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On the Singular Behaviour of Fluid in a Vertical Wedge

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Dedicated to the memory of Johannes Maul[1])

Es werden Lösungen der Gleichung für Kapillarflächen über Gebieten mit Ecken betrachtet. Dabei wird angenommen, daß die Ecke durch zwei Kurven begrenzt wird, die einen inneren Winkel 2α haben mit $0 < 2\alpha < \pi$ und $\alpha + \gamma < \pi/2$, wobei γ, $0 < \gamma < \pi/2$, der Kontaktwinkel zwischen der Fläche und der Containerwand ist. Es wird eine asymptotische Formel für Lösungen in der Umgebung der Ecke angegeben.

Исследуются решения уравнения капиллярности в областях с угловыми точками. Предполагается, что угловая точка окаймляется двумя кривыми имеющими внутренний угол 2α такой, что $0 < 2\alpha < \pi$ и $\alpha + \gamma < \pi/2$, где γ, $0 < \gamma < \pi/2$, является углом между контактной поверхностью и границей области. Доказывается асимптотическая формула для решений в окрестности угловой точки.

Solutions of capillary surface equation over domains with corners are considered. It is assumed that the corner is bounded by curves which make an interior angle 2α with $0 < 2\alpha < \pi$ and $\alpha + \gamma < \pi/2$, where γ, $0 < \gamma \leq \pi/2$ is the contact angle between the surface and the container wall. An asymptotic formula for the solutions near the corner is given.

1. Introduction. We consider the non-parametric capillary problem in the presence of gravity. One seeks a surface $S: u = u(x)$, $x = (x_1, x_2)$, defined over a bounded base domain $\Omega \subset \mathbb{R}^2$, such that S meets vertical cylinder walls over the boundary $\partial\Omega$ in a prescribed constant angle γ, $0 \leqq \gamma \leqq \pi/2$. The problem when a tube of cross-section Ω is placed into an infinite reservoir leads to the equations (see FINN [3])

$$\operatorname{div} Tu = \varkappa u \quad \text{in } \Omega, \tag{1.1}$$

$$\nu \cdot Tu = \cos\gamma \quad \text{on the smooth parts of } \partial\Omega, \tag{1.2}$$

where $Tu = Du/\sqrt{1 + |Du|^2}$, $\varkappa = \text{const} > 0$ and ν is the exterior unit normal on $\partial\Omega$. By Du we denote the gradient of u.

Let the origin $x = 0$ be a corner of Ω with the interior angle 2α satisfying $0 < 2\alpha < \pi$. We assume that the corner is bounded by two sufficiently regular curves and that each curve makes an angle α with the positive x_1-axis, see Figure 1.

In fact, it is enough that the curves belong to $C^{2,\mu}$ for some $\mu \in (0, 1)$. When the curves are lines near the origin, then CONCUS and FINN [2] have shown that u is unbounded at the origin if and only if $\alpha + \gamma < \pi/2$ holds. In this paper we are interested in this singular case. Thus, we suppose that $\alpha + \gamma < \pi/2$ in what follows. Let r, θ be polar coordinates centred at $x = 0$, set $k = \sin\alpha/\cos\gamma$ and define

$$u_0(r, \theta) = \left(\cos\theta - \sqrt{k^2 - \sin^2\theta}\right)/\varkappa k r. \tag{1.3}$$

[1]) See foot-note on p. 433.

Then, using a method of Concus and Finn, we have shown in [4] that

$$u(x) = u_0(r, \theta) + O(r^\varepsilon) \tag{1.4}$$

holds near the corner for an $\varepsilon > 0$ when the corner is bounded by lines near the origin. That is, $x_2 = \tan\alpha \cdot x_1$ is the upper curve and $x_2 = -\tan\alpha \cdot x_1$ the lower one which define the corner. The leading singular term $u_0(r, \theta)$ was discovered by CONCUS and FINN ([2] and [3, Theorem 5.5]). The expansion (1.4) shows that for fixed θ the function $u(r)$ is asymptotically a hyperbola. For $\theta = \pm\alpha$ one obtains the curves of contact on the container wall, compare FINN [3, Note 4, p. 131] with respect to an experiment performed by TAYLOR [5].

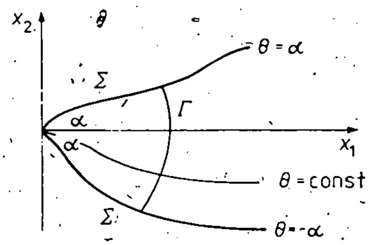

Fig. 1

The aim of this note is to obtain an expansion like (1.4) in the case when the corner is bounded by curves instead by lines. Under the stronger assumption $0 < \gamma < \pi/2$ we obtain by the same method that

$$u(x) = u_0(s, \theta) + q(\theta) + O(s^\varepsilon) \tag{1.5}$$

holds. Here s, θ denote curvilinear coordinates and q is the (unique) solution of a two-point boundary value problem for a regular second order ordinary differential equation, see the next sections.

Acknowledgement. I would like to thank Professor Robert Finn for initiating my interest in capillary problems and for useful discussions.

2. Curvilinear coordinates. We use curvilinear coordinates $x_1 = x_1(s, \theta)$ and $x_2 = x_2(s, \theta)$ $(-\alpha \leqq \theta \leqq \alpha;\ 0 \leqq s \leqq s_0,\ s_0$ small enough). Here $\theta =$ const yield the curves passing through the origin and s denotes the arc length on these curves measured from the origin, see Figure 1. More precisely, let

$$x_2 = f_1(x_1) = \tan\alpha \cdot x_1 + a_1 x_1^2 + O(x_1^3) \tag{2.1}$$

be the upper curve and

$$x_2 = f_2(x_1) = -\tan\alpha \cdot x_1 + a_2 x_1^2 + O(x_1^3) \tag{2.2}$$

the lower one which define the corner. We set

$$x_2(x_1, \theta) = \frac{1}{2}\left(1 + \frac{\tan\theta}{\tan\alpha}\right) f_1(x_1) + \frac{1}{2}\left(1 - \frac{\tan\theta}{\tan\alpha}\right) f_2(x_1) \tag{2.3}$$

and introduce the arc length instead of x_1 through

$$s = \int^{x_1} \sqrt{1 + x_{2,x_1}^2(\xi, \theta)}\, d\xi$$

which defines $x_1 = x_1(s, \theta)$ and $x_2 = x_2(s, \theta)$, where we denote $x_2(x_1(s, \theta), \theta)$ by $x_2(s, \theta)$ again. We find the coefficients g_1, g_2 in the expansions

$$x_1(s, \theta) = s \cos \theta + s^2 g_1(\theta) + O(s^3), \tag{2.4}$$

$$x_2(s, \theta) = s \sin \theta + s^2 g_2(\theta) + O(s^3) \tag{2.5}$$

as follows: Inserting (2.1) and (2.2) into (2.3) and then (2.4) for x_1, we obtain $x_2(s, \theta)$. Comparison of coefficients with (2.5) yields

$$g_2(\theta) = g_1(\theta) \tan \theta + 2^{-1} \cos^2 \theta \cdot G(a_1, a_2, \alpha, \theta), \tag{2.6}$$

where G is defined by

$$G(a_1, a_2, \alpha, \theta) = (1 + \tan \theta/\tan \alpha)\, a_1 + (1 - \tan \theta/\tan \alpha)\, a_2. \tag{2.7}$$

From (2.4), (2.5) and $x_s \cdot x_s = 1$ it follows that

$$g_1(\theta) = -g_2(\theta) \tan \theta \tag{2.8}$$

holds. Combining this equation with (2.6), we obtain

$$g_1(\theta) = -2^{-1} \sin \theta \cos^3 (\theta) \cdot G(a_1, a_2, \alpha, \theta) \tag{2.9}$$

and

$$g_2(\theta) = 2^{-1} \cos^4 \theta \cdot G(a_1, a_2, \alpha, \theta). \tag{2.10}$$

Set $x = (x_1, x_2)$ and $D = \det \begin{pmatrix} x_{1,s} & x_{2,s} \\ x_{1,\theta} & x_{2,\theta} \end{pmatrix}$. From (2.4), (2.5) we see that

$$x_\theta \cdot x_\theta = s^2 + 2e(\theta)\, s^3 + O(s^4),$$

$$x_s \cdot x_\theta = f(\theta)\, s^2 + O(s^3),$$

$$D = s + e(\theta)\, s^2 + O(s^3),$$

where

$$e(\theta) = -g_1'(\theta) \sin \theta + g_2'(\theta) \cos \theta,$$

$$f(\theta) = -g_1(\theta) \sin \theta + g_2(\theta) \cos \theta.$$

We mention that $e = f'$ holds because (2.8). Finally, we obtain from (2.9), (2.10) for f and e

$$f = (1/2) \cos^3 \theta \cdot G(a_1, a_2, \alpha, \theta), \tag{2.11}$$

$$e = -(3/2) \sin \theta \cos^2 \theta \cdot G(a_1, a_2, \alpha, \theta) + (\cos \theta/2 \tan \alpha)\, (a_1 - a_2), \tag{2.12}$$

where G is defined by (2.7).

3. The asymptotic formula. For $0 < \varrho < \varrho_0$, ϱ_0 small enough, we set $\Omega_\varrho = \Omega \cap B_\varrho$, $\sum_\varrho = (\partial\Omega \cap B_\varrho) \setminus \{0\}$ and $\Gamma_\varrho = \Omega \cap \partial B_\varrho$. Here B_ϱ denotes a disc with radius ϱ and the centre at the origin. The proof of the asymptotic formulas (1.4) and (1.5) is based on a method of Concus and Finn, see [3, proof of Theorem 5.5], which relies on the following comparison principle. We give here a special version which we need in our case. For the constant $\varkappa > 0$ let $Nv = \operatorname{div} Tv - \varkappa v$.

Theorem 3.1 (Concus and Finn [1]): *Suppose that $Nw \geqq Nv$ in Ω_ϱ, $v \geqq w$ on Γ_ϱ and $\nu \cdot Tv \geqq \nu \cdot Tw$ on Σ_ϱ hold. Then $v \geqq w$ in Ω_ϱ.*

With the abbreviation

$$R = D^2 + w_\theta^2 + x_\theta \cdot x_\theta w_s^2 - 2x_s \cdot x_\theta w_s w_\theta$$

we have in curvilinear coordinates s, θ

$$\operatorname{div} Tw = \frac{1}{|D|}\left[\left(\frac{x_\theta \cdot x_\theta w_s - x_s \cdot x_\theta w_\theta}{\sqrt{R}}\right)_s + \left(\frac{w_\theta - x_s \cdot x_\theta w_s}{\sqrt{R}}\right)_\theta\right]$$

and

$$\nu \cdot Tw = \begin{cases} (w_\theta - x_s \cdot x_\theta w_s)/\sqrt{R} & \text{on the upper curve } (\theta = \alpha), \\ (-w_\theta + x_s' \cdot x_\theta w_s)/\sqrt{R} & \text{on the lower curve } (\theta = -\alpha). \end{cases}$$

For

$$h(\theta) = \left(\cos\theta - \sqrt{k^2 - \sin^2\theta}\right)/k\varkappa, \tag{3.1}$$

where $k = \sin\alpha/\cos\gamma$, let $w = s^{-1}h(\theta) + q(\theta) - As^\lambda$ for a function $q \in C^2[-\alpha, \alpha]$, a constant $A \neq 0$ and a constant $\lambda > 0$. We define

$$Lq = \left(A_2(\theta)\, q'\right)' + A_1(\theta)\, q', \tag{3.2}$$

where $A_2 = h^2(h^2 + h'^2)^{-3/2}$ and $A_1 = 2hh'(h^2 + h'^2)^{-3/2}$, and set

$$\begin{aligned} F(\theta) = {} & -2(h^2 + h'^2)^{-3/2}(eh^3 + 2ehh'^2 + fh'^3) \\ & + [(h^2 + h'^2)^{-3/2}(fh^3 - eh'h^2)]' + eh(h^2 + h'^2)^{-1/2} \\ & - e[h'(h^2 + h'^2)^{-1/2}]' \end{aligned} \tag{3.3}$$

with e and f from (2.12) and (2.11). After some calculation, we obtain that

$$\operatorname{div} Tw = \varkappa s^{-1}h(\theta) + Lq + F + O(A\lambda s^\lambda) + O(s)$$

holds, provided that $\lambda \leqq 1$ and $|A|\,\lambda \leqq K_0$ are satisfied for a constant $K_0 > 0$. Hence, since $s^{-1}h(\theta) = w - q(\theta) + As^\lambda$, it follows

$$\operatorname{div} Tw = \varkappa w + \varkappa As^\lambda + O(A\lambda s^\lambda) + O(s) \tag{3.4}$$

if q is a solution of

$$Lq - \varkappa q + F = 0 \quad \text{on } (-\alpha, \alpha). \tag{3.5}$$

Again, after some calculation, one finds

$$\nu \cdot Tw = \cos\gamma - A\lambda hh'(h^2 + h'^2)^{-3/2} s^{1+\lambda} + O(s^2) \tag{3.6}$$

on the upper curve ($\theta = \alpha$) and

$$\nu \cdot Tw = \cos\gamma + A\lambda hh'(h^2 + h'^2)^{-3/2} s^{1+\lambda} + O(s^2) \tag{3.7}$$

on the lower one ($\theta = -\alpha$), provided that q satisfies the boundary conditions

$$q' + fh - eh' = 0 \quad \text{for} \quad \theta = -\alpha \text{ and } \theta = \alpha. \tag{3.8}$$

As above, we assume also here that $|A|\,\lambda \leqq K_0$ and $\lambda \leqq 1$ hold.

Lemma 3.1: *There exists a unique solution to the two-point boundary value problem* (3.5), (3.8).

Proof: It is enough to show that the homogeneous problem has only the solution $q = 0$. Let q_0 be a solution to the homogeneous problem associated to (3.5), (3.8) and u a solution to (1.1), (1.2), when the origin is a corner which is bounded by lines and each line makes an angle α with the positive x_1-axis. By the same argument as in [4] one finds $u = r^{-1}h(\theta) + q_0(\theta) + O(r^\varepsilon)$. Here r, θ denote polar coordinetes. Thus, since (1.4) holds too, it follows that $q_0(\theta) = 0$ on $[-\alpha, \alpha]$ ∎

Theorem 3.2: *Let u be a solution to* (1.1), (1.2) *and suppose that* $0 < 2\alpha < \pi$, $0 < \gamma < \pi/2$ *and* $\alpha + \gamma < \pi/2$. *Then, for an* $\varepsilon > 0$, $u = u_0(s, \theta) + q(\theta) + O(s^\varepsilon)$ *near the corner, where q is the solution to the boundary value problem* (3.5), (3.8) *and u_0 is defined through* (1.3).

Proof: Since $0 < \gamma < \pi/2$ holds it follows from the definition (3.1) of h that $h \in C^\infty[-\alpha, \alpha]$ and $h'(\alpha) = -h'(-\alpha) > 0$. Let $w = s^{-1}h(\theta) + q(\theta) - As^\lambda$, where the constant A is positive, then one obtains from (3.4), (3.6) and (3.7) by the same argument as in [4, proof of the theorem] that there are positive constants A, ϱ and λ not depending on the particular solution u considered such that $\operatorname{div} Tw - \varkappa w \geqq 0$ in Ω_ϱ, $w \leqq u$ on Γ_ϱ and $\nu \cdot Tw \leqq \cos\gamma$ on Σ_ϱ hold. Then, Theorem 3.1 implies $u \geqq u_0(s, \theta) + q(\theta) - As^\lambda$ in Ω_ϱ. By the same argument it follows $u \leqq u_0(s, \theta) + q(\theta) + As^\lambda$ for possibly other positive constants A, ϱ and λ. Here the comparison function $w = s^{-1}h(\theta) + q(\theta) + As^\lambda$, $A > 0$, is used. Thus, the theorem is proved ∎

Note added in proof. More recently, the correction term $q(\theta)$ was being calculated numerically by Dr. Berndt and Dr. Janassary from the University of Leipzig.

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