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Elliptic Oscillation Theory

V. B. Headley

Wir gebrauchen eine geeignete Darstellung von Gårdings Ungleichung, um Nichtoszillationskriterien für die allgemeine elliptische Differentialgleichung gerader Ordnung

$$\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^\alpha[A_{\alpha\beta}(x) D^\beta u] = 0 \qquad (x \in \Omega)$$

von bekannten Nichtoszillationskriterien für die häufiger betrachtete Gleichung $(-1)^m \Delta^m u + h(x) u = 0$ $(x \in \Omega)$ abzuleiten, wo Ω ein unbeschränktes offenes Gebiet des $\mathbb{R}^n$ ist.

Употребляя один из подходящих вариантов неравенства Гординга (Gårding) мы покажем, как можно вывести новые теоремы неосцилляции для общих эллиптических уравнений четного порядка

$$\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^\alpha[A_{\alpha\beta}(x) D^\beta u] = 0 \qquad (x \in \Omega)$$

из известных теорем неосцилляции для более часто изучаемого уравнения $(-1)^m \Delta^m u + h(x) u = 0$ $(x \in \Omega)$, где Ω является неограниченным открытым подмножеством $\mathbb{R}^n$.

Using an appropriate version of Gårding's inequality, we show how to deduce new non-oscillation theorems for the general even-order elliptic equation

$$\sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^\alpha[A_{\alpha\beta}(x) D^\beta u] = 0 \qquad (x \in \Omega)$$

from known non-oscillation theorems for the more frequently studied equation $(-1)^m \Delta^m u + h(x) u = 0$ $(x \in \Omega)$, where Ω is an unbounded, open subset of $\mathbb{R}^n$.

1. Introduction. Several writers (see, e.g., [2, 3, 5, 8, 9]), have obtained non-oscillation theorems for various forms of the elliptic partial differential equation

$$(-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha[a_{\alpha\beta}(x) D^\beta u] + a_0(x) u = 0 \qquad (x \in \Omega \subseteq \mathbb{R}^n)$$

in an unbounded open set Ω. In a recent paper [6], by using a version of Poincaré's inequality, the author obtained non-oscillation theorems for the more general equation

$$(-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha[A_{\alpha\beta}(x) D^\beta u] + \sum_{|\alpha|\leq m} B_\alpha(x) D^\alpha u = 0.$$

In the present paper, by using an appropriate version of Gårding's inequality, we will extend the results in [6] to the equation

$$Lu := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^\alpha[A_{\alpha\beta}(x) D^\beta u] = 0 \qquad (x \in \Omega \subseteq \mathbb{R}^n); \tag{1}$$

where the coefficient functions $A_{\alpha\beta}$ are real-valued and sufficiently smooth. (The multi-index notation employed here is the same as in [1].) Our main result is a comparison theorem, whose proof, based on a suitable version of Gårding's inequality, will show that every known non-oscillation theorem for the equation

$$(-1)^m \Delta^m u + h(x)\, u = 0 \qquad (x \in \Omega \subseteq \mathbb{R}^n) \tag{2}$$

gives rise to a new non-oscillation theorem for (1).

2. Definitions and preliminary results. Throughout this paper, G will denote any non-empty, open (possibly unbounded) subset of Ω. If k is any non-negative integer, we define the seminorm $|\cdot|_{k,G}$, the *weighted* seminorm $|\cdot|_{k,G,w}$, and the norm $\|\cdot\|_{k,G}$ as follows:

$$|u|_{k,G} = \Big[\sum_{|\alpha|=k} \int_G |D^\alpha u|^2\, dx\Big]^{1/2}, \tag{3}$$

$$|u|_{k,G,w} = \Big[\sum_{|\alpha|=k} \int_G (k!/\alpha!)\, |D^\alpha u|^2\, dx\Big]^{1/2}, \tag{4}$$

$$\|u\|_{k,G} = \Big[\sum_{j=0}^{k} |u|_{j,G}^2\Big]^{1/2}. \tag{5}$$

The definition of $|\cdot|_{k,G,w}$ is motivated by the following formula, which is valid for all real-valued Φ in $C_0^\infty(G)$:

$$\begin{aligned}(-1)^m \int_G \Phi \Delta^m \Phi\, dx &= (-1)^m \int_G \Phi \Big(\sum_{k=1}^{n} D_k^2\Big)^m \Phi\, dx\\ &= (-1)^m \int_G \Phi \sum_{|\alpha|=m} (m!/\alpha!)\, D^{2\alpha}\Phi\, dx\\ &= (-1)^m \sum_{|\alpha|=m} \int_G \Phi D^\alpha[(m!/\alpha!)\, D^\alpha \Phi]\, dx\\ &= \sum_{|\alpha|=m} \int_G (m!/\alpha!)\, |D^\alpha \Phi|^2\, dx.\end{aligned}$$

To compare the seminorms $|\cdot|_{m,G}$ and $|\cdot|_{m,G,w}$, we let

$$c_0 = \max\,\{m!/\alpha!\colon |\alpha| = m\}. \tag{6}$$

Then it is easily seen that

$$|u|_{m,G} \leqq |u|_{m,G,w} \leqq c_0^{1/2}\, |u|_{m,G}. \tag{7}$$

We also note that, in (3) and (5), when there is no danger of confusion, we omit the subscript G. Let $C_B^k(G) = \{u \in C^k(G)\colon \|u\|_{k,G} < \infty\}$, and let $H_k(G)$ and $H_k^0(G)$ denote the completions of $C_B^k(G)$ and $C_0^\infty(G)$, respectively, with respect to the norm $\|\cdot\|_{k,G}$.

If G is bounded, and if there exists a non-trivial function u in $H_m^0(G) \cap C^{2m}(G)$ such that (1) holds, then G is called a *nodal domain* for L or a nodal domain for (1). If for all positive r the region $\{x \in \Omega\colon |x| > r\}$ contains a nodal domain for L, then (1) is said to be *nodally oscillatory* (or *strongly oscillatory*) in Ω.

Using integration by parts, we can easily show that if G is any non-empty, open (possibly unbounded) subset of Ω, then for every real-valued Φ in $C_0^\infty(G)$ we have

$$\begin{aligned}\int_G \Phi L\Phi\, dx &= \sum_{|\alpha|=|\beta|=m} \int_G A_{\alpha\beta}(x)\, D^\alpha\Phi D^\beta\Phi\, dx + \int_G \Phi^2 A_{00}(x)\, dx\\ &+ \sum_{|\alpha|+|\beta|=1}^{2m-1} \int_G A_{\alpha\beta} D^\alpha \Phi D^\beta \Phi\, dx.\end{aligned} \tag{8}$$

The standard proof of the *global* version of Gårding's inequality [1: Theorem 7:6] now yields the following result.

Lemma 2.1: *Let E_0 denote the ellipticity constant of the differential operator L; in other words, let*

$$E_0 = \inf\left\{ \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\, \xi^{\alpha+\beta}\, |\xi|^{-2m} : 0 \neq \xi \in \mathbb{R}^n,\, x \in \Omega \right\}.$$

Suppose that the principal coefficients $A_{\alpha\beta}$ ($|\alpha| = |\beta| = m$) are uniformly continuous on Ω, and that the remaining coefficients $A_{\alpha\beta}$ ($|\alpha| + |\beta| \leqq 2m - 1$) are bounded and measurable on Ω. Let G be any non-empty, open subset of Ω. Then there exist constants $c_1 \in (0, \infty)$ and $c_2 \in [0, \infty)$ such that, for every real-valued Φ in $C_0^\infty(G)$, we have

$$\sum_{|\alpha|=|\beta|=m} \int_G A_{\alpha\beta}(x)\, D^\alpha\Phi D^\beta\Phi\, dx + \sum_{|\alpha|+|\beta|=1}^{2m-1} \int_G A_{\alpha\beta}(x) D^\alpha\Phi\, D^\beta\Phi\, dx \geqq c_1 E_0 \|\Phi\|_{m,G}^2 - c_2 |\Phi|_{0,G}^2.$$

The constant c_1 depends only on m and n; the constant c_2 depends only on m, n, E_0, $\sup\{|A_{\alpha\beta}(x)| : x \in \Omega;\ 1 \leqq |\alpha| + |\beta| \leqq 2m - 1\}$ and the modulus of continuity for the principal coefficients.

3. The main results. Using Lemma 2.1, we will first obtain a comparison theorem, which we can then employ to obtain new non-oscillation theorems for (1) from all known non-oscillation theorems for (2).

Theorem 3.1: *Let M be the differential operator defined by*

$$Mu = (-1)^m c_4\, \Delta^m u + [A_{00}(x) - c_2]\, u, \tag{9}$$

where

$$c_4 = c_1 E_0/c_0, \tag{10}$$

and c_0 is defined by (6). *If* (1) *is nodally oscillatory in Ω, then the differential equation*

$$Mu = 0 \tag{11}$$

is also nodally oscillatory in Ω.

Proof: If (1) is nodally oscillatory in Ω, then for every positive number r the region $\{x \in \Omega : |x| > r\}$ contains a nodal domain G for the differential operator L. Thus, there exists a non-trivial function u in $H_m{}^0(G) \cap C^{2m}(G)$ such that (1) holds. Furthermore, (8), Lemma 2.1, (9), integration by parts, (4), (7) and (10) imply that, for every Φ in $C_0^\infty(G)$, we have

$$\begin{aligned} &\int_G \Phi L\Phi\, dx - \int_G \Phi M\Phi\, dx \\ &\geqq [c_1 E_0 \|\Phi\|_m{}^2 - c_2 |\Phi|_0{}^2 + A_{00} |\Phi|_0{}^2] - [c_4 |\Phi|_{m,G,w}^2 + (A_{00} - c_2) |\Phi|_0{}^2] \\ &= c_1 E_0 \|\Phi\|_m{}^2 - c_4 |\Phi|_{m,G,w}^2 \geqq c_1 E_0 |\Phi|_m{}^2 - c_4 |\Phi|_{m,G,w}^2 \\ &\geqq [(c_1 E_0/c_0) - c_4] |\Phi|_{m,G,w}^2 = 0. \end{aligned} \tag{12}$$

Using (1), (12) and a continuity argument, we obtain $0 = \int_G uLu\, dx \geqq \int_G uMu\, dx$. Therefore, the smallest eigenvalue of the eigenvalue problem $Mv = \lambda v$, $v \in H_m{}^0(G) \cap C^{2m}(G)$ is non-positive. Hence, we can apply a known monotonicity principle [4] to show that G has a non-empty open subset G' such that zero is the smallest eigenvalue of the eigenvalue problem $Mw = \mu w$, $w \in H_m{}^0(G') \cap C^{2m}(G')$. Thus, we have shown that, for every positive number r, the equation $Mw = 0$ has a non-trivial solution w, with a nodal domain $G' \subset G \subset \{x \in \Omega : |x| > r\}$ ∎.

To illustrate how Theorem 3.1 may be employed to obtain new non-oscillation theorems for (1) from known non-oscillation theorems for (2), we now generalize the non-oscillation portion of [8: Theorem 1]. (In [7] we showed how to obtain new oscillation theorems for (1) from known oscillation theorems for (2).)

Theorem 3.2: *Consider the polynomial*

$$\prod_{j=0}^{m-1} \left[r + [(n - 2m + 4j)/2]^2\right] =: \sum_{k=0}^{m} b_k r^k .$$

If $n \geqq 2m$, *or if* $n < 2m$ *and* n *is odd, then* (1) *is nodally non-oscillatory in* Ω *if there exists a positive number* r_0 *such that for every* x *in the region* $\{x \in \Omega: |x| > r_0\}$ *we have*

$$[A_{00}(x) - c_2]/c_4 > -|x|^{-2m} \sum_{k=0}^{m} [(2k - 1)!!]\, b_k/4^k \log^{2k} |x| . \tag{13}$$

Proof: Suppose to the contrary that (1) is nodally oscillatory in Ω. Then it follows from Theorem 3.1 that (11) is nodally oscillatory in Ω, contrary to the fact, proved in [8: Theorem 1], that (11) is nodally non-oscillatory in Ω whenever (13) holds ∎

We invite the reader to formulate appropriate generalizations of other known non-oscillation criteria.

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VERFASSER:

Prof. Dr. VELMER B. HEADLEY
Department of Mathematics
Brock University
St. Catharines
Ontario, L2S 3A1, Canada