2p+2$, that the expansion of expression (11) contains the terms $(zz_0)^p J_1(z) f_{k-1}(z^{-1})$, $z_0(zz_0)^{p-1} J_2(z) f_{k-1}(z^{-1})$, $z_0{}^2(zz_0)^p J_2(z) f_{k-1}(z)$ and $(z_0 z)^p z_0 J_2(z) f_{k-1}(z^{-1})$. By similar considerations to those used for the case where k is odd it may be deduced that all of these terms are annihilated by the operator $D_{\mathbb{C}}{}^k$. By similar arguments to those used in the first part of this proof it now follows that the expression (11) is annihilated by the operator $D_{\mathbb{C}}{}^k$ when k is even ∎

Note: As the map inv: $K^{-1}(z_0, r)^\dagger \to K(z_0, r)^\dagger : z \mapsto z^{-1}$ is a diffeomorphism and the domain $K(z_0, r)$ is contractible within itself to a point it follows that in the cases where n is odd the function $J_k(z) f(z^{-1})$ is uniquely defined on $K^{-1}(z_0, r)^\dagger$.

Using Proposition 9 we may now deduce

Theorem 15: *Suppose that n is even, and that $U_{\mathbb{C}}$ is a domain lying in $\mathbb{C}^n \setminus N(0)$. Suppose also that $f\colon U_{\mathbb{C}} \to A_n(\mathbb{C})$ is a complex k-left regular function. Then*

$$\mathrm{inv}(f)\colon U_{\mathbb{C}}{}^{-1} \to A_n(\mathbb{C})\colon \mathrm{inv}\,(f)\,(z) = J_k(z)\, f(z^{-1})$$

is a complex k-left regular function, where $U_{\mathbb{C}}{}^{-1} = \{z \in \mathbb{C}^n \setminus N(0)\colon z^{-1} \in U_{\mathbb{C}}\}$.

Proof: Consider a sequence of Lie balls $\{K(z_q, r_q)^\dagger\}_{q=0}^\infty$ such that $\bigcup_q K(z_q, r_q)^\dagger = U_{\mathbb{C}}$. Then, on restricting f to $K(z_q, r_q)^\dagger$ for each q we have from Proposition 9 that $\mathrm{inv}(f)\colon K^{-1}(z_q, r_q)^\dagger \to A_n(\mathbb{C})$ is a complex k-left regular function. Moreover, $\bigcup_q K^{-1}(z_q, r_q)^\dagger = U_{\mathbb{C}}{}^{-1}$. As n is even $I_k(z) = G_l{}^\dagger(z)\, z^{k-1}$ is uniquely defined on $U_{\mathbb{C}}{}^{-1}$. Consequently, $\mathrm{inv}\,(f)\colon U_{\mathbb{C}}{}^{-1} \to A_n(\mathbb{C})$ is a uniquely defined conplex k-left regular function ∎

On combining Propositions 6, 7 and 9, and Theorems 14 and 15 and [24] it is straightforward to deduce

Theorem: 16 *Suppose that n is even, and that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix. Suppose also that the function $f\big((az+b)\,(cz+d)^{-1}\big)$ is complex k-left regular in the variable $(az+b)\,(cz+d)^{-1}$. Then the function $J_k(cz+d)\, f\big((az+b)\,(cz+d)^{-1}\big)$ is complex k-left regular in the variable z, where*

$$J_k(cz+d) = \big((cz+d)\,(cz\,\tilde{\mp}\,d)\big)^{(k-n)/2}, \quad k \text{ even}, \tag{14}$$

$$J_k(cz+d) = (cz\,\tilde{\mp}\,d)\,\big((cz+d)\,(cz\,\tilde{\mp}\,d)\big)^{(k-1-n)/2}, \quad k \text{ odd}. \tag{15}$$

As n is even, it may be observed from expressions (14) and (15) that the functions $J_k(cz+d)$ are uniquely determined over each domain in $\mathbb{C}^n$ for each k.

Definition 8: For each domain $U_{\mathbb{C}} \subseteqq \mathbb{C}^n$ the set

$$\{f\colon U^{\mathbb{C}} \to A_n(\mathbb{C})\colon f \text{ is a complex } k\text{-left regular function}\}$$

is denoted by $\Gamma_{l,k}(U_{\mathbb{C}})$.

It may be noted that this set is a right module over the algebra $A_n(\mathbb{C})$.

Theorem 17: *Suppose that n is even and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains lying in $\mathbb{C}^n$ such that for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Then the two modules $\Gamma_{l,k}(U_{1,\mathbb{C}}), \Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic via the linear map*

$$\vec{T}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: \Gamma_{l,k}(U_{2,\mathbb{C}}) \to \Gamma_{l,k}(U_{1,\mathbb{C}}): \vec{T}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(f)(z)$$
$$= J_k(cz + d)\, f\big((az + b)(cz + d)^{-1}\big).$$

We now return to consider the cases where n is odd. We begin with

Theorem 18: *Suppose that $U_{\mathbb{C}}$ is a domain lying in $\mathbb{C}^n \setminus N(0)$ and $U_{\mathbb{C}}$ is contractible within itself to a point. Then for each complex k-left regular function $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ the function $\operatorname{inv}(f): U_{\mathbb{C}}^{-1} \to A_n(\mathbb{C}): \operatorname{inv}(f)(z) = J_k(z) f(z^{-1})$ is complex k-left regular.*

Proof: As $U_{\mathbb{C}}$ is contractible within itself to a point it follows from the homeomorphism $\operatorname{inv}: U_{\mathbb{C}}^{-1} \to U_{\mathbb{C}}: z \mapsto z^{-1}$ that $U_{\mathbb{C}}^{-1}$ is also contractible within itself to a point. Consequently, $J_k^{\mathbb{C}}(z)$ is a well-defined function on $U_{\mathbb{C}}^{-1}$. The result now follows from the proof of Theorem 15 ∎

More generally we have

Theorem 19: *Suppose that n is odd and that $U_{\mathbb{C}} \subseteq \mathbb{C}^n$ is a domain which is contractible within itself to a point. Suppose also that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix such that the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: J(U_{\mathbb{C}}) \to U_{\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a diffeomorphism. Then, for $f: U_{\mathbb{C}} \to A_n(\mathbb{C})$ a complex k-left regular function the function*

$$J(f): J(U_{\mathbb{C}}) \to A_n(\mathbb{C}): J(f)(z) = J_k(cz + d)\, f\big((az + b)(cz + d)\big)^{-1}$$

is complex k-left regular, where

$$J_k(cz + d) = \begin{cases} \big((cz + d)(cz \mp d)\big)^{(k=n)/2} & \text{for } k \text{ even,} \\ (cz \mp d)\big((cz + d)(cz \mp d)^{(k-1-n)/2} & \text{for } k \text{ odd.} \end{cases}$$

Outline proof: As $U_{\mathbb{C}}$ is contractible within itself to a point, then it follows that $J(U_{\mathbb{C}})$ is also contractible within itself to a point. Consequently, the function $J_k(cz + d)$ is well defined on the domain $J(U_{\mathbb{C}})$. The result now follows from the same reasonings as used to establish Theorem 16 ∎

Theorem 19 also gives us

Theorem 20: *Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in $\mathbb{C}^n$ such that for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Suppose also that the domain $U_{1,\mathbb{C}}$ is contractible within itself to a point. Then the two modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic via the linear map*

$$\vec{T}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: \Gamma_{l,k}(U_{1,\mathbb{C}}) \to \Gamma_{l,k}(U_{2,\mathbb{C}}): \vec{T}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(f)(z)$$
$$= J_k(cz + d)\, f\big((az + b)(cz + d)\big).$$

In the cases where $c = 0$ the function $J_k(cz + d)$ is a constant. Consequently, we have

Proposition 10: *Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in $\mathbb{C}^n$ such that for some Clifford matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ the transformation $T_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \to (az + b)\, d^{-1}$ is a homeomorphism. Then the two modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic.*

In more general circumstances we have

Theorem 21: *Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in $\mathbb{C}^n$ such that for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c\tilde{c} \neq 0$, the transformation $T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Suppose also that for the inclusion map $i: U_{2,\mathbb{C}} \to \mathbb{C}^n \setminus N(-c^{-1}d)$ the first singular homology group homomorphism $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(-c^{-1}d)\, \mathbb{Z})$ is trivial. Then the function $J_k(cz + d)$ is well defined on $U_{2,\mathbb{C}}$, and the modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic.*

Proof: As $c\tilde{c} \neq 0$, then we have that $T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) = ac^{-1} + \lambda(cz\tilde{c} + d\tilde{c})^{-1}$, where $\lambda = a\tilde{d} - b\tilde{c}$. As the group homomorphism $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(cz\tilde{c} + d\tilde{c}), \mathbb{Z})$ is trivial it follows that for each closed loop $p: S' \to U_{2,\mathbb{C}}$ there is a continuous extension $\hat{p}: D \to \mathbb{C}^n \setminus N(-c^{-1}d)$ to the disc D. As the function $J_k(cz + d)$ may be uniquely defined on $\hat{p}(D) \subseteq \mathbb{C}^n \setminus N(-c^{-1}d)$, it follows that this function is well defined on each closed loop lying in $U_{2,\mathbb{C}}$. Consequently, we have that for each complex k-left regular function $f\big((az + b)(cz + d)^{-1}\big)$ defined on $U_{2,\mathbb{C}}$, the function $J_k(cz + d) f\big((az + b) \times (cz + d)^{-1}\big)$ is a well-defined complex k-left regular function on $U_{1,\mathbb{C}}$, and the modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic ∎

In order to describe what happens in more general circumstances we require the following result.

Proposition 11: *For each integer $n > 2$, we have that $H_1(\mathbb{C}^n \setminus N(0), \mathbb{Z}) \cong \mathbb{Z}$.*

Proof: As shown in [26] the space $\mathbb{C}^n \setminus N(0)$ can be homotopically deformed within itself to the set $B' = \{z \in \mathbb{C}^n: z = xe^{i\theta}: x \in S^{n-1}$ and $\theta \in [0, 2\pi]\}$. From the set B' we obtain the set $B'' = \big\{[z]: z \in B'$ and $[z] = \{z, -z\}\big\}$. Via the projection $p: B' \to B'': \mapsto [z]$ we have that B' is a double covering of B''. Moreover, it may be observed that the set B'' is a fibre bundle with base space RP^{n-1}, real projective $(n - 1)$-dimensional space, and fibre the circle, S^1.

Suppose now that $h': S^1 \to B'$ is a closed loop. Then on composing with the projection p we obtain a closed loop $h'': S' \to B''$. On composing this map with the fibre bundle projection $p_1: B'' \to RP^{n-1}$ we obtain a closed loop $h''': S^1 \to RP^{n-1}$. On identifying RP^{n-1} with the $(n - 1)$ dimensional disc K_{n-1}, with antipodal points on the boundary identified, it may be deduced, by elementary homotopy deformation arguments, that the loop $h'''(S^1)$ is either homotopic to a point or to a line segment passing through the centre of K_{n-1} and extending from a point on the boundary to its antipodal point. If this homotopy deforms $h'''(S^1)$ to a point $[k] \in RP^{n-1}$, then there is an induced homotopic deformation of $h'''(S^1)$, within B'', to a closed loop lying in the fibre, S^1, covering k. Furthermore, this homotopy deformation induces a homotopy deformation of $h'(S^1)$, within B', to a loop lying in the circle $\{ke^{i\theta}: 0 \leqq \theta \leqq 2\pi\} \subseteq B'$. It may also be observed that the winding number of this loop is even. If, on the other hand, $h'''(S^1)$ is homotopic to a line segment joining antipodal points, then by similar arguments to those used in the previous paragraph, it may be deduced that this deformation induces a deformation, of $h'(S^1)$ within B', to a loop which is homotopic to a loop which winds around $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ an even number of times and is joined to two semicircles. One of these semicircles is the set $\{ke^{i\theta}: 0 \leq \theta \leq \pi\}$ and the other is a semicircle lying in the sphere $S^{n-1} \subset \mathbb{R}^n$, and joining k to $-k$.

On considering the map $\pi': B' \to \mathbb{C}^n \setminus \{0\}: \pi'(z) = z^2$, it may be observed that of these loops only the ones which lie entirely in $\{ke^{i\theta}: 0 \leq \theta \leq 2\pi\}$ and having zero winding number are contractible to a point. On restricting the map π' to the set

$$\{ke^{i\theta}: 0 \leq \theta \leq \pi\} \cup \{k \cos\theta + l \sin\theta: l \in S^{n-1} \text{ and } lk + kl = 0, 0 \leq \theta \leq \pi\}$$

we obtain a loop in $\mathbb{C} \setminus \{0\}$ which winds once around the origin. As the sphere S^{n-1} is simply connected the result follows ∎

Observation 5: For the case where $n = 2$ it is straightforward to adapt the proof of Proposition 11 to show that $H_1(\mathbb{C}^2 \setminus N(0), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

As a consequence of Proposition 11 we have

Theorem 22: *Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in $\mathbb{C}^n$ such that for some Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c\bar{c} \neq 0$, the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Suppose also that for the inclusion map $i: U_{2,\mathbb{C}} \to \mathbb{C}^n \setminus N(-c^{-1}d)$ the group homomorphism $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(-c^{-1}d), \mathbb{Z})$ is such that the image set $H(i)(H_1(U_{2,\mathbb{C}}, \mathbb{Z}))$ comprises solely of even cycles (i.e. each closed loop has an even winding number). Then the function $J_k(cz + d)$ is well defined on $U_{2,\mathbb{C}}$, and the modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic.*

Proof: Suppose that l is a closed loop in $U_{2,\mathbb{C}}$. Then for the map $p: U_{2,\mathbb{C}} \to \mathbb{C} \setminus \{0\}: z \mapsto (z + c^{-1}d)^2$ the set $p(l)$ is a closed loop which winds around the origin an even number of times. Consequently, the function

$$\begin{aligned}\big((z + c^{-1}d)^{-1} (z + c^{-1}d)^{-1}\big)^{1/2} &= \big((cz + c^{-1}d)^{-1} (z + \tilde{d}\check{c}^{-1})^{-1}\big)^{1/2} \\ &= \big((z + c^{-1}d)^{-1} c^{-1}\bar{c}^{-1}(z + \tilde{d}\bar{c}^{-1})^{-1}\big)^{1/2} (c\bar{c})^{1/2} \\ &= \big((z + d)^{-1} (c\tilde{z} + d)\big)^{-1/2} (c\bar{c})^{1/2}\end{aligned}$$

is well defined on each such loop. It follows that $J_k(cz + d)$ is a welldefined function on $U_{2,\mathbb{C}}$. By similar arguments to those given in the proof of Theorem 2.1 it now follows that the modules $\Gamma_{l,k}(U_{1,\mathbb{C}})$ and $\Gamma_{l,k}(U_{2,\mathbb{C}})$ are isomorphic ∎

An example of a domain lying in $\mathbb{C}^n$ which satisfy the properties described in Theorem 22 and contains closed loops which are not homologous to zero in $\mathbb{C}^n \setminus N(-c^{-1}d)$ is $\mathbb{C}^{n,*} = \{z \in \mathbb{C}^n \setminus N(0): z \neq \lambda x + i\mu y: x, y \in S^{n-1}$, with $xy + yx = 0$, $\lambda, \mu \in \mathbb{R}$ and $|\lambda| > |\mu|\}$.

As shown in the proof of Proposition 11 the domain $\mathbb{C}^n \setminus N(0)$ possesses closed cycles whose winding number is odd. In order to deal with domains like $\mathbb{C}^n \setminus N(0)$ we require the following

Definition 9: Suppose that U is a domain in $\mathbb{C}^n$ such that with respect to some point $z_1 \in \mathbb{C}^n$ we have $U \subseteq \mathbb{C}^n \setminus N(z_1)$, and that for the inclusion map $i: U \to \mathbb{C}^n \setminus N(z_1)$ the group homomorphism $H_1(i): H_1(U, \mathbb{Z}) \to H_1(\mathbb{C}^n N(z_1), \mathbb{Z})$ is surjective. Then we denote the Riemann surface, which is a two-fold covering of U, by U^2, and we denote the right $A_n(\mathbb{C})$ module of complex k-left regular functions defined on U^2 by $\Gamma^2_{l,k}(U, A_n(\mathbb{C}))$.

It is now a straightforward consequence of the constructions given in the proof of Proposition 11 to deduce

Theorem 23: *Suppose that n is odd and that $U_{1,\mathbb{C}}$ and $U_{2,\mathbb{C}}$ are domains in $\mathbb{C}^n$ such that for same Clifford matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c\bar{c} \neq 0$, the transformation $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}: U_{1,\mathbb{C}} \to U_{2,\mathbb{C}}: z \mapsto (az + b)(cz + d)^{-1}$ is a homeomorphism. Suppose also that for the inclusion map $i: U_{2,\mathbb{C}} \to \mathbb{C}^n \setminus N(-d\bar{c})$ the group homomorphism $H_1(i): H_1(U_{2,\mathbb{C}}, \mathbb{Z}) \to H_1(\mathbb{C}^n \setminus N(d\bar{c}), \mathbb{Z})$ is surjective. Then the function $J_k(cz + d)$ is well defined on the Riemann surface $U^2_{1,\mathbb{C}}$ but not on $U_{1,\mathbb{C}}$, and the modules $\Gamma^2_{l,k}(U_{1,\mathbb{C}}, A_n(\mathbb{C}))$ and $\Gamma_{l,k}(U_{2,\mathbb{C}}, A_n(\mathbb{C}))$ are isomorphic. Moreover, the modules $\Gamma_{l,k}(U_{1,\mathbb{C}}, A_n(\mathbb{C}))$ and $\Gamma^2_{l,k}(U_{2,\mathbb{C}}, A_n(\mathbb{C}))$ are also isomorphic.*

On combining the results obtained in this section with Theorem 1 and Proposition 1′ we have

Theorem 24: *The set of linear differential operators whose solution spaces are invariant under conformal transformations in $\mathbb{C}^n$ is the set $C = \{\lambda D^k: \lambda \in \mathbb{C}\setminus\{0\}$ and $k \in \mathbb{Z}^+\}$.*

For $\lambda_1 D_{k_1}, \lambda_2 D_{k_2} \in C$ we may define the product $\lambda_1 D^{k_1} \times \lambda_2 D^{k_2} = \lambda_1\lambda_2 D^{k_1+k_2} \in C$. Under this product it may be observed that C is a semigroup which is canonically isomorphic to the semigroup $(\mathbb{C}\setminus\{0\}) \times \mathbb{Z}^+$.

Concluding remark: In this paper we have used Clifford analysis to classify linear, conformally invariant differential equations, and we have shown that each such equation possesses a homotopy invariant Cauchy integral formula. It follows that a large class of the results already known within Clifford analysis, and within potential theory, for the iterates of the Laplacian (e.g. [5, 6]), carry through to this context.

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