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Maximal Op*-Algebras on DF-Domains

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Sei D ein dichter Bereich in einem Hilbertraum und sei $\mathcal{L}^+(D)$ die maximale Op*-Algebra von Operatoren auf D. In der Arbeit wird die gleichmäßige Topologie τ_D auf $\mathcal{L}^+(D)$ für den Fall untersucht, daß D ein DF-Raum bezüglich der Graphtopologie ist. Als Hauptergebnis wird eine Charakterisierung der beschränkten Teilmengen von D und der Topologie τ_D durch beschränkte selbstadjungierte Operatoren in H gegeben. Insbesondere ist jede beschränkte Teilmenge von D in einem beschränkten Ellipsoid enthalten. Als Anwendung wird bewiesen, daß jeder Operator in $\mathcal{L}^+(D)$ durch beschränkte Operatoren approximiert werden kann.

Пусть D плотное подпространство в гильбертовом пространстве и пусть $\mathcal{L}^+(D)$ максимальная Op*-алгебра линейных операторов на D. В работе равномерная топология τ_D на $\mathcal{L}^+(D)$ исследуется в случае когда D является пространством типа DF относительно проективной топологии. Главный результат — характеризация ограниченных подмножеств пространства D и топологии τ_D с помощью сильных ограниченных самосопряженных операторов. В частности, каждое ограниченное подмножество пространства D содержится в некотором эллипсоиде. В применение доказано что каждый оператор в $\mathcal{L}^+(D)$ является пределом ограниченных операторов.

Let D be any dense domain in a Hilbert space and let $\mathcal{L}^+(D)$ be the maximal Op*-algebra of (possibly unbounded) linear operators. In this paper the uniform topology τ_D on $\mathcal{L}^+(D)$ is investigated for the case where D is a DF-space with respect to the graph topology. As a main result, a characterization of the bounded subsets of D and of the topology τ_D by strongly bounded selfadjoint operators is given. Especially, each bounded subset of D is contained in some bounded ellipsoid. This is applied to approximate the operators in $\mathcal{L}^+(D)$ by bounded ones.

1. Introduction

Among the non-normable topological *-algebras, the maximal *-algebra $\mathcal{L}^+(D)$ of (possibly unbounded) linear operators on a dense linear subspace D of some Hilbert space H is of special interest since this algebra and its subideals are used in quantum physics. Therefore, the structure of $\mathcal{L}^+(D)$ and of the domain D supporting the algebra has been studied extensively. But up to now, far reaching and deep results could only be proved in the case of a Fréchet domain D. Using the fact that the structure of $L^+(D)$ depends in some sense only of the structure of the bounded subsets of D, Kürsten [7] could generalize some essential results to so-called quasi-Fréchet domains.

But these methods fail completely for domains which are strong duals of non-normable Fréchet spaces. Examples of such domain will be given in Section 3. In the main part of this paper we will develop a totally new technique to attack this dual metric case and we will demonstrate the power of this technique in proving that any operator in $\mathcal{L}^+(D)$ is the τ_D-limit of a net of bounded operators. In the case of metrizable domains this was shown by Kürsten in [7]. As the key result in this paper appears Theorem 5.1. It states that every closed DF-domain admits a fundamental system of bounded sets which are all "ellipsoids". A refinement of the technique presented here could allow moreover a detailed study of several subideals of $\mathcal{L}^+(D)$ as nuclear or compact operators as it was done for the metric case in [4].

2. Notation and basic results

As usually, for any pair E and F of locally convex spaces we denote by $\mathcal{L}(E, F)$ the linear space of all linear continuous operators from E into F. Concerning the notion of maximal Op*-algebras we will follow [10]. First of all let us recall this definition and some well-known results. In all the following let H be any fixed Hilbert space and let D be any dense linear subspace of H. For any linear closable operator A in H we denote by $\bar{A}$, A^* and $D(A)$ its closure, adjoint and domain, respectively. The restriction of A^* to D will be denoted by A^+. For given D the maximal Op*-algebra associated to D is defined by

$$\mathcal{L}^+(D) = \{A \in \operatorname{End}(D) : A^* \text{ exists } D \subseteqq D(A^*) \ \ A^*(D) \subseteqq D\}.$$

Obviously, $\mathcal{L}^+(D)$ is a *-algebra. The graph topology t on D is defined by the system of all seminorms

$$p_A(d) = \|Ad\| \quad d \in D, \quad A \in \mathcal{L}^+(D).$$

This topology coincides with the projective topology on D defined by the mappings $A: D \to H$ for $A \in \mathcal{L}^+(D)$. Since the identity 1_H belongs to $\mathcal{L}^+(D)$, the canonical embedding $J: D \to H$ is t-$\|\cdot\|$-continuous. Moreover, any operator $A \in \mathcal{L}^+(D)$ is t-t-continous as a map from D into itself.

From now on we restrict ourself to *closed domains*, i.e., we suppose

$$D = \cap \{D(\bar{A}) : A \in \mathcal{L}^+(D)\} = \cap \{D(A^{**}) : A \in \mathcal{L}^+(D)\}.$$

The selfadjoint domains characterized by $D = \cap \{D(A^*) : A \in \mathcal{L}^+(D)\}$ appear as a special case of such domains. Since the graph topology t is even generated by the system of energetic norms

$$p_A^{\text{erg}}(d) = (\|Ad\|^2 + \|d\|^2)^{1/2}, \quad A \in \mathcal{L}^+(D),$$

and since the domains $D(\bar{A})$ are Hilbert spaces with respect to the energetic norm, it follows that D is a projective limit of Hilbert spaces. In particular, (D, t) is a semireflexive and complete locally convex space. Now, the following main questions can be posed:

1. To what extent does the topological structure of D reflect the structure of $\mathcal{L}^+(D)$ and vice versa?
2. What topologies should be introduced on $\mathcal{L}^+(D)$ and what about dense subsets and states?
3. What subalgebras does exist in $\mathcal{L}^+(D)$?

For the answer to these questions the introduction of the strong dual space D_b' of (D, t) proves useful.

To avoid antilinear mappings we introduce the complex conjugate space D^+ of D' by replacing the original scalar multiplication in D' by the new one $(\lambda, x) \to \bar{\lambda}x$. Since any vector $h \in H$ defines a continuous linear functional f_h on D by $\langle d, f_h\rangle = (d, h)_H$, we get linear continuous embeddings

$$J = J_1' J_1 : D \to H \to D_b^+.$$

If we consider the bipolar of D in D', we obtain

$$(J_1'J_1D)^{00} = \left(J_1^{-1}(J_1D)^0\right)^0 = \left(J_1^{-1}(0)\right)^0 = D'.$$

This shows that D is $\sigma(D', D)$-dense in D' by the bipolar theorem. Since D is semi-reflexive, it is even $\sigma(D', D'')$-dense in D'. But then Mazur's theorem shows that D is also dense in D_b' and D_b^+ with respect to the strong topology.

Proposition 2.1: *Every operator $A \in \mathcal{L}^+(D)$ admits a uniquely determined extension to some linear continuous operator $\tilde{A} \in \mathcal{L}(D_b^+, D_b^+)$.*

Proof: Define $\tilde{A}$ as the adjoint operator of $A^+ : D \to D$ with respect to the dual pair $\langle D, D' \rangle$. For any $d, d_0 \in D$ we have $\langle d_0, \tilde{A}d \rangle_{D,D'} = \langle A^+d_0, d \rangle_{D,D'} = (A^+d_0, d)_H = (d_0, Ad)_H = \langle d_0, Ad \rangle_{D,D'}$. This shows $\tilde{A}d = Ad$ for all $d \in D$. Since D is weakly dense in D_b', $\tilde{A}$ is the only (weakly-)continuous extension of A ∎

Let us consider now some subideals in the algebra $\mathcal{L}^+(D)$. Very small subideals can be obtained by the following method due to Timmermann [16]. Let $\mathcal{A}(H)$ be any ideal of operators in the algebra $\mathcal{L}(H)$ of all bounded linear operators on the Hilbert space H. Then the set

$$\mathcal{A}(D) = \{S \in \mathcal{L}^+(D) : \overline{XSY} \in \mathcal{A}(H) \text{ for all } X, Y \in \mathcal{L}^+(D)\}$$

is obviously a $*$-ideal in $\mathcal{L}^+(D)$. As above, $\overline{XSY}$ denotes the closure of the operator XSY. The ideals $\mathcal{A}(D)$ are very small, because they contain only bounded operators. Of special importance in this paper is the ideal

$$\mathcal{B}(D) = \{S \in \mathcal{L}^+(D) : \overline{XSY} \in \mathcal{L}(H) \text{ for all } X, Y \in \mathcal{L}^+(D)\},$$

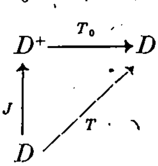
and we will mainly deal with such operators here. But the methods presented below can also be applied to the ideals $\mathcal{A}(D)$.

For closed domains D the ideal $\mathcal{B}(D)$ can be represented by [14, Chapter 3] as

$$\mathcal{B}(D) = \{S \in \mathcal{L}^+(D) : \overline{XS}, \overline{SY} \in \mathcal{L}(H) \text{ for all } X, Y \in \mathcal{L}^+(D)\}.$$

The following easy characterization of the operators of $\mathcal{B}(D)$ is very important.

Proposition 2.2: *An operator $T \in \mathcal{L}^+(D)$ belongs to $\mathcal{B}(D)$ if and only if there is an extension $T_0 \in \mathcal{L}(D_b^+, D)$ of T such that the following diagram commutes:*



Here J denotes the canonical embedding of D into D^+ introduced above. Using natural identifications we can express the proposition by the formula $\mathcal{B}(D) = \mathcal{L}(D_b^+, D)$.

Proof: We have to improve the construction of Proposition 2.1. Let $T \in \mathcal{B}(D)$ be given. In a first step we will prove $T^*(H) \subseteq D$ and $T^* \in \mathcal{L}(H, D)$. For fixed $h \in H$, $X \in \mathcal{L}^+(D)$ and $d \in D(X^*)$ we have $|(X^*d, T^*h)| = |(TX^*d, h)| \leq \|TX^*\| \, \|d\| \, \|h\|$. This shows $T^*h \in D(X^{**})$. Since D is assumed to be closed, this implies $T^*h \in D$. For any $Y \in \mathcal{L}^+(D)$ we get $p_Y(T^*h) = \|YT^*h\| \leq \|YT^*\| \, \|h\|$, and this proves the continuity of $T^* : H \to D$. Next, let us consider the adjoint operator $(T^*)' : D' \to H'$. It can be identified with some linear continuous operator $T_0 : D_b^+ \to H$. Let us prove that the range of T_0 is even contained in D. Fix $X \in \mathcal{L}^+(D)$, $d \in D(X^*)$ and $d' \in D^+$.

The continuity of d' means that there is some operator $Y \in \mathcal{L}^+(D)$ such that $|\langle d, d'\rangle| \leqq p_Y(d) = \|Yd\|$ for all $d' \in D$. So we get

$$|(X^*d, T_0d')_H| = |\langle T^*X^*d, d'\rangle_{D,D'}| \leqq \|YT^*X^*d\| \leqq \|YT^*X^*\| \, \|d\|.$$

This proves $T_0d' \in D(X^{**})$ for all $X \in \mathcal{L}^+(D)$. Hence $T_0D^+ \subseteqq D$. It remains to prove the continuity of $T_0\colon D_b^+ \to D$. Let $Y \in \mathcal{L}^+(D)$ be given. Then we have

$$p_Y(T_0d') = \|YT_0d'\|_H = \sup_{\|d\|\leqq 1} |(d, YT_0d')_H| = \sup_{\|d\|\leqq 1} |\langle T^*Y^+d, d'\rangle_{D,D'}|$$

$$\leqq \sup_{y\in T^*Y^+S_H} |\langle y, d'\rangle_{D,D'}| = \sup_{y\in M} |\langle y, d'\rangle| = p_{M^\circ}(d'),$$

where S_H is the unit ball in H and $M = T^*Y^+S_H$ is t-bounded in D because of the estimation $p_X(M) = \|XT^*Y^+S_H\| \leqq \|XT^*Y^+\|$. This proves $T_0 \in \mathcal{L}(D_b^+, D)$. Obvisously, T_0 coincides with T on D.

Conversely, let any $T_0 \in \mathcal{L}(D_b^+, D)$ be given. For fixed $X, Y \in \mathcal{L}^+(D)$ the product $YT_0\tilde{X}$ is a continuous map from D_b^+ into D. Especially, this is a continuous map from H into H. This proves $YT_0\tilde{X}|_H \in \mathcal{L}(H)$. ∎

There is a close connection between the set $\mathcal{B}(D)$ and the natural bornology of (D, t) for special domains. For Fréchet domains this was discovered in [4] and in [8]. In Section 4 we will treat the DF-case.

3. Selfadjoint DF-domains

In this section we present a general method to construct DF-domains. Let us recall the definition of DF-spaces. They have been introduced by Grothendieck to have a nice class containing the dual spaces of all F-spaces (F = Fréchet). Conversely, the strong duals of DF-spaces are F-spaces. But there exist DF-spaces without any predual. There are several different definitions of DF-spaces. Here we choose the following one (for equivalent conditions see [3]).

Definition 3.1: A locally convex space E is a *DF-space*, if it has a countable fundamental system of bounded subsets and if the intersection of any sequence of closed absolutely convex zero-neighbourhoods is a zero-neighbourhood provided that it absorbs all bounded subsets of E.

It is easy to see that every metrizable space with a countable fundamental system of bounded sets admits a bounded neighbourhood. So it must be normed. This implies that a non-normable DF-space cannot be metrizable. Now, let us start with the construction of DF-domains. This generalizes an example given in [9].

Let $\alpha = (\alpha_n)$ be any increasing sequence of positive real numbers satisfying

$$\lim_{n\to\infty} \ln n/\alpha_n \to 0. \tag{1}$$

Such sequences are called *nuclear exponent sequences of finite type*. The associated *power series space of finite type* is the space

$$\mathcal{X} = \Lambda_1(\alpha) = \left\{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |\varrho^{\alpha_n} a_n| < \infty \text{ for all } 0 < \varrho < 1\right\}.$$

This is a nuclear F-space with respect to the "normal" topology given by the seminorms $p_\varrho'(a) = \sum_n |\varrho^{\alpha_n} a_n|$, $0 < \varrho < 1$.

Proposition 3.1: *The following systems of seminorms are equivalent on* $\mathcal{X}$ $(0 < \varrho < 1)$:

(i) $p_\varrho'(a) = \sum_n |\varrho^{\alpha_n} a_n|$, (ii) $p_\varrho(a) = \left(\sum_n |\varrho^{\alpha_n} a_n|^2\right)^{1/2}$, (iii) $p_\varrho^\infty(a) = \sup_n |\varrho^{\alpha_n} a_n|$.

Proof: Clearly, we have $p_\varrho^\infty(a) \leqq p_\varrho(a) \leqq p_\varrho'(a)$ for all $a \in \mathbb{R}^\mathbb{N}$. To prove converse inequalities we first remark that, by (1), for every $0 < \mu < 1$ there is some number $n(\mu)$ such that $\ln n/\alpha_n + \ln \mu \leqq 0$ for all $n \geqq n(\mu)$. But this inequality is equivalent to $n\mu^{\alpha_n} \leqq 1$ for all $n \geqq n(\mu)$. Thus we get for $0 < \varrho < 1$ and $\mu = \sqrt{\varrho}$ the estimation

$$p_\varrho^\infty((na_n)) \leqq \sup_{n<n(\mu)} |n\varrho^{\alpha_n} a_n| + \sup_{n \geqq n(\mu)} |n\varrho^{\alpha_n} a_n|$$

$$\leqq n(\mu)\, p_\varrho^\infty(a) + \left(\sup_{n \geqq n(\mu)} |n\mu^{\alpha_n}|\right) p_\mu^\infty(a) \leqq c_\mu p_\mu^\infty(a), \tag{2}$$

where c_μ is some constant independent of a. This implies

$$p_\varrho'(a) = \sum_n |\varrho^{\alpha_n} a_n| = \sum_n n^{-2} \varrho^{\alpha_n} n^2 |a_n| \leqq 2 \sup_n |\varrho^{\alpha_n} n^2 a_n| \leqq 2c^2 p_\nu^\infty(a),$$

where $\nu = \varrho^{1/4}$ ∎

Corollary 3.2: *We have*

$$\mathcal{X} = \{a \in \mathbb{R}^\mathbb{N} : p_\varrho^\infty(a) < \infty \; \forall 0 < \varrho < 1\}$$

$$= \{a \in \mathbb{R}^\mathbb{N} : p_\varrho(a) < \infty \; \forall 0 < \varrho < 1\}.$$

Next, we will consider $\mathcal{X}$ as an algebra of diagonal operators on some subspace of l_2. Let $\mathcal{X}^\times$ be the Köthe dual of $\mathcal{X}$, i.e.

$$D := \mathcal{X}^\times = \{d \in \mathbb{R}^\mathbb{N} : \sum |d_n a_n| < \infty \text{ for all } a \in \mathcal{X}\}.$$

This is a complete locally convex space with respect to the normal topology given by

$$p_a'(d) = \sum |d_n a_n|, \quad a \in \mathcal{X}. \tag{3}$$

Since $l_2 \subseteqq \mathcal{X}$ we have $D = \mathcal{X}^\times \subseteqq l_2$. As in Corollary 3.2, this space D allows some other representations:

$$D = \{d \in \mathbb{R}^\mathbb{N} : p_a(d) = (\sum |d_i a_i|^2)^{1/2} < \infty,\ a \in \mathcal{X}\}$$

$$= \{d \in \mathbb{R}^\mathbb{N} : p_a^\infty(d) = \sup |d_i a_i| < \infty,\ a \in \mathcal{X}\}. \tag{4}$$

In fact, we have

$$p_a'(d) = \sum_n |d_n a_n| \leqq \left(\sum_n n^{-2}\right) \sup_n |d_n n^2 a_n| \leqq 2p_b^\infty(d),$$

where $b = (n^2 a_n)$. Since $a \in \mathcal{X}$, we get $b \in \mathcal{X}$ by (2).

Proposition 3.3: *The space D together with the equivalent systems of seminorms* (3) *and* (4) *is the strong dual space of the nuclear space $\mathcal{X}$. Especially, D is a nuclear DF-space contained in l_2.*

Proof: As a linear space, D is the topological dual of $\mathcal{X}$ with respect to the normal topology on $\mathcal{X}$ [6, § 30, 8]. It remains to be shown that the strong topology $b(D, \mathcal{X})$ on D coincides with the topology given by the seminorms (4). Clearly, this topology is weaker than the strong topology. On the other hand, let $M \subseteq \mathcal{X}$ be any p_ϱ^∞-bounded subset of $\mathcal{X}$. Then there is some constant c_ϱ such that $|a_n \varrho^{\alpha_n}| \leqq c_\varrho$ for all $n \in \mathbb{N}$ and all $a \in M$. Set $b_n = \sup \{|a_n| : a \in M\}$. Then we have $p_\varrho^\infty(b) \leqq c_\varrho$, thus $b \in \mathcal{X}$. But M is contained in the intervall $[-b, b]$. This proves

$$p_{M^\circ}(d) \leqq p_{[-b,b]^\circ}(d) = \sup_{a \in [-b,b]^\circ} |\sum d_i a_i| \leqq \sum |d_i b_i| = p_b'(d).$$

This completes the proof. ∎

Now we state the main result of this section.

Proposition 3.4: *For every nuclear exponent sequence, the space $D = \Lambda_1(\alpha)^\times$ is a selfadjoint domain in l_2 and the graph topology t on D coincides with the topology given by the seminorms (4). Especially, (D, t) is a nuclear DF-space.*

Proof: In a first step we consider the commutative algebra $\mathcal{X}$ as an operator algebra $\mathcal{A}$ on D by associating to each $a \in \mathcal{X}$ the diagonal operator $D_a = \sum_n a_n e_n \otimes e_n$. Here (e_n) denotes the canonical orthonormal basis in l_2. Since $\|D_a d\|^2 = \sum |a_n d_n|^2 = p_a(d)^2$, the operators D_a map D into itself, and the graph topology $t_{\mathcal{A}}$ generated by $\mathcal{A}$ on D coincides with the natural topology (4). Since D_a is a selfadjoint operator on the Hilbert space $l_2(a) = \{x : \sum |x_n a_n|^2 < \infty\}$, the domain $D = \cap \{l_2(a) : a \in \mathcal{X}\}$ is selfadjoint. It remains to be proven in a second step that $t_{\mathcal{A}} = t$, where t is the graph topology generated by the maximal Op*-algebra $\mathcal{L}^+(D)$ on D. To this end we use the following lemma due to KÜRSTEN [9] (recall that a sequence (e_n) is an unconditional basis in some locally convex space D, if for any $x \in D$ there are scalar coefficients α_n such that the net $\left\{\sum_{n \in I} \alpha_n e_n : I \subseteq \mathbb{N}, I \text{ finite}\right\}$ is convergent to x):

Let (e_n) be any orthonormal sequence in $D \subseteq H$. If (e_n) is an unconditional basis for some closed Op-algebra $\mathcal{A}$ on D then it is an unconditional basis for any closed Op*-algebra on D. Moreover, we have*

$$\sum_n |(x, e_n)|^2 \, \|Ae_n\|^2 < \infty \text{ for all } A \in \mathcal{L}^+(D) \text{ and all } x \in D.$$

Using this lemma we can show the coincidence of $t_{\mathcal{A}}$ and t on D. The canonical basis (e_n) in D is an unconditional basis for $(D, t_{\mathcal{A}})$, in fact, it is even an absolute basis by the definition of D and the topology (4). By the lemma we have

$$\sum |(d, e_n)| \, \|Ae_n\||^2 < \infty \text{ for all } d \in D,\ A \in \mathcal{L}^+(D).$$

Since $d = (\varrho^{\alpha_n}) \in D$ for all $0 < \varrho < 1$, this implies $a = (\|Ae_n\|) \in \mathcal{X}$ by Corollary 3.2. Since $Ad = \sum (d, e_n) Ae_n$, we finally obtain

$$p_A(d) = \|Ad\| = \|\sum (d, e_n) Ae_n\| \leqq \sum |(d, e_n)| \, \|Ae_n\| = p_a'(d).$$

This proves $t = t_{\mathcal{A}}$ ∎

4. Bounded Hilbert balls in domains

In this section we will give a characterization of the bounded Hilbert balls in domains. The set of all absolutely convex and bounded subsets of (D, t) will be denoted by $\mathfrak{B}(D)$. For every set $M \in \mathfrak{B}(D)$ the associated gauge functional is defined by

$$p_M(d) = \inf \{\varrho > 0 : d \in \varrho M\}.$$

If d is not in the linear hull of M then we put $p_M(d) = \infty$. The linear hull of M in D will be denoted by $D(M)$. If M is closed then $D(M)$ becomes a Banach space with respect to the norm p_M, and this space is continuously embedded into D. This is a consequence of the completenes of D (cf. [3, 1.3.4]). A set $M \in \mathfrak{B}(D)$ is called to be a *bounded Hilbert ball*, if its gauge functional p_M satisfies the parallelogram equation

$$p_M(x + y)^2 + p_M(x - y)^2 = 2\big(p_M(x)^2 + p_M(y)^2\big). \tag{5}$$

In this case $D(M)$ is a Hilbert space under p_M for closed sets M, and the associated scalar product will be denoted by $[x, y]$. It is very important that the Hilbert balls in D can be characterized in the following way.

Proposition 4.1: *For every closed and bounded Hilbert ball $M \subseteqq D$ there is some positive operator $T \in \mathscr{B}(D)$ sich that $M = T(S_H)$. Conversely, if T is any operator in $\mathscr{B}(D)$, then the set $M = T(S_H)$ is a bounded Hilbert ball.*

Proof: Let us start with the second statement. If $M = T(S_H)$, then its gauge functional can be computed as $p_M(y) = \inf \{\|x\| : y = Tx\}$. But the norm satisfies the parallelogram equation and this property transmits to the infimum. Furthermore, if any operator $A \in \mathscr{L}^+(D)$ is given, then we have $\|AM\| = \|ATS_H\| \leqq \|\overline{AT}\|$. This proves the boundedness of M in D. Conversely, let any closed and bounded Hilbert ball M in D be given. As above, we denote by $[\cdot, \cdot]$ the scalar product associated to p_M. Since M is even bounded in H, there is some constant $c > 0$ such that

$$[x, x] = p_M(x)^2 \geqq c\,\|x\|^2 \text{ for all } x \in M. \tag{6}$$

Let H_1 be the norm closure of $D(M)$ in H. Since every p_M-Cauchy sequence in $D(M)$ is convergent in $D(M)$, the form $[\cdot, \cdot]$ is even closed in the sense of [5]. By [5, Thm. 2.33] there is some positive operator W in H_1 such that $D(M) \subseteqq D(W) \subseteqq H_1$ and $[x, y] = (Wx, Wy)_H$ for every $x, y \in (M)$. Especially, we have

$$p_M(x) = \|Wx\| \text{ for all } x \in D(M). \tag{7}$$

From $N(W) + \overline{R(W)} = H_1$ and $N(W) = 0$ by (6) we obtain $\overline{R(W)} = H_1$. By (7), the map $W: D(M) \to H_1$ is a p_M-$\|\cdot\|$-isometry. Hence, $R(W)$ is norm-complete in H_1, since $D(M)$ is complete. This shows $R(W) = H_1$ Since $N(W) = 0$, the inverse operator $W^{-1}: H_1 \to D(M)$ exists and is an isometry, too. Especially, we have $W^{-1}(S_{H_1}) = M$. Now, define $T \in \mathscr{L}(H)$ by $T(x_1 \oplus x_2) = W^{-1}x_1$ for $x_1 \oplus x_2 \in H_1 \oplus (H - H_1)$. Then we have $T(S_H) = M$ and $T = T^* \geqq 0$. Since M is t-bounded, the sets $YTS_H = YM$ is norm-bounded for all $Y \in \mathscr{L}^+(D)$. Therefore, YT has a bounded closure in H. Since $TY \subseteqq (YT^*)^*$, the same is true for TY. This proves $T \in \mathscr{B}(D)$ ∎

5. Bounded subsets in DF-domains

As already mentioned in the introduction, the characterization of the bounded subsets plays a key role for numerous questions concerning the structure of D and of $\mathscr{L}^+(D)$. The main result in this section is the following theorem.

Theorem 5.1: *Let (D, t) be any closed DF-domain and let M be any subset of D. Then the following conditions are equivalent:*

(i) *M is t-bounded.*
(ii) *There is some bounded Hilbert ball containing M.*
(iii) *There is some operator $T \in \mathcal{B}(D)$ such that $M \subseteq T(S_H)$.*

For F-spaces a similar result has been proved in [8]. In view of Proposition 4.1 it is sufficient to show that every t-bounded absolutely convex and closed subset of D is contained in some bounded Hilbert ball of D. But before we need some deep results from the theory of operator ideals in Banach spaces. Let E and F be any Banach spaces. An operator $T \in \mathcal{L}(E, F)$ is called to be a *Hilbert operator* if there is a factorization $T = SR$ with $R \in \mathcal{L}(E, H)$ and $S \in \mathcal{L}(H, F)$, where H is a Hilbert space. The set of all Hilbert operators from E into F will be denoted by $\mathcal{H}(E, F)$. It can be shown [12] that the quantity

$$|||T||| = \inf \|R\| \, \|S\|$$

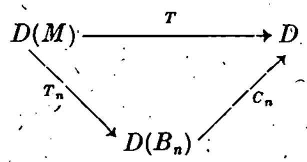
defines a norm on $\mathcal{H}(E, F)$. Here the infimum is taken over all possible factorizations of T through some Hilbert space. It is easy to see that for operators $T \in \mathcal{L}(E, F)$ with $\dim R(T) = d < \infty$ the inequality $|||T||| \leqq d \, \|T\|$ holds true.

The next proposition is crucial for the proof of Theorem 5.1. Roughly spoken, this proposition states that Hilbert operators can be characterized by its finite-dimensional parts. Let dim (E) be the set of all finite-dimensional subspaces of E. For $M \in$ dim (E) we denote by J_M the canonical embedding of M into E. Analogously, let codim (F) be the set of all subspaces of F of finite codimension. For $N \in$ codim (F) let Q_N be the canonical map from F onto the factor space F/N.

Proposition 5.2: *Let E and F be any Banach spaces. An operator $T \in \mathcal{L}(E, F)$ is a Hilbert operator if and only if there is a constant c depending only on T such that $|||Q_N T J_M||| \leqq c$ holds true for all $M \in$ dim (E) and all $N \in$ codim (F).*

The proof of this proposition can be found in [12, 19.3.7/8]. The idea runs as follows. The finite-dimensional operators $Q_N T J_M$ for $N \in$ codim (F) and $M \in$ dim (E) admit uniformly bounded factorizations through Hilbert spaces. Using the ultraproduct technique one can reconstruct the operator T from its finite-dimensional parts. But the ultraproduct of Hilbert spaces is again a Hilbert space. This yields the desired factorization of T. Now we are ready to prove the theorem.

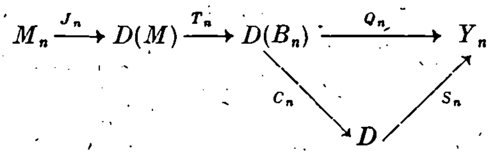
Proof of Theorem 5.1: Let M be any t-bounded, absolutely convex and closed subset of D. The linear hull $D(M)$ of M in D is a Banach space with respect to the gauge functional p_M. Suppose for the moment that there would be another bounded, absolutely convex and closed subset M_1 of D containing M such that the embedding map $T: D(M) \to D(M_1)$ factorizes through some Hilbert space H_1 as $T = RS$. Then the set $R(S_{H_1})$ would be a bounded Hilbert ball in $D(M_1) \subseteq D$, and (ii) would follow from $M = T(M) = RS(M) \subseteq \|S\| \, R(S_{H_1})$. Se we have reduced the proof of the theorem to the existence of such a set M_1. Let us suppose now that such a set M_1 would not exist. Since D is supposed to be a DF-space, there is a countable fundamental system (B_n) of closed, absolutely convex and bounded subsets. We may suppose $M \subseteq B_1$ and $2B_n \subseteq B_{n+1}$. The linear hull $D(B_n)$ of B_n in D is a Banach space with respect to the norm p_{B_n}. Since $M \subseteq B_n$ for all $n \in \mathbb{N}$, the canonical embedding T: $D(M) \to D$ factors through the canonical embeddings $C_n: D(B_n) \to D$ according to the diagram



By assumption, non of the maps T_n is a Hilbert map. According to Proposition 5.2 there are subspaces $M_n \in \dim(D(M))$ and $N_n \in \operatorname{codim}(D(B_n))$ such that the operators

$$Q_nT_nJ_n \colon M_n \hookrightarrow D(M) \to D(B_n) \twoheadrightarrow D(B_n)/N_n$$

satisfy $|||Q_nT_nJ_n||| > 2n$. For abbreviation we set $Y_n = D(B_n)/N_n$. Put $\varepsilon = d_n^{-1}$, where $d_n = \operatorname{rank} J_n = \dim R(J_n)$. Now, we use Proposition 4.3.11 of [3]. This result states that for every $\varepsilon > 0$ there is a linear continuous operator S_n in the non-commutative diagram



such that $\|S_nC_n\| \leqq 2$ and $\|(S_nC_n - Q_n)\, T_nJ_n\| < \varepsilon$. This means that up to ε the operator S_n is a bounded lifting of Q_n on the finite-dimensional subspace $T_nJ_n(M_n) \subseteqq D$. Now, we can estimate the Hilbert norm of the finite-dimensional operators as follows:

$$\begin{aligned} |||Q_nT_nJ_n - S_nTJ_n||| &= |||(Q_n - S_nC_n)\, T_nJ_n||| \\ &\leqq \|(Q_n - S_nC_n)\, T_nJ_n\|\, d_n \leqq 1 . \end{aligned}$$

This proves

$$\begin{aligned} |||S_nTJ_n||| &= |||Q_nT_nJ_n - (Q_nT_nJ_n - S_nTJ_n)||| \\ &\geqq |||Q_nT_nJ_n||| - |||Q_nT_nJ_n - S_nTJ_n||| \geqq 2n - 1 \geqq n . \end{aligned}$$

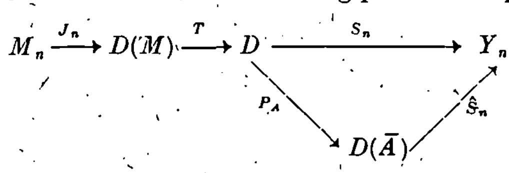
We pull back now the closed unit balls S_{Y_n} of the spaces Y_n to D by the definition

$$V \doteq \cap \{S_n^{-1}(S_{Y_n}) \colon n \in \mathbb{N}\} .$$

Let us prove that V is a t-neighbourhood in D. In view of the continuity of the S_n and of Definition 3.1 it remains to be shown that V absorbs each B_k. Since the finite intersection $\cap \{S_n^{-1}(S_{Y_n}) \colon n \leqq k\}$ is a t-neighbourhood, it absorbs B_k. But for $n > k$ we have $2B_k \subseteqq B_n$ and $S_n(B_n) = S_nC_n(B_n) \subseteqq 2S_{Y_n}$ because of $\|S_nC_n\| \leqq 2$. This means $B_k \subseteqq S_n^{-1}(S_{Y_n})$. Therefore, V absorbs B_k. This shows that V is a t-neighbourhood in D. But then there exists some operator $A \in \mathcal{L}^+(D)$ such that

$$p_V(d) \leqq p_A^{\mathrm{erg}}(d) = (\|Ad\|^2 + \|d\|^2)^{1/2} \tag{8}$$

for all $d \in D$. So we have the following product of operators:



where P_A is the continuous embedding of D into the domain $D(\bar{A})$ of $\bar{A}$ equipped with the norm p_A^{erg}, and where $\hat{S}_n$ is some operator satisfying $\hat{S}_n P_A = S_n$. By (8), $\hat{S}_n$ is uniquely determined and of norm $\|\hat{S}_n\| \leqq 1$. Finally, we can replace Y_n by some factor space of $D(\bar{A})$. Indeed, the space $N_n = \ker \hat{S}_n$ is of finite codimension in $D(\bar{A})$ and we get a factorization of $\hat{S}_n$ through the quotient map $Q_n' : D(\bar{A}) \to D(\bar{A})/N_n$ as $\hat{S}_n = \hat{\hat{S}}_n Q_n'$. The uniquely determined operator $\hat{\hat{S}}_n : D(\bar{A})/N_n \to Y_n$ is of the norm $\|\hat{\hat{S}}_n\| = \|\hat{S}_n\| \leqq 1$. This way we have constructed operators

$$Q_n' P_A T J_n : M_n \to D(M) \to D(\bar{A}) \to D(\bar{A})/N_n$$

such that

$$\begin{aligned} n < |||S_n T J_n||| &= |||\hat{S}_n P_A T J_n||| = |||\hat{\hat{S}}_n Q_n' P_A T J_n||| \\ &\leqq \|\hat{\hat{S}}_n\| \, |||Q_n' P_A T J_n||| \leqq |||Q_n' P_A T J_n|||. \end{aligned}$$

This implies $P_A T \notin \mathcal{H}(D(M), D(\bar{A}))$ by Proposition 5.2, but this contradicts the fact that $D(\bar{A})$ is a Hilbert space. Thus we are done ∎

6. The uniform topology τ_D and its characterization for DF-domains

There are several possibilities to introduce natural topologies on $\mathcal{L}^+(D)$. One of the most important among these is the so-called uniform topology τ_D. This topology was introduced by Lassner in [10] and it was intensively studied in the past by several authors. Concerning the case of F-domains we refer once more to [8]. The topology τ_D is given by the system of all seminorms

$$p_M(A) = \sup \{|(Ad_1, d_2)| : d_1, d_2 \in M\}, \qquad A \in \mathcal{L}^+(D),$$

where M runs over a basis of the absolutely convex and t-bounded subsets of D. The embedding $D \subseteqq H \subseteqq D_b^+$ leads to the embedding $\mathcal{L}^+(D) \subseteqq \mathcal{L}(D, D_b^+)$. In this context, the topology τ_D appears as the restriction of the bounded open topology on $\mathcal{L}(D, D_b^+)$. The results of Section 5 allow a characterization of τ_D by the subalgebra $\mathcal{B}(D)$ of $\mathcal{L}^+(D)$.

Theorem 6.1: *Let D be any closed DF-domain. Then the uniform topology τ_D on $\mathcal{L}^+(D)$ is given by the system of all seminorms*

$$p_T(A) = \|\overline{TAT}\|, \; A \in \mathcal{L}^+(D), \; T \in \mathcal{B}(D), \; T \geqq 0.$$

Proof: The statement follows directly from Theorem 5.1 and Proposition 4.1 ∎

The foregoing theorem allows the application of the technique developped for the metric case in [8] and [9] also in the DF-case.

Proposition 6.2: *Let D by any closed DF-domain. For every $X \in \mathcal{L}^+(D)$ and for every τ_D-continuous seminorm p there is some orthogonal projection $P \in \mathcal{B}(D)$ such that $p(X - PXP) \leqq 1$.*

Proof: By Theorem 6.1 we may assume $p = p_T$ for some $T \in \mathcal{B}(D)$, $T \geqq 0$. Let $T = \int \lambda \, dE_\lambda$ be the spectral representation of T. First of all, we prove that for every $\varepsilon > 0$ the projection

$$P_\varepsilon = \int_\varepsilon^\infty dE_\lambda$$

belongs to $\mathscr{B}(D)$. To this aim we introduce the operator

$$R_\varepsilon = \int_\varepsilon^\infty \lambda^{-1}\, dE_\lambda .$$

Then we have $R_\varepsilon \in \mathscr{L}(H)$ and $P_\varepsilon = TR_\varepsilon$. Since $P_\varepsilon(H) = TR_\varepsilon(H) \subseteqq D$, we get $P_\varepsilon \in \mathscr{L}^+(D)$. But $P_\varepsilon: H \to D$ is even $\|\cdot\|$-t-continuous. Indeed, for every $A \in \mathscr{L}^+(D)$ we have

$$p_A(P_\varepsilon h) = \|AP_\varepsilon h\| = \|ATR_\varepsilon h\| \leqq \|\overline{AT}\|\, \|R_\varepsilon\|\, \|h\| .$$

Therefore, the adjoint operator $P_\varepsilon^+: D^+ \to H$ exists, and $P_\varepsilon^+ P_\varepsilon \in \mathscr{L}(D^+, D)$ is an extension of P_ε. This proves $P_\varepsilon \in \mathscr{B}(D)$ by Proposition 2.2. Define $Q_\varepsilon = 1 - P_\varepsilon$ and let $X \in \mathscr{L}^+(D)$ be given. Then we obtain

$$\begin{aligned} p_T(X - P_\varepsilon X P_\varepsilon) &= \|T(X - P_\varepsilon X P_\varepsilon)\, T\| = \|TXT - TP_\varepsilon X P_\varepsilon T\| \\ &= \|Q_\varepsilon TXT + P_\varepsilon TXTQ_\varepsilon\| \\ &\leqq \|Q_\varepsilon T\|\, \|XT\| + \|P_\varepsilon\|\, \|TX\|\, \|TQ_\varepsilon\| \leqq \varepsilon\|XT\| + \|TX\|\, \varepsilon \leqq 1 \end{aligned}$$

for sufficiently small $\varepsilon > 0$ ∎

Corollary 6.3: *Let D be any closed DF-domain. Then the set $\mathscr{B}(D)$ is τ_D-dense in $\mathscr{L}^+(D)$.*

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