Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (5) 1990, S. 415–428

Potential Flow Past a Circular Cylinder with Permeable Surface and Nonlinear Filtration Law

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Dedicated to Prof. Dr. H. Beckert on the occasion of his 70th birthday

Es wird die ebene Potentialströmung um einen Kreiszylinder mit poröser Oberfläche bei zwei Klassen von nichtlinearen Filtrationsgesetzen behandelt, indem das Problem entsprechend auf eine nichtlineare singuläre Integralgleichung und auf ein lineares Riemann-Hilbert-Problem für eine quasilineare Gleichung vom Beltrami-Typ zurückgeführt wird. Für das Problem wird dann ein allgemeiner topologischer Existenzsatz bewiesen.

Задача плоского потенциального обтекания круглого цилиндра с пористой оболочкой для двух классов нелинейных законов фильтрации сводится соответственно к нелинейному сингулярному интегральному уравнению и к линейной задаче Римана-Гильберта для квазилинейного уравнения типа Бельтрами. Доказывается общая топологическая теорема существования для этой задачи.

The plane potential flow past a circular cylinder with porous surface for two classes of nonlinear filtration laws is dealt with by reducing the problem to a nonlinear singular integral equation and to a linear Riemann-Hilbert problem for a quasilinear equation of Beltrami type, respectively. A general topological existence theorem for the problem will be proven.

Introduction

The problem of plane potential flow of an inviscid incompressible fluid past a circular cylinder with porous surface has been dealt with by Baičorov in 1951, 1952 (cf. [6]). He proposed two models for this problem supposing that either the tangential velocity of the fluid on the inner surface of the cylinder is zero or the pressure of the fluid is constant there. Both models of Baičorov were investigated from the mathematical viewpoint in our paper [6]. In [13] the second author proposes another model for this problem constructing also the whole flow inside the cylinder as a potential flow and do not making a special assumption about the flow on the inner surface of the cylinder. In the case of a linear filtration law the problem is reduced to a linear singular integral equation of Hilbert type which is solvable in closed form.

In the present paper we consider this model for a general nonlinear filtration law. After briefly restating the mathematical model, we prove three existence theorems under different assumptions. First the problem is reduced to a nonlinear singular integral equation of Hilbert type for which a solution is constructed by means of successive approximation. This method works for nearly linear filtration laws and may also be used for computing the solution of the problem. Further, the problem is equivalent to a nonlinear Riemann-Hilbert problem for a holomorphic function in the unit disk. In the second proof the last problem is transformed to a linear Riemann-Hilbert problem for a quasi-linear equation of Beltrami type. Adopting an approach by VINOGRADOV [10], we prove the existence of a solution to this problem in the case of filtration laws with a linear minorant and majorant. Then a third nonconstructive general existence proof will be given by means of the Leray-Schauder degree and homotopy theory. Using recent results of the first author about strongly nonlinear Riemann-Hilbert problems, we will obtain the existence of a solution for filtration laws with linear majorant alone, including the case of a semipermeable surface of the cylinder.

1. Statement of the problem

As in [13] we consider the plane steady irrotational flow of an inviscid incompressible fluid with density ϱ past a circular cylinder with radius a and permeable (porous or perforated) surface. We look for the velocity field $\mathfrak{v}(z)$, $z = x + iy$, in a cross-section plane of the flow with the components v_x, v_y and the amount q. It is assumed that the flow at infinity is parallel to the x-axis and has the speed q_∞. The pressure field in the flow is denoted by $p(z)$ and the pressure at infinity by p_∞. We neglect the influence of gravity. Further we suppose that the flows outside and inside the cylinder are potential flows with prescribed mass and vorticity distributions in a closed domain in the interior of the cylinder, inducing a total flux Q through the cylinder surface Γ and a circulation J around Γ inside the cylinder. Besides we also assume a circulation $\bar{J}$ outside the cylinder. (In [13] for simplicity $J = \bar{J}$ was taken.)

The complex velocity function $W = v_x - iv_y$ possesses the decomposition (cf. [13])

$$W(z) = \begin{cases} W_0(z) + q_\infty\left(1 - \dfrac{a^2}{z^2}\right) - \dfrac{\mathrm{i}D}{2\pi z} + \dfrac{a}{z} + w(z) & \text{if } |z| > a, \\ W_0(z) + \dfrac{a}{z}\, w(z) & \text{if } |z| < a, \end{cases} \tag{1.1}$$

where $D = \bar{J} - J$, $W_0(z)$ is the complex velocity function due to the prescribed mass and vorticity distributions, and $w(z)$ is a sectionally holomorphic function satisfying the conditions

$$w(0) = 0, \qquad w(z) = O(1/z) \text{ at infinity}. \tag{1.2}$$

In particular

$$W_0(z) = -(2\pi z)^{-1}(Q + \mathrm{i}J) \tag{1.3}$$

if there are a sink of strength Q and a vortex of strength J at the origin.

As in [13] we assume that the filtration velocity V in direction of the inner normal on Γ obeys a filtration law

$$V = F(s, p_a - p_i) \qquad \text{on } \Gamma\colon z = a\,e^{is}, \tag{1.4}$$

where the function $F(s, p)$ is continuous and non-decreasing with respect to the difference $p = p_a - p_i$ of the pressure on the outer and inner surface of Γ satisfying $F(s, 0) = 0$, and is 2π-periodic and (piecewise-) continuous with respect to the polar angle s. By means of Bernoulli's equation and the continuity of the normal component of the velocity through Γ this condition leads to the basic relation

$$\Phi(s) = A(s) - F\left(s, c + C(s)\left[q_\infty \sin s - \frac{D}{4\pi a} + \Psi(s)\right]\right) \tag{1.5}$$

for the real and imaginary parts Φ, Ψ of the inner and the outer boundary values

$$w^+(a\,e^{is}) = \Phi(s) - \mathrm{i}\Psi(s), \quad w^-(a\,e^{is}) = \Phi(s) + \mathrm{i}\Psi(s)$$

of w on Γ, where $\Psi = H\Phi$ and

$$\int_{-\pi}^{\pi} \Phi(s)\, ds = \int_{-\pi}^{\pi} \Psi(s)\, ds = 0 \tag{1.6}$$

in virtue of (1.2) with the Hilbert operator

$$H\Phi(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\sigma) \cot \frac{\sigma - s}{2}\, d\sigma.$$

Here c is an unknown pressure constant and $C(s) = 2\varrho[B(s) - q_\infty \sin s + D/4\pi a]$, where $A(s)$, $B(s)$ are the normal and tangential component of the prescribed velocity field W_0. In particular, in the case (1.3) we have $A(s) = Q/2\pi\bar{a}$, $B(s) = J/2\pi a$.

Remark 1: If the functions Φ, Ψ and the unknown constant c are determined, the velocity and pressure distributions on Γ are given by the formulae (39), (40) of [13], where in the expressions for v_s^- and p_a the term $D/2\pi a$ has to be added to B.

As shown in [13] there exists at most one solution of the flow problem with continuous velocity and pressure distributions on Γ if the function $F(s, p)$ is continuous and possesses a positive continuous derivative $F_p(s, p)$ with respect to p.

2. Reduction to a singular integral equation

The basic relation (1.5) with $\Psi = H\Phi$ represents a nonlinear singular integral equation for Φ with an unknown constant c and the additional condition (1.6) for Φ. Applying the Hilbert operator H to (1.5), we obtain an analogous integral equation for Ψ. Especially in the case of (1.3) with $J = \bar{J} = 0$ we have $A(s) = Q/2\pi a$, $B(s) = 0$ and the equation for Ψ takes the simple form

$$\Psi = -H\big[F\big(s, c + C(s)\,\{q_\infty \sin s + \Psi(s)\}\big)\big] \tag{2.1}$$

with $C(s) = -2\varrho q_\infty \sin s$, and the constant c is determined by the condition

$$\int_{-\pi}^{\pi} F\big(s, c + C(s)\,\{q_\infty \sin s + \Psi(s)\}\big)\, ds = Q/a \tag{2.2}$$

following from (1.5) by integration over Γ. The condition (1.6) for Ψ is automatically fulfilled by (2.1).

In the sequel we consider the equations (2.1), (2.2) for functions $F(s, p)$ of the form

$$F(s, p) = Kp + F_0(s, p) \tag{2.3}$$

with a positive constant K and a function F_0 which is 2π-periodic and continuous with respect to s and fulfils a Lipschitz condition with respect to p:

$$|F_0(s, p_1) - F_0(s, p_2)| \leqq N\, |p_1 - p_2|, \quad N = \text{const.} \tag{2.4}$$

Since $F_0(s, 0) = 0$, from (2.4) we further obtain

$$|F_0(s, p)| \leqq N\, |p|. \tag{2.5}$$

The substitution $\Psi_1(s) = \Psi(s) + q_\infty \sin s$ leads to the equation for Ψ_1

$$\Psi_1 - \tilde{M}H[\sin s\, \Psi_1] = q_\infty \sin s - H[F_0], \tag{2.6}$$

where $M = 2\varrho K q_\infty$, $F_0 = F_0(s, c - M_0 \sin s\, \Psi_1(s))$, $M_0 = 2\varrho q_\infty$ together with the condition

$$2\pi K c = Q/a + M \int_{-\pi}^{\pi} \sin s\, \Psi_1\, ds - \int_{-\pi}^{\pi} F_0\, ds. \tag{2.7}$$

Considering (2.6) as a linear singular integral equation with known right-hand side, from [12: Appendix] and [4] after some lengthy calculations we obtain

$$\Psi_1 = \Psi_0 + a(s)\, F_0 - b_1(s)\, H[b_2 F_0]\,(s), \tag{2.8}$$

where

$$\Psi_0 = q_\infty \frac{\sin s}{\sqrt{1 + M^2} - M \cos s}, \qquad a(s) = \frac{M \sin s}{1 + M^2 \sin^2 s},$$

$$b_1(s) = \frac{\sqrt{1 + M^2} + M \cos s}{1 + M^2 \sin^2 s}, \qquad b_2(s) = \frac{\sqrt{1 + M^2} - M \cos s}{1 + M^2 \sin^2 s}.$$

Analogously the condition (2.7) reads

$$c = c_0 - \frac{1}{2\pi K} \int_{-\pi}^{\pi} b_2(s)\, F_0\, ds \quad \text{with} \quad c_0 = \frac{1}{K}\left[\frac{Q}{2\pi a} + \frac{M q_\infty}{1 + \sqrt{1 + M^2}}\right]. \tag{2.9}$$

Finally, we set $\chi = \Psi_1 - \Psi_0$ and $C = c - c_0$ and write the equations (2.8), (2.9) in fixed-point form

$$\{\chi, C\} = T\{\chi, C\} \tag{2.10}$$

with the operator

$$T\{\chi, C\} = \begin{cases} a(s)\, F_0 - b_1(s)\, H[b_2 F_0] \\ -(2\pi K)^{-1} \int_{-\pi}^{\pi} b_2(s)\, F_0\, ds. \end{cases} \tag{2.11}$$

The functions a, b_1, b_2 satisfy the inequalities

$$|a(s)| \leqq M/(1 + M^2), \qquad |b_k(s)| \leqq M + \sqrt{1 + M^2} \qquad (k = 1, 2). \tag{2.12}$$

In the following we are looking for solutions $\Psi_1 \in L_2(-\pi, \pi)$, i.e., $\{\chi, C\} \in E := L_2(-\pi, \pi) \times \mathbb{R}$ of (2.10) by means of Banach's fixed-point theorem. Corresponding solutions of our flow problem are said to be *generalized solutions*. Obviously, in virtue of (2.5) the operator T maps the space E into itself. Further, for two pairs $\{\chi_k, C_k\} \in E$ with $p_k := c_0 + C_k - M_0 \sin s\, [\Psi_0 + \chi_k]$ $(k = 1, 2)$ by (2.4) we have

$$|F_0(s, p_1) - F_0(s, p_2)| \leqq N\, |C_1 - C_2| + N M_0\, |\chi_1 - \chi_2|$$

and, for the norm $\|\cdot\|$ in $L_2(-\pi, \pi)$,

$$\|F_0(s, p_1) - F_0(s, p_2)\| \leqq \varrho(\chi_1 - \chi_2, C_1 - C_2) \tag{2.13}$$

with

$$\varrho(\chi, C) = \sqrt{2\pi}\, N\, |C| + N M_0\, \|\chi\|. \tag{2.14}$$

Besides, by (2.12) for any $h \in L_2(-\pi, \pi)$ there holds the estimation

$$\|b_1 H[b_2 h]\| \leqq \tilde{M}^2\, \|h\|, \qquad \tilde{M} = M + \sqrt{1 + M^2}. \tag{2.15}$$

In virtue of (2.12), (2.13), (2.15) for the first component T_1 of $T\{\chi, C\}$ we obtain

$$\|T_1\{\chi_1, C_1\} - T_1\{\chi_2, C_2\}\| \leqq A_0 \varrho(\chi_1 - \chi_2, C_1 - C_2) \tag{2.16}$$

with $A_0 = \tilde{M}^2 + M/(1 + M^2)$.

Further, by Schwarz' inequality, for any $h \in L_2(-\pi, \pi)$ there holds

$$\left| \int_{-\pi}^{\pi} b_2 h \, ds \right|^2 \leqq 2\pi \sqrt{1 + M^2} \, \|h\|^2.$$

Hence, taking (2.13) into account, for the second component T_2 of $T\{\chi, C\}$ we have

$$|T_2\{\chi_1, C_1\} - T_2\{\chi_2, C_2\}| \leqq \left(A_1/\sqrt{2\pi}\right) \varrho(\chi_1 - \chi_2, C_1 - C_2) \tag{2.17}$$

with $A_1 = (1 + M^2)^{1/4}/K$.

Introducing in E the equivalent norm $\|\{\chi, C\}\|_E = \varrho(\chi, C)$, the estimations (2.16), (2.17) imply $\|T\{\chi, C\}\|_E \leqq \varkappa \, \|\{\chi, C\}\|_E$ with $\varkappa = N[M_0 A_0 + A_1] = N_0 A(M)$, where $N_0 = N/K$ and $A(M) = M\left(\tilde{M}^2 + M/(1 + M^2)\right) + (1 + M^2)^{1/4}$. Banach's fixed point theorem now yields

Theorem 1: *If $\varkappa < 1$ with $\varkappa$ defined by* (2.19), *there exists a uniquely determined generalized solution of the flow problem in the case of* (1.3) *with $\bar{J} = J = 0$ and* (2.3) *with* (2.4). *The corresponding equation* (2.10) *can be solved by the method of successive approximations.*

Remark 2: The constants M and N_0 are dimensionless, therefore also χ is a dimensionless constant. In the example of Baičorov (cp. [6]) M has the value $M = 0.13296$ with $A(M) = 1.19509$. Hence in this example the condition $\varkappa < 1$ is fulfilled if $N_0 < 0.83675$. Generally, since $A(M) > 1$, there must be always $N_0 < 1$.

Remark 3: The condition $\varkappa < 1$ is fulfilled if for fixed M the constant N_0 is sufficiently small. If $F_0(s, p)$ satisfies a Hölder-Lipschitz condition of the form

$$|F_0(s_1, p_1) - F_0(s_2, p_2)| \leqq N_0[q_\infty |s_1 - s_2|^\alpha + K |p_1 - p_2|], \quad 0 < \alpha < 1,$$

implying $|F_0(s, p)| \leqq N |p|$, $N = N_0 K$, for sufficiently small N_0 there also exists a Hölder continuous solution $\Psi \in C^\alpha[-\pi, \pi]$. This can be proved by means of Schauder's fixed point theorem applied to the operator T on the convex compact set

$$\mathcal{K} = \{\{\chi, C\} \in F := C[-\pi, \pi] \times \mathbb{R} : |\chi(s)| \leqq R, |C| \leqq S, \\ |\chi(s_1) - \chi(s_2)| \leqq R_0 |s_1 - s_2|^\alpha\}$$

with fixed positive R, R_0, S (cf. [6]). If then also the condition $\varkappa < 1$ is fulfilled, the iteration sequence $\{\chi_{n+1}, C_{n+1}\} = T\{\chi_n, C_n\}$ $(n \in \mathbb{N}_0)$ with $\{\chi_0, C_0\} \in \mathcal{K}$ converges in the space F towards the unique solution of (2.10) in $\mathcal{K}$, i.e., the sequence $\{\chi_n\}$ is uniformly convergent (cf. [3, 6]).

We finally remark that in an analogous way the equation (2.10) can be investigated in balls of Hölder spaces if the function $F_0(s, p)$ only satisfies a (with respect to p) local Hölder-Lipschitz condition

$$|F_0(s_1, p_1) - F_0(s_2, p_2)| \leqq N_0[q_\infty |s_1 - s_2|^\alpha + K(R) |p_1 - p_2|]$$

for $|p_1|$, $|p_2| \leqq R$, $R > 0$. This is fulfilled for instance if $F_0(s, p)$ behaves like a polynomial or an exponential function in p (cp. the investigation of Baičorov's integral equation in [6]).

3. Reduction to a Riemann-Hilbert problem

The basic relation (1.5) is equivalent to the following nonlinear Riemann-Hilbert problem for the holomorphic function $\omega(\zeta)$, $\zeta = z/a$ with $\operatorname{Re} \omega(e^{is}) = \Phi(s)$ in the unit disk $|\zeta| < 1$.

Problem P. Find a holomorphic function $\omega(\zeta)$ in $|\zeta| < 1$ and a real constant c such that the boundary condition

$$\operatorname{Re} \omega(e^{is}) = A(s) - F\left(s, c + C(s)\left[q_\infty \sin s - \frac{D}{4\pi a} - \operatorname{Im} \omega(e^{is})\right]\right) \tag{3.1}$$

on $|\zeta| = 1$, $\zeta = e^{is}$, and the additional condition $\omega(0) = 0$ are fulfilled.

In the sequel for convenience we write z instead of ζ for the points in the unit disk G with $t = e^{is}$ for the points on the circumference Γ and also use the notations $w(z)$ and $W(z)$ for certain functions in the unit disk independently from their meaning above.

We assume that $F = F(p)$ depends on p alone and possesses a continuous derivative F' which fulfils the inequality

$$0 < K_1 \leqq F'(p) \leqq K_2 < \infty, \qquad p \in \mathbb{R}, \tag{3.2}$$

with positive constants K_1, K_2. Let F_1 be the inverse function to F. Then (3.1) is equivalent to the condition

$$c + C(s)\left[q_\infty \sin s - \frac{D}{4\pi a} - \operatorname{Im} \omega(e^{is})\right] = F_1\big(A(s) - \operatorname{Re} \omega(e^{is})\big). \tag{3.3}$$

We now introduce the new unknown function

$$W(z) = \big(H(z)/\varrho_0(z)\big)\left[F_1\big(u(z) + \bar{A}\big) - F_1(\bar{A}) + iv(z) - w_0(z)\right], \tag{3.4}$$

where $\bar{A} = Q/2\pi a$, the mean value of the function A, the holomorphic function $w(z) = u(z) + iv(z) := S[u](z)$ is the Schwarz integral of the boundary function $u(t) = A(s) - \bar{A} - \operatorname{Re} \omega(t)$ on Γ, and $w_0(z)$ is the uniquely determined holomorphic function in G satisfying $w_0(0) = 0$ and the boundary condition $u_0(t) - C(s) v_0(t) = f(s) + c_0$ on Γ, where

$$f(s) = f_0(s) - F_1(\bar{A}) \quad \text{with } f_0(s) = C(s)\left[q_\infty \sin s - D/4\pi a + HA(s)\right]$$

and c_0 is some real constant. The function w_0 is given by the formula (cf. [2: Chap. IV, § 29.3] or [12: Appendix])

$$w_0(z) = \varrho_0(z)\, S[e^{H\mu_0} g_0],$$

where

$$\varrho_0(z) = \exp\big(-i\Omega_0(z)\big), \qquad \Omega_0(z) = S[\mu_0](z),$$

$$\mu_0(s) = \arctan C(s), \qquad \bar{\mu}_0 = (2\pi)^{-1} \int_{-\pi}^{\pi} \mu_0(s)\, ds,$$

$$g_0(s) = \frac{f(s) + c_0}{\sqrt{1 + C^2(s)}}, \quad c_0 = -\frac{1}{2\pi \cos \bar{\mu}_0} \int_{-\pi}^{\pi} \frac{e^{H\mu_0(s)}}{\sqrt{1 + C^2(s)}} f(s)\, ds.$$

Finally, $H(z)$ is the positive real-valued harmonic function in G with the boundary values $H(t) = \sqrt{1 + C^2(s)} \exp\big(-H\mu_0(s)\big)$ on Γ. The function W is solution of a generalized Beltrami-type equation of the form

$$\partial W/\partial \bar{z} + \mu(z, W)\, \overline{\partial W/\partial z} + d(z, W) = 0 \tag{3.5}$$

with a continuous function μ satisfying the uniform ellipticity condition $|\mu(z, W)| \leqq \gamma < 1$, where $\gamma = \max\big((K_2 - 1)/(K_2 + 1),\ (1 - K_1)/(1 + K_1)\big)$ and d is a continuous function which is uniformly bounded in W. Further, W fulfils the boundary condition

$$\operatorname{Re} W(t) = D \quad \text{on } \Gamma \tag{3.6}$$

with an unknown real constant $D = c - c_0$ and the side condition

$$W(0) = 0. \tag{3.7}$$

Boundary value problems for quasi-linear first-order elliptic systems of the form (3.5) and related form have been dealt with in the literature for a long time. But it seems that the case of boundary conditions with a free constant D like (3.6) and the side condition (3.7) has not been studied. In the sequel we apply a method of VINOGRADOV [10] in a modified form to the problem (3.5)—(3.7).

For the unknown function W we use the integral representation of [10] with an additional term

$$W(z) = T\varphi := -\frac{1}{\pi} \iint_G \left[\frac{\varphi(\zeta)}{\zeta - z} + \frac{z\overline{\varphi(\zeta)}}{1 - z\bar{\zeta}}\right] d\xi\, d\eta + \frac{1}{\pi} \iint_G \frac{\varphi(\zeta)}{\zeta}\, d\xi\, d\eta, \qquad \zeta = \xi + i\eta, \tag{3.8}$$

where $\varphi \in L_p(G)$, $p > 2$. The ansatz (3.8) automatically fulfils the conditions (3.6), (3.7). It reduces the differential equation (3.5) to the nonlinear two-dimensional singular integral equation

$$\varphi + \mu(z, T\varphi)\, \overline{P\varphi} + d(z, T\varphi) = 0, \tag{3.9}$$

where P is the same singular integral operator as in [10]:

$$P\varphi = \frac{\partial T\varphi}{\partial z} = -\frac{1}{\pi} \iint_G \left[\frac{\varphi(\zeta)}{(\zeta - z)^2} + \frac{z\bar{\zeta}\,\overline{\varphi(\zeta)}}{(1 - z\bar{\zeta})^2}\right] d\xi\, d\eta - \frac{1}{\pi} \iint_G \frac{\overline{\varphi(\zeta)}}{1 - z\bar{\zeta}}\, d\xi\, d\eta$$

(cf. also [5: Part 1, Chap. 9]).

The equation (3.9) can now be treated in an analogous way as in [10] by means of Schauder's fixed-point theorem. The only difference is the proof of the uniqueness of the solution to the corresponding homogeneous linear equation to (3.5) with the conditions (3.6), (3.7). But also this proof can be performed along the usual lines. Additionally, one has only to take into account the minimum property of harmonic functions when the general solution of the differential equation with the boundary condition (3.6) is subjected to the side condition (3.7).

Applying now Schauder's fixed point theorem to the equation (3.9) like in [10], we obtain the existence of a solution $\varphi \in L_p(G)$ for some $p > 2$ to (3.9) which by (3.8) yields a solution $W \in W_p^1(G)$ of (3.5)—(3.7). From this the existence of a solution $\omega \in W_p^1(G)$ for some $p > 2$ of the nonlinear Riemann-Hilbert problem P follows. Lastly this yields a Hölder-continuous solution pair Φ, Ψ (with Hölder exponent $\alpha = (p - 2)/p$) of the basic relation (1.5).

Theorem 2: *The flow problem with a filtration law* (1.4), *where* $F = F(p)$ *is a continuously differentiable function satisfying* (3.2) *possesses a uniquely determined solution with Hölder continuous velocity and pressure distributions.*

Remark 4: VINOGRADOV [10] assumes that the coefficient satisfies a Lipschitz condition with respect to W which is fulfilled for the equation (3.5) if $F'(p)$ satisfies a Lipschitz condition, too. But this assumption (and the analogous one for d) is only needed for the proof of uniqueness of the solution, and for our problem a general uniqueness theorem for only continuously differentiable functions F has already been proved in [13] as is pointed out above.

Remark 5: If F has a Lipschitz continuous derivative, also the combination of the imbedding method with Newton's method like in BEGEHR and HSIAO [1] should be applicable to the problem (3.5)–(3.7).

4. A general existence theorem

In this section we investigate the solvability of the Riemann-Hilbert problem P for the determination of the holomorphic function ω and the real constant c from a more general viewpoint.

After substituting $w = u + iv = i\omega$, the boundary condition (3.1) takes the form

$$v(e^{is}) = A(s) - F\big(s, c + E(s) + C(s)\, u(e^{is})\big), \tag{4.1}$$

with the abbreviation $E(s) = C(s)\,(q_\infty \sin s - D/4\pi a)$. The additional condition $\omega(0) = 0$ is equivalent to

$$w(0) = 0. \tag{4.2}$$

Let us start with explaining the philosophy of our method. The Riemann-Hilbert problem (4.1) with $c \in \mathbb{R}$ fixed has a solution set $\{w_d^c\}_{d\in\mathbb{R}}$ which can be parametrized by a real parameter d. Therefore we are left with balancing the two parameters c and d to satisfy the additional condition (4.2). In order to do this we apply the geometric approach developed in [7–9] to get an idea of the influence of c and d to $w_d^c(0)$. In this way we shall determine the range $\{w_d^c(0): c, d \in \mathbb{R}\}$ and decide whether the condition (4.2) can be fulfilled or not.

According to the assumptions in [7] the function F is required to be monotone in p and to satisfy the following hypotheses:

The function F together with its derivatives $\partial F/\partial s$, $\partial F/\partial p$, $\partial^2 F/\partial s\,\partial p$ is continuous with respect to both arguments. Thereby the (finite) limits

$$F_p^{\pm}(s) = \lim_{p\to\pm\infty} \partial F(s,p)/\partial p, \qquad \lim_{p\to-\infty} p^{-2}\, \partial F(s,p)/\partial s = 0 \tag{4.3}$$

should exist uniformly with respect to s. In particular, there is a positive decreasing function g tending to zero at infinity such that

$$|\partial F(s,p)/\partial p - F_p^{\pm}(s)| \leqq g(|p|). \tag{4.4}$$

Moreover we suppose that

$$F_p^+(s) = F_p^-(s) \tag{4.5}$$

and write $F_p(s)$ instead of $F_p^{\pm}(s)$. We assume that the function $F_p(s)$ is continuously differentiable.

Remark 6: The condition (4.5) looks somewhat artificial from the physical viewpoint but it is essential for the given existence proof. Nevertheless we conjecture that Theorem 3 remains valid without this assumption.

The solvability of the problem significantly depends on the following quantities:

$$F^{\pm}(s) := \lim_{p\to\pm\infty} F(s,p), \quad \overline{F}^{\pm} := \frac{1}{2\pi}\int_{-\pi}^{\pi} F^{\pm}(s)\,ds, \quad \overline{A} := \frac{1}{2\pi}\int_{-\pi}^{\pi} A(s)\,ds.$$

Here it is allowed that the values of $\overline{F}^{\pm}$ are $+\infty$ or $-\infty$, respectively.

To stress the geometric nature of the Riemann-Hilbert problem we define the curves

$$M_s^c = \{u + \mathrm{i}v : v = A(s) - F(s, c + E(s) + C(s)\,u)\}$$
$$(c \in \mathbb{R},\ s \in [-\pi, \pi])$$

and write the boundary condition (4.1) in the form

$$w(\mathrm{e}^{\mathrm{i}s}) \in M_s^c. \tag{4.6}$$

The curves M_s^c are smooth open curves whose shape depends on the sign of the function $C(s)$ as is illustrated in Fig. 1. For fixed s the family of curves $\{M_s^c\}_{c\in\mathbb{R}}$ covers the domain of the (u, v)-plane bounded from above by the straight line $v = A(s) - F^-(s)$ and from below by $v = A(s) - F^+(s)$. For $c_1 > c_2$ the curve $M_s^{c_1}$ lies below $M_s^{c_2}$.

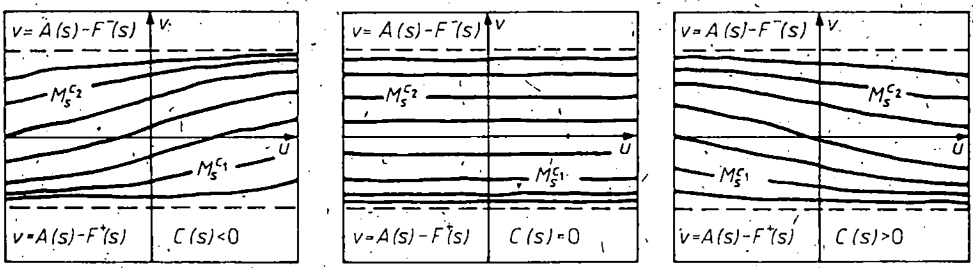

Fig. 1

We check the assumptions (i)—(v) in [7: Sect. 3]. For this end the curves M_s^c are parametrized by

$$M_s^c = \{u + \mathrm{i}v = \sigma + \mathrm{i}\mu^c(s,\sigma) : \sigma \in \mathbb{R}\},$$

where $\mu^c(s,\sigma) = A(s) - F(s, c + E(s) + C(s)\,\sigma)$. The smoothness assumptions (i)—(iii) follow from $E, C \in C^1[-\pi, \pi]$ and from the continuity of $\mu^c, \partial\mu^c/\partial s, \partial\mu^c/\partial\sigma, \partial^2\mu^c/\partial s\,\partial\sigma$. To prove (iv) we calculate the quantities $\alpha^{\pm}, \beta^{\pm}$ (cf. [7: p. 219]). Obviously,

$$\alpha_c^{\pm}(s) := \lim_{\sigma\to\pm\infty} \operatorname{Re} \partial\mu^c(s,\sigma)/\partial\sigma = 1,$$

$$\beta_c^{\pm}(s) := \lim_{\sigma\to\pm\infty} \operatorname{Im} \partial\mu^c(s,\sigma)/\partial\sigma = \begin{cases} -C(s)\,\partial F(s,+\infty)/\partial p & \text{as } C(s) > 0,\\ 0 & \text{as } C(s) = 0,\\ -C(s)\,\partial F(s,-\infty)/\partial p & \text{as } C(s) < 0, \end{cases} \tag{4.7}$$

i.e., by (4.5),

$$\beta_c^{\pm}(s) = -C(s)\,F_p(s). \tag{4.8}$$

Further we have to show the uniformity of the limits (4.7). Let $M := \sup |\partial F/\partial p|$. On account of (4.4) and (4.8) for any $\varepsilon > 0$ the estimate

$$\left|\frac{\partial \mu^c}{\partial \sigma}(s,\sigma) - \beta_c^{\pm}(s)\right| = |C(s)| \left|\frac{\partial F}{\partial p}\left(s, c + E(s) + C(s)\,\sigma\right) - F_p(s)\right| < \varepsilon$$

holds, whenever

$$|C(s)| < \frac{\varepsilon M}{2}$$

or

$$|C(s)| \geqq \frac{\varepsilon M}{2} \quad \text{and} \quad g\left(|\sigma| \frac{\varepsilon M}{2} - \sup_s |c + E(s)|\right) < \frac{\varepsilon}{\sup_s |C(s)|},$$

respectively. This proves the uniformity of the limits (4.7). Finally, the assumption (v), i.e., the continuous differentiability of $\beta_c^{\pm}$, follows from (4.8) and from the continuous differentiability of $\beta_c^{\pm}$, follows from (4.8) and from the continuous differentiability of F_p and C.

Now we can state some results about the solvability of problem (4.6).

Lemma 1: *The following statements are true.*

(i) *Under the above assumptions, for any $c, d \in \mathbb{R}$ the problem (4.6) possesses exactly one solution w_d^c with boundary values from the Sobolev space $W_2^1(\Gamma)$ which fulfils the side condition*

$$u_d^c(1) = d. \tag{4.9}$$

(ii) *If $d_1 > d_2$, then*

$$u_{d_1}^c(e^{is}) > u_{d_2}^c(e^{is}) \text{ for all } s \in [-\pi, \pi].$$

(iii) *For all $c \in \mathbb{R}$ we have*

$$\lim_{d\to+\infty} \inf_s u_d^c(e^{is}) = +\infty, \qquad \lim_{d\to-\infty} \sup_s u_d^c(e^{is}) = -\infty.$$

(iv) *The mapping $\mathbb{R}^2 \to W_2^1(\Gamma)$: $\{c, d\} \mapsto w_c^d$ is continuous.*

To prove Lemma 1 we remark that the index $\varkappa$ of the considered Riemann-Hilbert problem is zero. Hence the assertions (i)—(iii) immediately follow from Theorem 3 in [7].

We merely sketch the basic ideas for proving (iv). According to [11] the boundary value problems (4.6), (4.9) are reduced to fixed point equations for compact operators N_d^c acting in W_2^1. Each operator has exactly one fixed point with Leray-Schauder fixed point index 1. Since the operators N_d^c depend continuously on c and d (in some sense), it follows from the stability of the fixed point index (with respect to a suitable chosen small ball; cf. [14]) that the fixed points of N_d^c are continuous functions of c and d. This implies (iv). For more details we refer to [9: Proof of Theorem 2], where an analogous result was derived for a similar problem ∎

Our concern is now to find values of the parameters c and d so that the corresponding solution w_d^c of the Riemann-Hilbert problem (4.6), (4.9) given by Lemma 1 satisfies the condition $w_d^c(0) = 0$. For this end at first we vary d. For any fixed $c \in \mathbb{R}$ we define curves L_c by $L_c = \{w_d^c(0) : d \in \mathbb{R}\}$. In virtue of Lemma 1/(iv) and the mean value formula

$$w_d^c(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_d^c(e^{is})\, ds \tag{4.10}$$

the curves L_c are Jordan curves. Additionally, by Lemma 1/(iii), any vertical intersects each curve L_c. Due to Lemma 1/(ii) we have

$$u_{d_1}^c(0) - u_{d_2}^c(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [u_{d_1}^c(e^{is}) - u_{d_2}^c(e^{is})]\, ds > 0 \tag{4.11}$$

if $d_1 > d_2$. Further, taking into account the definition of the curves M_s^c and the continuous differentiability of the function F, we obtain the existence of constants K_c such that

$$|v_{d_1}^c(e^{is}) - v_{d_2}^c(e^{is})| \leqq K_c[u_{d_1}^c(e^{is}) - u_{d_2}^c(e^{is})].$$

This implies the estimates

$$\begin{aligned} |v_{d_1}^c(0) - v_{d_2}^c(0)| &\leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} |v_{d_1}^c(e^{is}) - v_{d_2}^c(e^{is})|\, ds \\ &\leqq \frac{K_c}{2\pi} \int_{-\pi}^{\pi} [u_{d_1}^c(e^{is}) - u_{d_2}^c(e^{is})]\, ds \\ &= \frac{K_c}{2\pi} [u_{d_1}^c(0) - u_{d_2}^c(0)]. \end{aligned} \tag{4.12}$$

From (4.11) and (4.12) we conclude the existence of Lipschitz-continuous functions λ_c such that $L_c = \{u + iv \in \mathbb{C} : v = \lambda_c(u)\}$. These functions λ_c satisfy the estimates

$$\bar{A} - \bar{F}^+ \leqq \lambda_{c_1}(u) \leqq \lambda_{c_2}(u) \leqq \bar{A} - \bar{F}^-, \qquad u \in \mathbb{R}, \tag{4.13}$$

if $c_1 > c_2$, where $\bar{A}$, $\bar{F}^+$, $\bar{F}^-$ are the above defined mean values of the functions A, F^+, F^-, respectively. The outer inequalities immediately follow by integrating the estimates (cf. Fig. 1)

$$A(s) - F^+(s) \leqq v_d^c(e^{is}) \leqq A(s) - F^-(s)$$

over Γ. To prove the middle inequality in (4.13) we introduce the following notations. We write $w \geqq M^c$ if for the continuous holomorphic function w the values $w(e^{is})$ lie above or on the curves M_s^c for any $s \in [-\pi, \pi]$. If additionally $w(e^{is})$ lies actually above M_s^c for at least one s, we denote this by $w > M^c$. Analogously we write $w(0) \geqq L_c$ if the value $w(0)$ lies above or on the curve L_c and $w(0) > L_c$ means that $w(0)$ lies above L_c.

Lemma 2: *Let w be a holomorphic function with boundary values from $W_2^1(\Gamma)$. If $w \geqq M^c$, then $w(0) \geqq L_c$. Further, if $w \geqq M^c$ and $w(0) \in L_c$, then $w(e^{is}) \in M_s^c$ for all s, i.e., $w = w_d^c$ for some d.*

The proof of Lemma 2 will be given below.

Now $c_1 > c_2$ yields $w_d^{c_2} \geqq M^{c_1}$ since the curves $M_s^{c_2}$ lie above or on $M_s^{c_1}$ for any s. From Lemma 2 we conclude $w_d^{c_2}(0) \geqq L_{c_1}$. This implies $\lambda_{c_1}(u) \leqq \lambda_{c_2}(u)$ for $u = \operatorname{Re} w_d^{c_2}(0)$. Vaying d over $\mathbb{R}$ we get the middle inequality in (4.13).

If the function $F(s, p)$ is even strictly monotone in p for some s, we have $w_d^{c_2} > M^{c_1}$ and consequently the second part of Lemma 2 yields $w_d^{c_2}(0) > L_{c_1}$. This implies the strict inequalities

$$\bar{A} - \bar{F}^+ < \lambda_{c_1}(u) < \lambda_{c_2}(u) < \bar{A} - \bar{F}^-, \quad u \in \mathbb{R}. \tag{4.14}$$

At last we investigate the behaviour of the curves L_c as $c \to \pm\infty$. For any $u \in \mathbb{R}$ the relations

$$\lim_{c\to+\infty} \lambda_c(u) = \bar{A} - \bar{F}^+, \qquad \lim_{c\to-\infty} \lambda_c(u) = \bar{A} - \bar{F}^- \tag{4.15}$$

hold. We only prove this for bounded functions F^+, F^-, otherwise the proof has to be modified in an obvious way. For any $\varepsilon > 0$ we choose a real-valued 2π-periodic function h which is continuously differentiable and satisfies the inequalities

$$A(s) - F^+(s) < h(s) < A(s) - F^+(s) + \varepsilon.$$

Further we define the holomorphic functions w_d by their boundary values

$$w_d(e^{is}) = Hh(s) + ih(s) + d, \qquad d \in \mathbb{R}.$$

Since Re w_d is bounded, we have $w_d \geqq M^c$ for sufficiently large c. Taking into account that Re $w_d(0) = d$, Im $w_d(0) < \bar{A} - \bar{F}^+ + \varepsilon$, from Lemma 2 and (4.13) we obtain the inequality

$$\bar{A} - \bar{F}^+ \leqq \lambda_c(d) \leqq \bar{A} - \bar{F}^+ + \varepsilon.$$

Since ε is an arbitrary positive number this yields the first relation in (4.15) with $u = d$. The second estimate can be proved in an analogous manner.

Closing the preparations for the general existence theorem for the flow problem we still have to prove Lemma 2. Let w be a holomorphic function with boundary values from $W_2^1(\Gamma)$ which satisfies $w \geqq M^c$. We must show that either $w = w_d^c$ for some d or $w(0) > L_c$. Evidently the point $w(0)$ can lie above, on, or below the curve L_c. In the first case $w(0) > L_c$ there is nothing to prove. In the second case $w(0) \in L_c$ there exists a real number d with $w(0) = w_d^c(0)$, i.e., the function $w_0 = w - w_d^c$ satisfies $w_0(0) = 0$. From the geometry of the curves M_s^c it can be seen that the boundary values of w_0 belong to the complex plane slitted along the negative imaginary axis:

$$w_0(e^{is}) \in S \cup \{0\}, \qquad S = \{w \in \mathbb{C} \setminus \{0\} : \arg w \neq -\pi/2\}. \tag{4.16}$$

If w_0 is not a constant, then in virtue of $w_0(0) = 0$ there must exist a point z_0 in a neighborhood of the origin such that $w_0(z_0) = -i\varepsilon$ with a sufficiently small positive ε. From (4.16) we see that the winding number of the function $w_0 + i\varepsilon$ is zero. But then the function $w_0 + i\varepsilon$ can not vanish at z_0. Therefore w_0 is a constant function, $w_0 \equiv w_0(0) = 0$, i.e. $w = w_d^c$. In the third case there exists a positive number δ with $w(0) + i\delta \in L_c$. As in the proof of the second case this implies $w + i\delta = w_d^c$ for some d. But this is impossible because $w(e^{is}) = w_d^c(e^{is}) - i\delta$ would lie below the curve M_s^c in contradiction to the assumption $w \geqq M^c$.

Remark 7: In the paper [8] of the first author it is shown, that Lemma 2 can be extended to more general situations.

We are now ready to formulate the main existence theorem.

Theorem 3: *Let the function F satisfy the above assumptions. Then the following assertions about the solvability of the Riemann-Hilbert problem* (4.1), (4.2) *for the holomorphic function w with boundary values from* $W_2^1(\Gamma)$ *and the real constant c hold*:

(i) *If either* $\bar{A} > \bar{F}^+$ *or* $\bar{A} < \bar{F}^-$, *then there exists no solution.*

(ii) *If* $\bar{F}^- < \bar{A} < \bar{F}^+$, *then the problem is solvable. The function w is uniquely determined.*

(iii) *If the function* $F(s, p)$ *is strictly monotone in p for at least one value of s, then the function w and the constant c are uniquely determined. A solution exists if and only if* $\bar{F}^- < \bar{A} < \bar{F}^+$.

Proof: 1. According to (4.13) the curves L_c lie completely in the strip $T = \{u + iv \in \mathbb{C} : u \in \mathbb{R}, \bar{A} - \bar{F}^+ \leqq v \leqq \bar{A} - \bar{F}^-\}$. This proves (i).

2. In virtue of Lemma 1/(iv) the mapping $\mathbb{R}^2 \to T : \{c, d\} \mapsto w_d^c(0)$ is continuous. Further, by Lemma 1/(ii), (iii), the curves L_c run from $-\infty$ to $+\infty$. Finally, by (4.13), the curves L_c depend monotonically on c and by (4.15) they extend up to the both boundaries of the strip T. Therefore the curves L_c completely exhaust the interior int T of T. That means, since $0 \in \operatorname{int} T$ in case (ii), there exists a function w_d^c satisfying $w_d^c(0) = 0$.

Let further $w_{d_1}^{c_1}$ and $w_{d_2}^{c_2}$ be two solutions of the problem. If $c_1 = c_2$, then Lemma 1/(ii) implies $d_1 = d_2$. If on the other hand $c_1 > c_2$, then we have $w_{d_2}^{c_2} \geqq M^{c_1}$ (cp. Fig. 1) and $w_{d_2}^{c_2}(0) \in L_{c_1}$ since $w_{d_2}^{c_2}(0) = 0 = w_{d_1}^{c_1}(0) \in L_{c_1}$. Hence Lemma 2 implies $w_{d_2}^{c_2} = w_d^{c_1}$ for some d. But $w_d^{c_1}$ must be equal to $w_{d_1}^{c_1}$ as we just proved. Consequently we have $w_{d_2}^{c_2} = w_{d_1}^{c_1}$.

3. Let the function $F(s_0, p)$ be strictly monotone in p for some $s_0 \in [-\pi, \pi]$. By (4.14) the curves L_c lie completely in the interior of the strip T. Hence solutions cannot exist if $\bar{A} = \bar{F}^-$ or $\bar{A} = \bar{F}^+$.

If $w_{d_1}^{c_1}$ and $w_{d_2}^{c_2}$ are two solutions of the problem with $c_1 > c_2$, then we have $w_{d_2}^{c_2} \geqq M^{c_1}$ and again $w_{d_2}^{c_2}(0) \in L_{c_1}$. Hence, by Lemma 2, the relation $w_{d_2}^{c_2}(e^{is}) \in M_s^{c_1}$ holds for all s. But $w_{d_2}^{c_2}(e^{is_0})$ lies above $M_{s_0}^{c_1}$. This contradiction shows that $c_1 = c_2$ ∎

Remark 8: In case of a semi-permeable surface (where $F(s, p) = 0$ if $p < 0$, for instance) only the statements (i), (ii) of Theorem 3 apply. In particular, if $0 < \bar{A} < \bar{F}^+$, we have a unique solution w of (4.1), (4.2), but the pressure constant c may not be uniquely determined. This seems plausible also from the physical point of view (cf. also the remark to the uniqueness theorem in [13]).

Remark 9: Comparing the obtained existence theorems for the flow problem we point out that in the Theorems 1 and 2 we always have $\bar{F}^- = -\infty$, $\bar{F}^+ = +\infty$, so that under the assumptions of these theorems the problem is solvable for any value of $\bar{A} = Q/2\pi a$. In contrast to this, for finite $\bar{F}^-$, $\bar{F}^+$ the existence condition in Theorem 3 is a natural restriction to the magnitude of the total mass flux through the surface of the cylinder.

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Manuskripteingang: 10. 10. 1988

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