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Integration by Means of Riemann Sums in Banach Spaces I[1])

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G. Birkhoff [2] hatte 1938 auf dem Weg über modifizierte Riemannsche Summen eine Theorie der Integration in Banachräumen entwickelt. In der vorliegenden Arbeit verallgemeinern wir die Birkhoffsche Integration, indem wir ihr eine beliebige (in natürlicher Weise gerichtete) Menge von „μ-Zerlegungen" anstelle der Menge aller „μ-Zerlegungen" des gegebenen Maßraums (F, μ) zugrunde legen. Unsere Technik beim Arbeiten mit unendlichen Riemannschen Summen und Limites von gefilterten Familien solcher Summen verwendet vorteilhaft die 1-Punkt-Vervollständigungen gewisser partieller universeller Algebren, die von G. Grimeisen in [13] diskutiert wurden.

В 1938 году Г. Биркгоф [2] создал теорию интегрирования в банаховых пространствах через модифицированные суммы Римана. В этой статье мы обобщаем метод интегрирования Биркгофа тем, что положим ему в основу произвольное, естественным образом направленное множество „μ-разложений" вместо множества всех „μ-разложений" заданного пространства с мерой (F, μ). Наша техника работы с бесконечными суммами Римана и пределами фильтрующихся семейств таких сумм употребляет с пользой одноточечные дополнения некоторых частичных универсальных алгебр, которые ранее обсуждались Г. Гримейсеном [13].

In 1938, G. Birkhoff [2] developed a theory of integration in Banach spaces which uses the approach via modified Riemann sums. In our paper, we generalize Birkhoff's integration by basing it on an arbitrary set of "μ-partitions" (being directed in a natural way) instead of the set of all "μ-partitions" of the underlying measure space (F, μ). Our technique of working with infinite Riemann sums and limits of filtered families of such sums takes advantage of the 1-point completions of certain partial universal algebras as discussed by G. Grimeisen in [13].

If one leaves, in the theory of integration, ordered spaces (as spaces, where the integrands assume their values), *in particular* two methods to define an integral $\int f\,d\mu$ of a function f defined on a measure space, say (F, μ), into the space, for the following a Banach space E, seem to be appropriate:

a) Approximate, first, f by means of a sequence (f_n) of certain step functions f_n (whatever one means by "approximation"); secondly, define $\int f\,d\mu$ as the limit (w. r. to the norm topology) of the sequence of the integrals $\int f_n\,d\mu$, those being defined in a natural way [see Dinculeanu [4, p. 120] (where "approximation" refers to "pointwise convergence μ-almost everywhere"), there in a much more general situation than here, or Zaanen [28] (definition of the Bochner integral on p. 219, where "approximation" refers to "convergence in mean")].

b) Specify, first, a system $\mathfrak{X}$ of countable partitions (called μ-partitions) of F (being in some way compatible with the measure μ); define Riemann sums belonging to f, each $\mathfrak{x} \in \mathfrak{X}$ and each choice function $\varphi \in \mathsf{P}_{X \in \mathfrak{x}} X$, and introduce $\int f\,d\mu$ as an appro-

[1]) Der abschließende Teil II dieses Beitrages wird in Kürze ebenfalls in dieser Zeitschrift erscheinen.

priate limit of Riemann sums. [*Implicitly*, this method is used by BIRKHOFF [2], where $\mathfrak{X}$ consists of all μ-partitions of F, but the Riemann sums are replaced by sums which we call later Birkhoff sums (see Definition 5 and Remark 6), *explicitly*, e.g. by KURZWEIL [21], MAWHIN [23], MCSHANE [25] (in a less general situation, where μ need not have all properties of a measure) and SION [27] (in a much more general situation), where (in all four references) $\mathfrak{X}$ consists of certain finite μ-partitions of F. (For more references for the generalized Riemann integral, see MAWHIN [23] and MCSHANE [25].)]

In the present paper, the authors apply the method b) and allow (the "μ-partition system") $\mathfrak{X}$ to be arbitrary except for being directed by a natural relation $\leqq$ between the μ-partitions of F (for μ-partitions $\mathfrak{x}$, $\mathfrak{y}$ of F, $\mathfrak{x} \leqq \mathfrak{y}$ means intuitively that $\mathfrak{y}$ is "finer" than $\mathfrak{x}$, see Definition 4); this leads us to an integral ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$, which, in the case that $\mathfrak{X}$ is the class of all μ-partitions of F, extends the Bochner integral and coincides essentially with the Birkhoff integral (see [2, Definition 4]). In order to handle limits and unconditional (infinite) sums (in particular infinite Riemann sums) in a convenient way, we use a technique prepared in [13]. Indeed, we do not work in the "partial algebras" $(E, \lim_\tau)$, $(E, {}^{\mathrm{u}}\sum_I)$ (where ${}^{\mathrm{u}}\sum_I$ and τ denote the unconditional summation of mappings $\chi\colon I \to E$ (I countable and non-empty) and the norm topology on E, respectively) but in their 1-point completions $(\mathfrak{E}, (\lim_I)^\wedge)$ and $(\mathfrak{E}, ({}^{\mathrm{u}}\sum_I)^\wedge)$ (see [13, p. 123]). For the next, let $f\colon F \to \mathfrak{E}$ (instead of $f\colon F \to E$). Actually, our integral ${}^{\mathfrak{X}}\!\int \cdot\, d\hat{\mu}$ maps $\mathfrak{E}^F := \{g \mid g\colon F \to \mathfrak{E}\}$ into $\mathfrak{E}$, while ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$ induces an F-ary partial operation $({}^{\mathfrak{X}}\!\int \cdot\, d\mu)^\vee$ in E (see [13, p. 121]). A representation of this paper in the classical terminology (working implicitly in *partial* instead of *full* algebras) would be possible but would turn out to be very cumbersome: On many places, quantifiers like "for almost all μ-partitions of E" (whatever this means precisely) and phrases like "if $x (\in E)$ is the limit resp. the unconditional sum of ..., then x is the limit resp. the unconditional sum of ..." would be, then, unavoidable.

As another unusual aspect of our approach to integration, we raise the question, how the integral ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$ behaves under a change of $\mathfrak{X}$. [Even in the numerical analysis, the similar (much more special) question arises, which set of "subdivisions" of F suffices in order to obtain a "good" approximation (via a limit of a sequence of Riemann sums) of an integral, which might be defined by means of "all subdivisions" of F, whatever "subdivision" might mean for the special integration (see Remark 10 in Part II [6] of this paper).] The answer (see Theorem 4) is, that, given two μ-partition systems $\mathfrak{X}'$ and $\mathfrak{X}''$ of F, $\mathfrak{X}' \subseteqq \mathfrak{X}''$ implies ${}^{\mathfrak{X}'}\!\int f\, d\mu \subseteqq {}^{\mathfrak{X}''}\!\int f\, d\mu$. In other words: if $\mathfrak{X}' \subseteqq \mathfrak{X}''$, then the partial operation $({}^{\mathfrak{X}'}\!\int \cdot\, d\mu)^\vee$ in E is a restriction of the partial operation $({}^{\mathfrak{X}''}\!\int \cdot\, d\mu)^\vee$ in E, i.e., then the integration w.r. to $\mathfrak{X}'$ is stronger than that w.r. to $\mathfrak{X}''$.

Furthermore, ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$ is a "pointwise" integral (for this notion, see [12, p. 94], there for an integration based on the extended real line instead of E), i.e. (in particular) if ${}^{\mathfrak{X}}\!\int f\, d\mu$ is non-empty, then ${}^{\mathfrak{X}}\!\int f\, d\mu$ is the limit (in a precise sense) of the family $\big(R(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot))\big)_{\mathfrak{x} \in \mathfrak{X}}$ of Riemann sums for each (choice function) $\varphi \in \mathsf{P}_{(\mathfrak{x}, X) \in S} X$, where $S = \{(\mathfrak{x}, X) \mid \mathfrak{x} \in \mathfrak{X} \text{ and } X \in \mathfrak{x}\}$, the φ's being considered as "points" (see Theorem 11 in Part II). While in the definition of ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$, the choice functions (i. e. the elements of $\bigcup_{\mathfrak{x} \in \mathfrak{X}} \mathsf{P}_{X \in \mathfrak{x}} X$) are involved in the limit process (see Definition 5), this result says that "the choice functions can be drawn out from the limit process". This fact allows to represent each iterated integral as a limit of a certain family of iterated Riemann sums (without using anything about product measures) (see Theorem 14 in Part II). [Also this result has some relationship to a situation, one encounters even in the numerical analysis when one evaluates iterated Riemann integrals.] Proving Theorem

14, one needs to interchange limits with unconditional summation without having available uniform convergence; this interchange can be handled by means of Theorem 7 in [13]. Exactly the treatment of iterated integrals was the first motivation to introduce our integration via Riemann sums (method b)).

Because of its length, we divide this paper in Part I and Part II. So far this introduction dealt with the whole paper.

In the present Part I, § 0 contains a collection of basic definitions and simple facts proved elsewhere or following immediately from the definitions. In § 1, a certain extension LIM of the usual limit operation w.r. to the norm topology in E to the class of all filtered families in $\mathfrak{P}E$ is studied. For so-called "τco-nested" nets $(g, I, \leqq)$ in $\mathfrak{P}E$, LIM $(g, I, \leqq)$ can be characterized in a nice way (Proposition 4). In §2, a certain extension of the unconditional summation in E to all sequences in $\mathfrak{P}E$ (which occurs already in BIRKHOFF's paper [2]) is investigated, especially the question, whether and in which sense the operation assigning to each subset of E its closed convex cover can be interchanged with this extended summation (the answer being given in Proposition 15). §3 contains the definitions of Riemann sums, Birkhoff sums and of the $(\mu, \mathfrak{X})$-integral. Certain families of Birkhoff sums turn out to be τco-nested (Corollary to Proposition 21). Based on this result, firstly, the monotonicity of the integral ${}^{\mathfrak{X}}\!\!\int f\, d\mu$ w.r. to $\mathfrak{X}$ can be proved (Theorem 4), and, secondly, the relationship between Birkhoff's integral in [2] and the $(\mu, \mathfrak{X})$-integral can be cleared up (Remark 7).

This paper (Part II included) has a long history: A part of it was ready, before first-mentioned author started to work on his dissertation for the Dr. rer. nat. (see ERBEN [5]). After he had cleared up essential open questions, both authors agreed to write a common paper on the "Representation of the Bochner integral by means of Riemann sums". In the final version of this paper (for which we chose another title) as it is submitted now for publication, special cases of results of the mentioned dissertation (being unpublished yet) are included.

A remark of the second-mentioned author: Unfortunately, Dr. Erben was hindered by professional duties to participate in the procedure of writing down this paper. Besides his agreement to use freely the results of his dissertation, he provided me with sketches of proofs for the Propositions 13, 15 and the essential Proposition 21, which is the key for the results in § 4.

Remark on some generalization: After a suitable modification of the notions of summation and of integration, many results of this Part I and some of Part II [6] of this paper (at least all those indicated by references to the paper [5]) can be established (under certain suppositions) for locally convex spaces instead of Banach spaces (as it has already been done by ERBEN in [5]). The authors have refrained to develop this paper in such a generality, since, in such a framework, the preparations being necessary would require much more space than here in order to discuss additional notions (which could be avoided here) in an adequate way. The authors hope to present such a generalization of the integration theory developed here in another publication.

§ 0 Terminology

In this section, we collect some definitions and facts mostly presented elsewhere, in order to facilitate the reading of the present paper.

a) Let E be a set. Then, throughout this paper, $\mathfrak{E}$ denotes the set $\{X \mid X \subseteqq E$ and card $X \leqq 1\}$. Let I be a non-empty set and Ω an *I-ary partial operation in E*, i.e. a mapping from E^I into E [see b) below]. Then $\Omega^\wedge$ denotes the I-ary operation in $\mathfrak{E}$, defined by

$$\Omega^\wedge \varphi = \left\{x \in E \mid \exists \psi \in \mathop{\mathrm{P}}_{i \in I} \varphi(i)\ (x = \Omega\psi)\right\} \quad \text{for all} \quad \varphi \in \mathfrak{E}^I,$$

where $\mathsf{P}_{i \in I}\, \varphi(i)$ denotes the Cartesian product of φ. Conversely, if Θ is an I-ary operation in $\mathfrak{E}$ having, for all $\varphi \in \mathfrak{E}^I$, the property

$$\text{if } \Theta\varphi \neq \emptyset, \quad \text{then} \quad \varphi(i) \neq \emptyset \quad \text{for all} \quad i \in I,$$

then the I-partial operation $\Theta^\vee$ in E is defined by letting, for all $\psi \in E^I$ and all $x \in E$,

$$(\psi, x) \in \Theta^\vee \quad \text{if and only if} \quad \{x\} = \Theta(\psi^\sim),$$

where $\psi^\sim(i) := \{\psi(i)\}$ for all $i \in I$. One has $(\Omega^\wedge)^\vee = \Omega$ and $(\Theta^\vee)^\wedge = \Theta$ (for a more general statement, see Theorem 5 in [13]). The I-ary algebra $(\mathfrak{E}, \Omega^\wedge)$ is called the *1-point completion of* the I-ary partial algebra (E, Ω). (See [13, p. 123].) We define a mapping $e: \{\{x\} \mid x \in E\} \to E$ by letting $e(\{x\}) = x$ for all $x \in E$. The mapping e is one-to-one, and e^{-1} is an isomorphism of the partial algebra (E, Ω) into the algebra $(\mathfrak{E}, \Omega^\wedge)$ (for a more general statement, see Theorem 6 in [13, p. 123]).

b) For any relation (in particular, for each mapping) R, Dmn R denotes the *domain*, Rng R the *range of* R. Given classes A and B, we distinguish between mappings f *from* A into B (Dmn $f \subseteqq A$) and mappings f *on* A into B (Dmn $f = A$, symbol $f: A \to B$). B^A denotes the class of all $f: A \to B$. Often, we speak of a "*family* $(f(a))_{a \in A}$ *in* B" instead of a "mapping $f: A \to B$". For each set M, $\Phi(M)$ denotes the class of all filtered families in M, a *filtered family in* M being defined to be an ordered triple $(g, K, \mathfrak{b})$ consisting of a non-empty set K, a mapping $g: K \to M$ and a filter $\mathfrak{b}$ on K. Let (E, τ) be a Hausdorff space, τ its topology. (We use the word "topology" as a synonym for "closure operator".) Then, $\mathfrak{B}_\tau: E \to \mathfrak{P}\mathfrak{P}E$ denotes the *neighborhood operator induced by* τ, $\lim_\tau$ (or lim) the *limit operation induced by* τ (which is a mapping *from* $\Phi(E)$ into E). The mapping $\lim_\tau^\wedge$ (or $\lim^\wedge$) *on* $\Phi(\mathfrak{E})$ into $\mathfrak{E}$ is defined by

$$\lim{}^\wedge (g, K, \mathfrak{b}) = \Big\{x \mid x \in E \text{ and, for some } B \in \mathfrak{b} \text{ and some } f \in \mathsf{P}_{k \in B}\, g(k),\ x = \lim (f, B, \mathfrak{b}_B)\Big\}$$

for all $(g, K, \mathfrak{b}) \in \Phi(\mathfrak{E})$, where $\mathfrak{b}_B$ denotes the "trace" $\mathfrak{b}_B = \{C \cap B \mid C \in \mathfrak{b}\}$ of the filter $\mathfrak{b}$ in B. (Most often, we write $\lim (g, K, \mathfrak{b}) \left(= {}^{\mathfrak{b}}\lim_{k \in K} g(k)\right)$ instead of $\lim^\wedge (g, K, \mathfrak{b})$ $\left(= {}^{\mathfrak{b}}\lim^\wedge_{k \in K} g(k)\right)$. Furthermore, we omit the superscript "$\mathfrak{b}$" in the expression ${}^{\mathfrak{b}}\lim_{k \in K} g(k)$, if $K = \mathbb{N}$ and $\mathfrak{b} = \mathcal{F}\mathbb{N}$ ($\mathbb{N}$ being directed in the natural way). But we refrain from doing so if K is an arbitrary directed set and $\mathfrak{b} = \mathcal{F}K$.) Obviously, one has for all $(g, K, \mathfrak{b}) \in \Phi\, \mathfrak{E}$

$$\lim{}^\wedge (g, K, \mathfrak{b}) = \lim{}^\wedge (g \mid B, B, \mathfrak{b}_B) \quad \text{for all } B \in \mathfrak{b}. \tag{0.1}$$

In a certain sense, $(E, \lim)$ can be considered as an $\mathfrak{M}$-ary partial algebra (where $\mathfrak{M}$ denotes the class of all filtered sets) and $(\mathfrak{E}, \lim^\wedge)$ as an $\mathfrak{M}$-ary algebra, called the *1-point completion of* $(E, \lim)$. (See [13, p. 120–123].) — If $\mathfrak{a}$ is a filter on a set I and $P(i)$ is a set-theoretic formula (see MONK [26, p. 15]), where i occurs as a free variable, then we say "$P(i)$ *holds for* $\mathfrak{a}$-*almost all* $i \in I$" if there is an $A \in \mathfrak{a}$ such that $P(i)$ holds for all $i \in I$. Furthermore, $\mathcal{S}\mathfrak{a}$ denotes the set $\{C \mid C \subseteqq I$ and, for all $A \in \mathfrak{a}, C \cap A \neq \emptyset\}$, being called the *grill associated to the filter* $\mathfrak{a}$.

c) Let $(E, \|\cdot\|)$ be a Banach space, $\|\cdot\|$ its norm. Then τ denotes its topology induced by $\|\cdot\|$. $\mathcal{K}$ denotes the field of scalars of E. Without any danger of confusion, we denote by 0 the number zero and the zero vector of E as well. For each $\varepsilon \in \mathbb{R}$ with

$\varepsilon > 0$, the symbol $B_\varepsilon(0)$ stands for the set $\{x \in E \mid \|x\| < \varepsilon\}$. For intuitive reasons, we write the scalar multiplication as a right-side multiplication: We extend the addition $+$ in E and the scalar multiplication by letting, for all $X, Y \in \mathfrak{P}E$ and all $\alpha \in \mathcal{K}$, $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ and $X\alpha = \{x\alpha \mid x \in X\}$, and accordingly we define $X - Y$. Furthermore, we extend the mapping $\|\cdot\|$ to a mapping $\|\cdot\|^*$: $\mathfrak{P}E \to \mathfrak{P}\mathbb{R}$ by letting, for all $X \in \mathfrak{P}E$,

$$\|X\|^* = \{\|x\| \mid x \in X\},$$

where we omit the superscript $*$ in the sequel. In several situations we refer to the Banach space $(\mathbb{R}, |\cdot|)$ with the usual norm $|\cdot|$. Then, tacitly, all the agreements made for $(E, \|\cdot\|)$ will be applied in the special case. The letter $\mathfrak{R}$ will then denote the set $\{X \mid X \subseteq \mathbb{R} \text{ and } X \leqq 1\}$, replacing $\mathfrak{E}$ in the special case. — Given mappings g, h: $F \to \mathfrak{E}$, $j: F \to \mathcal{K}$ and $\alpha \in \mathcal{K}$, the mappings $g + h$, $g - h$, $g\alpha$, gj: $F \to \mathfrak{E}$ and $\|g\|$: $F \to \mathfrak{R}$ are defined pointwise; the same agreement holds if $\mathfrak{E}$ and $\mathfrak{R}$ are replaced by E and $\mathbb{R}$, repectively. — The relation $\leqq$ on $\mathbb{R}$ will be extended on $\mathfrak{P}\mathbb{R}$ in two different ways: Let, for all $X, Y \in \mathfrak{P}\mathbb{R}$,

$$X \leqq^* Y \text{ iff, for all } (x, y) \in X \times Y,\ x \leqq y;$$

$$X \leqq_0 Y \text{ iff, for each } x \in X, \text{ there is a } y \in Y \text{ such that } x \leqq y.$$

[The relations $\leqq^*$ and $\leqq_0$ are reflexive; $\leqq_0$ is transitive but not antisymmetric; $\leqq^*$ fails to be transitive or antisymmetric.] Instead of $X \leqq^* Y$ we write also $X \leqq Y$; if Y is a singleton $\{y\}$, often $X \leqq y$ stands for $X \leqq Y$ and $X \leqq_0 y$ stands for $X \leqq_0 Y$. — All agreements made here for the relation $\leqq$ shall be valid also for the relation $<$ on $\mathbb{R}$.

d) Given a mapping φ, we denote by $\mathsf{P}\varphi$ the Cartesian product $\mathsf{P}_{k \in K} \varphi(k)$, where $K = \operatorname{Dmn} \varphi$ (see MONK [26, p. 55]). Furthermore, we define two mappings $\mathscr{P}$ and $\mathcal{Q}$ on $\mathcal{V}$ (= class of all sets) into $\mathcal{V}$ by letting, for each set A,

$$\mathscr{P}A = \mathsf{P}_{a \in A}\, a$$

and

$$\mathcal{Q}A = \mathsf{P}_{(a,b) \in S}\, b, \text{ where } S := \mathscr{S}A (= \mathsf{S}_{a \in A}\, a) \quad \text{(see e) below)}$$

(for the notation, see [12, p. 74], where we used β and α instead of the symbols $\mathscr{P}$ and $\mathcal{Q}$). We call $\mathscr{P}$ and $\mathcal{Q}$ the *choice operator of the first* and that *of the second kind*, respectively. In accordance with usual conventions for mappings defined on a relation, we agree to denote, for each $\varphi \in \mathcal{Q}A$ and each $x \in A$, by $\varphi(x, \cdot)$ the mapping defined by $\varphi(x, \cdot)(y) = \varphi(x, y)$ for all $y \in x$. One has $\varphi(x, \cdot) \in \mathscr{P}x$. (Recall the definition of a set (see MONK [26, p. 14]).) — For later use, we observe that, for sets A, B with $A \subseteq B$, the following holds: If $\varphi \in \mathcal{Q}A$, then there is a $\psi \in \mathcal{Q}B$ such that $\psi \mid \mathscr{S}A = \varphi$. If $\varphi \in \mathcal{Q}B$, then $\varphi \mid \mathscr{S}A \in \mathcal{Q}A$.

e) Let $(K_i)_{i \in I}$ be a family of sets K_i, its domain I being a set. Then, $\mathsf{S}_{i \in I} K_i$ denotes the set $\{(i, k) \mid i \in I \text{ and } k \in K_i\}$, which is called the *direct sum of* $(K_i)_{i \in I}$. We define a mapping $\mathscr{S}$ on $\mathcal{V}$ into $\mathcal{V}$ by letting, for each set A,

$$\mathscr{S}A = \mathsf{S}_{a \in A}\, a.$$

Let $\mathfrak{a}$ be a filter on I and $(\mathfrak{b}_i)_{i \in I}$ a family of filters $\mathfrak{b}_i$ on the sets K_i. Then, ${}^{\mathfrak{a}}\mathsf{S}_{i \in I}\, \mathfrak{b}_i$ denotes the $\mathfrak{a}$-*filtered sum of* $(\mathfrak{b}_i)_{i \in I}$, being defined to be the filter on $\mathsf{S}_{i \in I} K_i$ generated by the filterbase $\left\{ \underset{i\in A}{\mathsf{S}}\, \varphi(i) \mid A \in \mathfrak{a} \text{ and } \varphi \in \underset{i\in A}{\mathsf{P}}\, \mathfrak{b}_i \right\}$. (Use of Satz 13 in [8]). In the special case that $\mathfrak{b}_i = \mathfrak{b}$ (fixed), thus $K_i = \bigcup \mathfrak{b}$ holds for all $i \in I$, ${}^{\mathfrak{a}}\underset{i\in I}{\mathsf{S}}\, \mathfrak{b}_i$ is called the *ordinal product* $\mathfrak{a} \otimes \mathfrak{b}$ *of* $\mathfrak{a}$ *and* $\mathfrak{b}$. Furthermore, we recall the notation of the *cardinal product* $^*\underset{i\in I}{\mathsf{P}}\, \mathfrak{b}_i$ of the filters $\mathfrak{b}_i$ and its special case $^*\mathfrak{b}^I$ (if $\mathfrak{b}_i = \mathfrak{b}$ for all $i \in I$) used in [13, p. 116] and agree to denote (in the special case $I = \{1, 2\}$) the *cardinal product of* $\mathfrak{b}_1$ *and* $\mathfrak{b}_2$, which is a filter on $K_1 \times K_2$, by $\mathfrak{b}_1 (^*\times)\, \mathfrak{b}_2$.

f) For the notion of a "directed set", we use the terminology used in KELLEY's book [18, p. 65]. Given a directed set $(M, \leqq)$, the filter on M generated by the filterbase $\{K(x) \mid x \in M\}$, where $K(x) := \{y \in M \mid x \leqq y\}$ for each $x \in M$, is called the *filter of perfinality on* M *w.r.t.* $\leqq$ and will be denoted, in general, by $\mathcal{F}(M, \leqq)$ or $\mathcal{F}M$. In particular, if I is a non-empty set, the set $\mathfrak{e}I := \{K \mid K \text{ finite and non-empty subset of } I\}$ together with the relation $\subseteqq |\mathfrak{e}I$ is a directed set. We denote the filter of perfinality on $\mathfrak{e}I$, $\mathcal{F}(\mathfrak{e}I)$, by $\mathfrak{e}^0 I$. The statement (0.2) formulated next follows immediately from the definitions:

$$\textit{If } (M, \leqq) \textit{ is a directed set and } K \in \mathcal{F}(M, \leqq), \textit{ then } (K, \leqq | K) \textit{ is a directed set, and one has } \mathcal{F}(K, \leqq | K) = \big(\mathcal{F}(M, \leqq)\big)_K. \tag{0.2}$$

g) Let again E be a Banach space. Then the unconditional summation ${}^{\mathrm{u}}\sum_I$, where I is a non-empty countable set is an I-ary partial operation in E. In this paper, we write consistently $\sum_I$ instead of ${}^{\mathrm{u}}\sum_I$ and, for each $\varphi \in \operatorname{Dmn} \sum_I$, we write also $\sum_{i\in I} \varphi(i)$ instead of $\sum_I \varphi$. Furthermore, for each $\varphi \in \mathfrak{E}^I$, we write also $\sum_{i\in I} \varphi(i)$ instead of $\sum_I{}^\wedge \varphi$, any confusion being avoided by the choice of φ. One has the following relationship between $\sum_I{}^\wedge$ and $\lim{}^\wedge$ (see [13, p. 128]):

$$\text{For each } \varphi \in \mathfrak{E}^I, \quad \textstyle\sum_I{}^\wedge \varphi = {}^{\mathfrak{e}^0 I}\!\lim\limits_{K\in\mathfrak{e}I}{}^\wedge \sum\limits_{i\in K} \varphi(i). \tag{0.3}$$

Next, we collect some simple assertions (which, except for (0.4) and (0.9), are just translations of classical facts into our language). For those assertions, let $(I, \mathfrak{a})$ be a filtered set, $(g, I, \mathfrak{a}) \in \Phi\mathfrak{E}$, $(h, I, \mathfrak{a}) \in \Phi\mathfrak{E}$, $(A_k)_{k\in K}$, $(B_k)_{k\in K}$ families in $\mathfrak{E}$ with non-empty countable domain K, and $\alpha \in \mathcal{K}$. Then, (0.4)–(0.13) hold:

$$\textit{If } g(i) \subseteqq h(i) \textit{ for } \mathfrak{a}\textit{-a.a. (i.e., for } \mathfrak{a}\textit{-almost all) } i \in I, \textit{ then } {}^{\mathfrak{a}}\lim_{i\in I} g(i) \subseteqq {}^{\mathfrak{a}}\lim_{i\in I} h(i). \tag{0.4}$$

$${}^{\mathfrak{a}}\lim_{i\in I} g(i) + {}^{\mathfrak{a}}\lim_{i\in I} h(i) \subseteqq {}^{\mathfrak{a}}\lim_{i\in I} \big(g(i) + h(i)\big). \tag{0.5}$$

$$\Big({}^{\mathfrak{a}}\lim_{i\in I} g(i)\Big)\, \alpha \subseteqq {}^{\mathfrak{a}}\lim_{i\in I} \big(g(i)\, \alpha\big). \tag{0.6}$$

$$\Big\| {}^{\mathfrak{a}}\lim_{i\in I} g(i) \Big\| \subseteqq {}^{\mathfrak{a}}\lim_{i\in I} \|g(i)\|. \tag{0.7}$$

$$\textit{If } g(i) \leqq h(i) \textit{ for } \mathfrak{a}\textit{-a.a. } i \in I, \textit{ then } {}^{\mathfrak{a}}\lim_{i\in I} g(i) \leqq {}^{\mathfrak{a}}\lim_{i\in I} h(i), \textit{ provided that } E = \mathbb{R} \textit{ (then } \mathfrak{E} = \mathfrak{R}). \tag{0.8}$$

$$\textit{If } A_k \subseteqq B_k \textit{ for all } k \in K, \textit{ then } \sum_{k\in K} A_k \subseteqq \sum_{k\in K} B_k. \tag{0.9}$$

$$\sum_{k\in K} A_k + \sum_{k\in K} B_k \subseteqq \sum_{k\in K} (A_k + B_k) \quad \text{(see [13, Theorem 16])}. \tag{0.10}$$

$$\Big(\sum_{k\in K} A_k\Big)\alpha \subseteq \sum_{k\in K}(A_k\,\alpha). \tag{0.11}$$

$$\Big\|\sum_{k\in K} A_k\Big\| \leqq \sum_{k\in K}\|A_k\|. \tag{0.12}$$

If $A_k \leqq B_k$ *for all* $k \in K$, *then* $\sum_{k\in K} A_k \leqq \sum_{k\in K} B_k$, (0.13)

provided that $E = \mathbb{R}$ (*then* $\mathfrak{E} = \mathfrak{R}$).

Throughout this paper, we will use the agreements and facts, collected in this section, often tacitly. In the set-theoretic terminology, we follow MONK's book [26] almost completely. Deviating from [26], e.g., we denote the cardinality of a set X by card X. Up to some exceptions (concerning the axiom of choice), we will never mention the use of axioms of set theory.

§ 1 Sums of set sequences, limits of filtered families of sets

For the whole paper, let $(E, \|\cdot\|)$ be a Banach space. We recall the agreements in § 0.

Definition 1 (cf. BIRKHOFF [2, p. 362], and [5, 2.1.2]): Let I be a non-empty countable set. Let $\sum_I^{\mathrm{r}}$ be the mapping assigning to each $\varphi \in (\mathfrak{P}E)^I$ the set

$$\textstyle\sum_I^{\mathrm{r}} \varphi = \{x \in E \mid \mathbf{P}\varphi \subseteq \mathrm{Dmn} \sum_I \text{ and, for some } \psi \in \mathbf{P}\varphi,\ x = \sum_I \psi\}.$$

Instead of $\sum_I^{\mathrm{r}} \varphi$ we write also $\sum_{i\in I} \varphi(i)$, which notation does not give rise to confusion (see Remark 1/c).

Remark 1: The most natural extension of the I-ary partial operation $\sum_I$ in E to an I-ary operation in $\mathfrak{P}E$ seems to be the mapping $\sum_I^*$ being defined by

$$\textstyle\sum_I^* \varphi = \{\sum_I \psi \mid \psi \in (\mathrm{Dmn} \sum_I) \cap \mathbf{P}\varphi\} \quad \text{for all} \quad \varphi \in (\mathfrak{P}E)^I.$$

Then, one has for all $\varphi \in (\mathfrak{P}E)^I$

$$\textstyle\sum_I^{\mathrm{r}} \varphi = \sum_I^* \varphi \text{ in the case that } \mathbf{P}\varphi \subseteq \mathrm{Dmn} \sum_I \text{ and } \sum_I^{\mathrm{r}} \varphi = \emptyset \text{ else.}$$

For this reason, $\sum_I^{\mathrm{r}} \varphi$ could be called the "reduced sum" of φ. Of course, the following statements (a)—(c) hold:

(a) If I is finite, then $\sum_I^{\mathrm{r}} = \sum_I^*$.

(b) $\sum_I^{\mathrm{r}} \varphi \subseteq \sum_I^* \varphi$ for all $\varphi \in (\mathfrak{P}E)^I$.

(c) $\sum_I^{\mathrm{r}} \mid \mathfrak{E}^I = \sum_I^* \mid \mathfrak{E}^I = \sum_I{}^{\wedge}$.

Definition 2: Let LIM_τ denote the mapping on $\Phi(\mathfrak{P}E)$ into $\mathfrak{P}E$ defined by letting, for each $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$ and each $x \in E$,

$$x \in \mathrm{LIM}_\tau\,(f, I, \mathfrak{a}) \text{ iff, } \forall\, U \in \mathfrak{V}_\tau x, \quad \emptyset \neq f(i) \subseteq U \quad \text{for} \quad \mathfrak{a}\text{-a.a. } i \in I.$$

Instead of LIM_τ we write mostly LIM, instead of LIM $(f, I, \mathfrak{a})$ also ${}^{\mathfrak{a}}\mathrm{LIM}_{i\in I} f(i)$.

Clearly, this definition can be formulated in every topological space. Since τ is a Hausdorff topology, the first assertion in the following proposition holds (for the definition of lim $\inf_\tau$ see [13, p. 116]).

Proposition 1: *One has* card $\mathrm{LIM}_\tau\,(f, I, \mathfrak{a}) \leqq 1$ *and* $\mathrm{LIM}_\tau\,(f, I, \mathfrak{a}) \subseteq \liminf_\tau\,(f, I, \mathfrak{a})$ *for all* $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$, *furthermore* $\mathrm{LIM}_\tau\,(f, I, \mathfrak{a}) = \lim\,(f, I, \mathfrak{a})$ *for all* $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{E})$.

If $X \subseteqq E$, then co X denotes the *convex cover of* X and aco X the *absolutely convex cover of* X (as defined, e.g., in KÖTHE's book [19, p. 173]); we consider co and aco as mappings on $\mathfrak{P}E$ into $\mathfrak{P}E$. Recall that $\tau\colon \mathfrak{P}E \to \mathfrak{P}E$, and that τX denotes the closure of X. We define τco, τaco, and coτ to be the mappings $\tau \circ$ co, $\tau \circ$ aco, and co $\circ\, \tau$, respectively.

The following statements (1.1)–(1.9) (which are listed for later use without any intention to reach completeness) hold for all $X, Y \subseteqq E$ and all $\alpha \in \mathcal{K}$:

$$\mathrm{co}(X + Y) = \mathrm{co}\, X + \mathrm{co}\, Y. \tag{1.1}$$

$$\tau X + \tau Y \subseteqq \tau(X + Y). \tag{1.2}$$

$$\tau X + \tau Y = \tau(X + Y) \text{ if } X \text{ or } Y \text{ is } \tau\text{-compact}. \tag{1.3}$$

$$\mathrm{co}\,(\tau\mathrm{co}\, X) = \tau\mathrm{co}\, X. \tag{1.4}$$

$$\tau\mathrm{co}\, X + \tau\mathrm{co}\, Y \subseteqq \tau\mathrm{co}(X + Y). \tag{1.5}$$

$$\mathrm{aco}\, X \subseteqq \mathrm{co}\left(X \cup (-X) \cup (Xj) \cup (-Xj)\right) \quad \text{if} \quad \mathcal{K} = \mathbb{C}, \text{ where } j = \sqrt{-1},$$

$$\text{and } \mathrm{aco}\, X \subseteqq \mathrm{co}\left(X \cup (-X)\right) \quad \text{if} \quad \mathcal{K} = \mathbb{R}. \tag{1.6}$$

If K *is a non-empty finite set,* $\varphi\colon K \to X$ *and* $\lambda\colon K \to \mathcal{K}$ *with*

$$\sum_{k\in K} |\lambda(k)| \leqq 1, \quad \text{then} \quad \sum_{k\in K} \varphi(k)\,\lambda(k) \in \mathrm{aco}\, X. \tag{1.7}$$

$$(\mathrm{co}\, X)\,\alpha = \mathrm{co}\,(X\alpha). \tag{1.8}$$

$$(\tau X)\,\alpha = \tau(X\alpha). \tag{1.9}$$

Next, we return to the discussion of LIM_τ.

Proposition 2: *Let* $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$. *Then,* (a) *and* (b) *are equivalent:*

(a) $\mathrm{LIM}\,(f, I, \mathfrak{a}) \neq \emptyset$.

(b) *For all* $\varepsilon > 0$, *there is an* $A \in \mathfrak{a}$ *such that* $\emptyset \neq \|f(i) - f(k)\| < \varepsilon$ *holds for all* $(i, k) \in A \times A$.

Proof: 1. Proving (a) $\Rightarrow$ (b), one uses the triangle inequality. 2. Assume (b). Then, there is an $A \in \mathfrak{a}$ such that $\emptyset \neq \|f(i) - f(j)\| < 1$, thus $f(i) \neq \emptyset$, holds for all $(i, j) \in A \times A$. Let $\chi \in \mathsf{P}(f \mid A)$. By (b), for each $\varepsilon > 0$, there is a $C(\varepsilon) \in \mathfrak{a}_A$ (for the notation, see §0/b)) such that

(1) $\|\emptyset \neq f(i) - f(j)\| < \varepsilon/2$ for all $(i, j) \in C(\varepsilon) \times C(\varepsilon)$,

hence

(2) $\|\chi(i) - \chi(j)\| < \varepsilon/2$ for all $(i, j) \in C(\varepsilon) \times C(\varepsilon)$.

Since E is complete, there is an $x \in E$ such that $x = \lim_\tau(\chi, A, \mathfrak{a}_A)$.

Let $\varepsilon > 0$. Then, in view of (2), one has

(3) $\|\chi(i) - x\| \leqq \varepsilon/2$ for all $i \in C(\varepsilon)$.

Let $i \in C(\varepsilon)$ and $y \in f(i)$. Then, by (1) and (3), one has $\|y - x\| < \varepsilon$, therefore, by the choice of y, $\emptyset \neq \|f(i) - \{x\}\| < \varepsilon$ holds for all $i \in C(\varepsilon)$. Thus, by the choice of ε, $x \in \mathrm{LIM}\,(f, I, \mathfrak{a})$, and so (a) ∎

If $(I, \leqq)$ is a directed set, M is a set and $f\colon I \to M$, then we call the filtered family $(f, I, \mathcal{F}(I, \leqq))$ a *net in M* and denote it by $(f, I, \leqq)$.

For the remainder of this section, let $(f, I, \leqq)$ be a net in $\mathfrak{P}E$.

Definition 3: $(f, I, \leqq)$ is called to be τco-*nested* if (a) holds:

(a) For all $i, j \in I$, if $i \leqq j$ and $f(i) \neq \emptyset$, then $\emptyset \neq f(j) \subseteqq \tau\text{co}\, f(i)$.

Remark 2: By (1.4), $(f, I, \leqq)$ is τco-nested if and only if (b) and (c) hold:

(b) For all $i, j \in I$; if $i \leqq j$ and $f(i) \neq \emptyset$, then $f(j) \neq \emptyset$.
(c) For all $i, j \in I$, if $i \leqq j$, then $\tau\text{co}\, f(j) \subseteqq \tau\text{co}\, f(i)$.

Condition (c) says that the mapping $\tau\text{co} \circ f$ is a homomorphism of the directed set $(I, \leqq)$ into the partially ordered set $(\mathfrak{P}E, (\subseteqq^{-1}) \mid \mathfrak{P}E)$. This fact is the author's motivation for choosing the adjective "τco-nested".

There are many trivial examples of τco-nested nets $(f, I, \leqq)$, which are not "nested", i.e., which do not have the property that, for all $i, j \in I$, $i \leqq j$ implies $f(j) \subseteqq f(i)$. For a non-trivial example, see a certain net (of "Birkhoff sums") occuring, e.g., in the Corollary to Proposition 21.

Proposition 3: *Let $(f, I, \leqq)$ be τco-nested. Then the statements* (a) *and* (b) *are equivalent:*

(a) $\text{LIM}\,(f, I, \leqq) \neq \emptyset$.
(b) *For all $\varepsilon > 0$, $\emptyset \neq \|f(i) - f(i)\| < \varepsilon$ holds for some $i \in I$.*

Proof: 1. (a) $\Rightarrow$ (b) follows from Proposition 2 by means of the triangle inequality.

2. Assume (b). Let $\varepsilon > 0$. Choose i as in (b). Let $j, k \in I$ with $i \leqq j$ and $i \leqq k$. Then, one obtains

$$0 \neq \|f(j) - f(k)\| \subseteqq \|\tau\text{co}\, f(i) - \tau\text{co}\, f(i)\| \subseteqq \left\|\tau\text{co}\ \big(f(i) - f(i)\big)\right\| \leqq \varepsilon,$$

where one uses in this order: (b) and (b) in Remark 2; (c) in Remark 2; (1.5), (1.8), (1.9) and the right side inequality occuring in (b), saying that $f(i) - f(i) \subseteqq B_\varepsilon(0)$. In view of Proposition 2 and of the choice of ε, we have showed (a) ∎

Proposition 4: *Let $(f, I, \leqq)$ be τco-nested. Then, for every $x \in E$,* (a) *and* (b) *are equivalent:*

(a) $x \in \text{LIM}\,(f, I, \leqq)$.
(b) *For all $U \in \mathfrak{B}_\tau x$, $\emptyset \neq f(i) \subseteqq U$ holds for some $i \in I$.*

Proof: 1. (a) $\Rightarrow$ (b) follows immediately from the definition of LIM. 2. Assume (b). Let $V \in \mathfrak{B}_\tau x$; we choose $\varepsilon > 0$ such that $U := B_\varepsilon(x)$ and $\tau U \subseteqq V$. Choose i as in (b). Let $i \leqq j \in I$. Then, by (b) in Remark 2, one has $\emptyset \neq f(j) \subseteqq \tau U$, since $f(i) - \{x\} \subseteqq B_\varepsilon(0)$ implies $\tau\text{co}\,\big(f(i) - \{x\}\big) = \tau\text{co}\,\big(f(i)\big) - \{x\} \subseteqq \tau B_\varepsilon(0)$ (use of (1.1) and (1.3)). Thus (a) ∎

Proposition 5: *Let $(f, I, \leqq)$ be τco-nested. Then, one has* (a):

(a) *If $K \subseteqq I$ and K is directed by $\leqq \mid K$ ($=$ restriction of $\leqq$ to K), then* $\text{LIM}\,(f \mid K, K, \leqq \mid K) \subseteqq \text{LIM}\,(f, I, \leqq)$.

Proof: The net $(f \mid K, K, \leqq \mid K)$ is τco-nested. Applying Proposition 4 twice, one gets the asserted inclusion ∎

Proposition 6: *Let $(f, I, \leqq)$ be τco-nested. Then* (a) *implies* (b):

(a) $\text{LIM}\,(f, I, \leqq) \neq \emptyset$.
(b) $\text{LIM}\,(f, I, \leqq) = \bigcap \big\{\tau\text{co}\big(f(i)\big) \mid i \in I \text{ and } \tau\text{co}\big(f(i)\big) \neq \emptyset\big\}$.

Proof: For abbreviation, let $g(i) = \tau\text{co}(f(i))$ for all $i \in I$ and $K = \{i \in I \mid g(i) \neq \emptyset\}$ $(= \{i \in I \mid f(i) \neq \emptyset\})$. Assume (a). Then, by Proposition 1, one has card $\text{LIM}(f, I, \leqq) = 1$, and, by Definition 2, K is non-empty. — Let $x = e(\text{LIM}(f, I, \leqq))$ (for the terminology, see §0/a)). **1.** Since $(f, I, \leqq)$ is τco-nested, $(K, \leqq \mid K)$ is a directed set, and for all $i, j \in K$, $i \leqq j$ implies $g(j) \subseteqq g(i)$. Choose $\varphi \in \mathsf{P}_{i \in K}\, g(i)$. Then $(\varphi, K, \leqq \mid K)$ is a net in E. **2.** Let, for the remainder of this proof, $\varepsilon > 0$. By Definition 2, there is an $i_0 \in I$ such that

(1) $\emptyset \neq f(i) \subseteqq B_{\varepsilon/2}(x)$ holds for all $i \in I$ with $i_0 \leqq i$.

Therefore,

(2) $g(i) \subseteqq B_\varepsilon(x)$ holds for all $i \in K$ with $i_0 \leqq i$.

3. Let $i_1 \in K$. Since $i_0 \in K$ (by (1)), there exists (by Part 1 of this proof) an $i_2 \in K$ such that $i_0, i_1 \leqq i_2$. By (2) and the choice of φ,

(3) $\varphi(i) \in g(i) \subseteqq B_\varepsilon(x)$ holds for all $i \in K$ with $i_2 \leqq i$.

In view of the choice of ε, we obtain $x = \lim_\tau (\varphi, K, \leqq \mid K)$, thus $x \in g(i_1)$ by (3), since $g(i) \subseteqq g(i_1)$ for all $i \in K$ with $i_1 \leqq i$, and $g(i_1)$ is τ-closed. **4.** Let $y \in \bigcap \{g(i) \mid i \in K\}$. Then, (since $i_0 \in K$) one has $y \in g(i_0)$, thus (by (2)) $y \in B_\varepsilon(x)$, therefore (by the choice of ε) $y = x$, and so (by the choice of y) $\bigcap \{g(i) \mid i \in K\} \subseteqq \text{LIM}(f, I, \leqq)$ ∎

§ 2 The interplay between sums of set sequences and the closed convex covers of sets

For this section, let I be a countable non-empty set and $(A_i)_{i \in I}$, $(B_i)_{i \in I}$ be families in $\mathfrak{P}E$. Preparing the discussion of the subject indicated in the headline, we collect some simple properties of the summation of set sequences.

Proposition 7: *The following assertions (a) and (b) are true:*

(a) $\sum_{i \in I} A_i$ *is non-empty if and only if* $\mathsf{P} A_i \subseteqq \text{Dmn} \sum_I$ *and* A_i *is non-empty for all* $i \in I$.

(b) *If* $\sum_{i \in I} B_i$ *is non-empty and* $\emptyset \neq A_i \subseteqq B_i$ *holds for all* $i \in I$, *then* $\sum_{i \in I} A_i$ *is non-empty.*

Proposition 8: *If* $\sum_{i \in I} A_i$ *is non-empty and* $\emptyset \neq K \subseteqq I$, *then* $\sum_{i \in K} A_i$ *is non-empty.*

Proof: Use of Theorem 17/b in [13] and Proposition 7/a ∎

Proposition 9: *If* $\sum_{i \in I} \|A_i\|$, *where the summation is (taken in the sense of Definition 1, but) related to the Banach space* $(\mathbb{R}, |\cdot|)$, *is non-empty, then* $\sum_{i \in I} A_i$ *is non-empty.*

Proof: Use of the fact that absolute convergence implies unconditional convergence (Theorem 14 in [13]) ∎

Proposition 10 (cf. [5, 2.5.6]): *Let* $\sum_{i \in I} A_i$ *and* $\sum_{i \in I} B_i$ *be non-empty. Then, one has* (a) *and* (b):

(a) $\sum_{i \in I} A_i + \sum_{i \in I} B_i = \sum_{i \in I} (A_i + B_i)$.

(b) $\left(\sum_{i \in I} A_i\right) \alpha = \sum_{i \in I} (A_i \alpha)$ *for all* $\alpha \in \mathcal{K}$.

Proof: Use Proposition 7/a, (0.10) and (0.11) ∎

For the next considerations (in Lemma 1 and Proposition 11), let $(L_k)_{k\in K}$ be a family in $\mathfrak{P}I$ being one-to-one (as a mapping) such that $\{L_k \mid k \in K\}$ is a partition of I.

Lemma 1: *If K is finite and $A_i \in \mathfrak{C}$ for all $i \in I$, then* $\sum_{i\in I} A_i = \sum_{k\in K} \sum_{i\in L_k} A_i$.

Proof: Use of (0.3) and Theorem 17/b in [13] ∎

Proposition 11: *One has* (a) *and* (b):

(a) *If* $\sum_{i\in I} A_i$ *is non-empty, then* $\sum_{i\in I} A_i = \sum_{k\in K} \sum_{i\in L_k} A_i$.

(b) *If K is finite, then* $\sum_{i\in I} A_i = \sum_{k\in K} \sum_{i\in L_k} A_i$.

Proof: For abbreviation, we set, for each $k \in K$, $M_k = \sum_{i\in L_k} A_i$.

Ad (a): Assume $\sum_{i\in I} A_i \neq \emptyset$. Then, for each $\varphi \in \mathsf{P}_{k\in K} M_k$, there exists a $\chi_\varphi \in \mathsf{P}_{i\in I} A_i$ (we fix one) such that $\emptyset \neq \sum_{i\in I} \{\chi_\varphi(i)\} = \sum_{k\in K} \{\varphi(k)\}$. (Indeed, given φ, there is a family $(\psi_k)_{k\in K}$ with $\psi_k \in \mathsf{P}_{i\in L_k} A_i$ and $\{\varphi(k)\} = \sum_{i\in L_k} \{\psi_k(i)\}$ for all $k \in K$. Define χ_φ by $\bigcup_{k\in K} \psi_k$. Then, use $\sum_{i\in I} A_i \neq \emptyset$ and Theorem 19 in [13].) Therefore, $\mathsf{P}_{k\in K} M_k \subseteq \operatorname{Dmn} \sum_k$. Since $\sum_{i\in I} A_i \neq \emptyset$, the set M_k is non-empty for all $k \in K$ (by Proposition 8). Thus, by Theorem 19 in [13], one obtains $\sum_{i\in I} A_i \subseteq \sum_{k\in K} M_k$. — Conversely, let $x \in \sum_{k\in K} M_k$. Then, there is a $\varphi \in \mathsf{P}_{k\in K} M_k$ such that $\{x\} = \sum_{k\in K} \{\varphi(k)\}$. Therefore, one has $x \in \sum_{k\in K} A_i$, since $\sum_K \varphi = \sum_I \chi_\varphi \in \sum_{i\in I} A_i$.

Ad (b): By means of Proposition 7/a and Lemma 1, one obtains that $\sum_{k\in K} M_k \neq \emptyset$ implies $\sum_{i\in I} A_i \neq \emptyset$. Therefore, (b) holds by (a) ∎

For later use, we note here a simple consequence of Proposition 11:

$$\textit{Let } K \textit{ be a non-empty proper subset of } I. \textit{ Let } A_i = \{0\} \textit{ for all } i \in I \setminus K. \textit{ Then } \sum_{i\in I} A_i = \sum_{i\in K} A_i. \tag{2.1}$$

Proposition 12 (cf. [5, 2.5.7 and 2.2.6)]: *If* $\sum_{i\in I} A_i$ *and* $\sum_{i\in I} B_i$ *are non-empty, then* $\sum_{i\in I} (A_i \cup B_i)$ *is non-empty.*

Proof: **1.** Since, by the premise, the sets $\mathsf{P}_{i\in I} A_i$ and $\mathsf{P}_{i\in I} B_i$ are non-empty, $\mathsf{P}_{i\in I} (A_i \cup B_i)$ is non-empty. Choose, for the next, $\varphi \in \mathsf{P}_{i\in I} A_i$ and $\psi \in \mathsf{P}_{i\in I} B_i$. **2.** Let $\chi \in \mathsf{P}_{i\in I} (A_i \cup B_i)$, $K := \{i \in I \mid \chi(i) \in A_i\}$ and $L := I \setminus K$. If $K = \emptyset$ or $K = I$, then one has (by the premise) $\chi \in \operatorname{Dmn} \sum_I$. Assume $\emptyset \neq K \neq I$. Then, by the premise, the mappings $(\chi \mid K) \cup (\varphi \mid L)$ and $(\chi \mid L) \cup (\psi \mid K)$ are members of $\operatorname{Dmn} \sum_I$, therefore, by Theorem 17/b in [13], $(\chi \mid K) \in \operatorname{Dmn} \sum_K$ and $(\chi \mid L) \in \operatorname{Dmn} \sum_L$. By Lemma 1, one obtains, therefore, $\chi \in \operatorname{Dmn} \sum_I$ ∎

Lemma 2: *Let I be infinite. If $\varphi \in \operatorname{Dmn} \sum_I$, then, for all $\varepsilon > 0$, there is a $K_0 \in \mathfrak{e}I$ such that* $\left\| \sum_{i\in K_2\setminus K_1} \varphi(i) \right\| < \varepsilon$ *holds for all $K_1, K_2 \in \mathfrak{e}I$ with $K_0 \subseteq K_1 \subseteq K_2$ and $K_1 \neq K_2$.*

Proof: Use of (0.3) ∎

Proposition 13 (see BIRKHOFF [2, p. 362], and [5, 2.4.3]): *Let I be infinite and $A_i \neq \emptyset$ for all $i \in I$. Then the following statements* (a) *and* (b) *are equivalent:*

(a) $\sum_{i \in I} A_i \neq \emptyset$.

(b) *For each $\varepsilon > 0$, there exists a $K \in \mathrm{e}I$ such that, for all $L \in \mathrm{e}(I \setminus K)$, $\left\| \sum_{i \in L} A_i \right\| < \varepsilon$.*

Proof: 1. Concluding from (b) to (a), one uses Proposition 7/a, the completeness of E and (0.3). 2. Assume (a) and non (b). Then, there is an $\varepsilon > 0$ and a family $(L_i)_{i \in I}$ in $\mathrm{e}I$ such that $L_i \cap L_j = \emptyset$ for all $i, j \in I$ with $i \neq j$, furthermore (since all A_i are non-empty) a family $(\varphi_i)_{i \in I}$ with $\varphi_i \in \mathsf{P}_{j \in L_i} A_j$ for all $i \in I$ such that

(1) $\varepsilon \leqq \left\| \sum_{j \in L_i} \varphi_i(j) \right\|$ for all $i \in I$.

(We have used that card $I = \aleph_0$ and applied recursion.) Let $M := \bigcup_{i \in I} L_i$. Choose $\psi \in \mathsf{P}_{i \in I} A_i$ and define $\varphi = \left(\bigcup_{i \in I} \varphi_i\right) \cup \left(\psi \mid (I \setminus M)\right)$. Then $\varphi \in \mathsf{P}_{i \in I} A_i$, thus $\varphi \in \operatorname{Dmn} \sum_I$ by (a). Now, choose K_0 as in Lemma 2. Since K_0 is finite and I infinite, there is an $i \in I$ such that $K_0 \cap L_i = \emptyset$. Define $K_1 = K_0$ and $K_2 = K_0 \cup L_i$. Then, with these data, the inequality in Lemma 2 contradicts (1) ∎

Corollary: *Let I be infinite and $\sum_{i \in I} A_i$ be non-empty. Then, for all $\varepsilon > 0$, there is a $K \in \mathrm{e}I$ such that $\left\| \sum_I \varphi - \sum_{i \in L} \varphi(i) \right\| < \varepsilon$ holds for all $L \in \mathrm{e}I$ with $K \subseteqq L$ and all $\varphi \in \mathsf{P}_{i \in I} A_i$, therefore $\sum_{i \in I \setminus L} A_i \subseteqq B_\varepsilon(0)$ holds for all such L.*

Proof: Let $\delta > 0$ and choose K as in Proposition 13/b for $\varepsilon = \delta/2$. Let $\varphi \in \mathsf{P}_{i \in I} A_i$ and $K \subseteqq L \in \mathrm{e}I$. Then, one obtains, using Lemma 1, (0.3), (0.7), and Proposition 13/b together with $\mathrm{e}(I \setminus L) \subseteqq \mathrm{e}(I \setminus K)$ in this order:

$$\left\| \sum_I \varphi - \sum_{i \in L} \varphi(i) \right\| = \left\| \sum_{I \setminus L} \left(\varphi \mid (I \setminus L)\right) \right\| = \left\| {}^{\mathfrak{a}}\lim_{N \in B}{}_{\tau} \sum_{i \in N} \varphi(i) \right\|$$
$$= {}^{\mathfrak{a}}\lim_{N \in B}{}_{\sigma} \left\| \sum_{i \in N} \varphi(i) \right\| < \delta,$$

where $\mathfrak{a} := \mathrm{e}^0(I \setminus L)$, $B := \mathrm{e}(I \setminus L)$ and σ denotes the Euclidean topology of $\mathbb{R}$ ∎

Proposition 14 (cf [5, 2.6.1 and 2.6.2]): *Let $\sum_{i \in I} A_i$ be non-empty. Then,* (a) *is equivalent to* (b) *for all $\varepsilon > 0$, therefore, one has* (c):

(a) *There is an $\alpha \in \mathbb{R}$ such that $\sum_{i \in I} A_i \subseteqq B_\varepsilon(0)\, \alpha$.*

(b) *There is a $\beta\colon I \to \mathbb{R}$ such that $A_i \subseteqq B_\varepsilon(0)\, \beta(i)$ holds for all $i \in I$.*

(c) *$\sum_{i \in I} A_i$ is bounded if and only if A_j is bounded for all $j \in I$.*

Proof (cf. [5, loc. cit.]): The assertion is clear if I is finite. Let I be infinite. 1. Assume (a). Let $j \in I$; choose $\varphi \in \mathsf{P}_{i \in I} A_i$ (use of $\mathsf{P}_{i \in I} A_i \neq \emptyset$). Then $X_j := \sum_{i \in I \setminus \{j\}} \{\varphi(i)\}$ is a singleton, say $\{x_j\}$, since $\sum_{i \in I} A_i \neq \emptyset$. By (a) and Lemma 1, one has $A_j \subseteqq \sum_{i \in I} A_i - X_j \subseteqq B_\varepsilon(0)\, \alpha - X_j$. Define $\beta(j) = \alpha + (1/\varepsilon)\, \|x_j\|$; then $A_j \subseteqq B_\varepsilon(0)\, \beta(j)$. Therefore (b). 2. Assume (b). Since $\sum_{i \in I} A_i \neq \emptyset$, there is (by the Corollary to Proposition 13) a $K \in \mathrm{e}I$ such that $\sum_{i \in I \setminus K} A_i \subseteqq B_\varepsilon(0)$. Thus (by Proposition 11/b and (b)) $\sum_{i \in I} A_i = \sum_{i \in I \setminus K} A_i + \sum_{i \in K} A_i \subseteqq B_\varepsilon(0)\, \alpha$, where $\alpha = 1 + \sum_{i \in K} |\beta(i)|$. Therefore (a) ∎

Proposition 15 (cf. [5, 2.9.5 and 2.9.6]): *Assume* $\sum_{i\in I} A_i \neq \emptyset$. *Then, one has* (a) *and* (b):

(a) $\emptyset \neq \sum_{i\in I} \tau\mathrm{co}\, A_i \subseteq \tau\mathrm{co} \sum_{i\in I} A_i$ (see BIRKHOFF [2, p. 363]).

(b) $\sum_{i\in I} \tau\mathrm{aco}\, A_i \neq \emptyset$.

Proof: For finite I, (a) is clear by (1.5), while (b) is then trivial. Let I be infinite.

Ad (a): 1. Let $\varepsilon > 0$. By Prop. 13, there exists a $K \in \mathrm{e}I$ such that $\sum_{i\in L} A_i \subseteq B_{(\varepsilon/2)}(0)$, thus (by (1.5))

$$\sum_{i\in L} \tau\mathrm{co}\, A_i \subseteq \tau B_{(\varepsilon/2)}(0) \subseteq B_\varepsilon(0) \text{ holds for all } L \in \mathrm{e}(I \setminus K);$$

therefore, since $A_i \neq \emptyset$ for all $i \in I$, one has, by the choice of ε and Proposition 13 (applied to $(\tau\mathrm{co}\, A_i)_{i\in I}$ instead of $(A_i)_{i\in I}$), the inequality $\sum_{i\in I} \tau\mathrm{co}\, A_i \neq \emptyset$. 2. Let $\varepsilon > 0$. For abbreviation, we define C_0, $C_4 - C_6$ and, for all $L \in \mathrm{e}I$, $C_1(L) - C_3(L)$ as follows:

$$C_0 = \sum_{i\in I} \tau\mathrm{co}\, A_i, \qquad C_4 = \sum_{i\in I} A_i + B_{2\varepsilon}(0),$$

$$C_5 = \tau\left(\mathrm{co}\left(\sum_{i\in I} A_i\right) + B_{2\varepsilon}(0)\right), \quad C_6 = \tau\mathrm{co}\left(\sum_{i\in I} A_i\right) + B_{3\varepsilon}(0),$$

$$C_1(L) = \left(\sum_{i\in L} \tau\mathrm{co}\, A_i\right) + B_\varepsilon(0), \quad C_2(L) = \tau\mathrm{co}\left(\sum_{i\in L} A_i\right) + B_\varepsilon(0),$$

$$C_3(L) = \sum_{i\in L} A_i + B_\varepsilon(0).$$

Then, the chain $C_1(L) \subseteq C_2(L) \subseteq \tau\mathrm{co}\, C_3(L)$ holds for all $L \in \mathrm{e}I$ by (1.5). Proving the chain $\tau\mathrm{co}\, C_4 = C_5 \subseteq C_6$, one uses (1.1) and the definition of τ by means of $\|\cdot\|$. By the Corollary to Proposition 13, $C_0 \subseteq C_1(L)$ holds for e^0I-almost all $L \in \mathrm{e}I$ (use of $C_0 \neq \emptyset$), and $C_3(L) \subseteq C_4$ holds for e^0I-almost all $L \in \mathrm{e}I$, since $\sum_{i\in I} A_i \neq \emptyset$. Summarizing, one obtains $C_0 \subseteq C_6$, since e^0I is a filter. By the choice of ε, we have proved the validity of the sign $\subseteq$ in (a).

Ad (b). Using (1.6) for each $i \in I$ (replacing there X by A_i), and the monotonicity of τ w.r. to $\subseteq$, furthermore the Propositions 10/b and 12, finally (a) [applied to $(A_i \cup (-A_i))_{i\in I}$ instead of $(A_i)_{i\in I}$ in the case $\mathcal{K} = \mathbb{R}$ and similarly (see (1.6)) in the case $\mathcal{K} = \mathbb{C}$], one obtains (b) by means of Proposition 7/b ∎

§ 3 μ-partition systems, Riemann sums, Birkhoff sums, $(\mu, \mathfrak{X})$-integral

For the remainder of this paper, let (F, μ) be a non-empty measure space with a measure μ on a σ-algebra $\mathfrak{S} \subseteq \mathfrak{P}F$, $\mathrm{Dmn}\,\mu = \mathfrak{S}$ (for the terminology, see BAUER [1, p. 16]: "σ-Algebra", or HALMOS [15, p. 24]: "σ-ring $\mathfrak{S} \subseteq \mathfrak{P}E$ such that $F \in \mathfrak{S}$"). Each countable partition $\mathfrak{x}$ of F with the properties $\mathfrak{x} \subseteq \mathrm{Dmn}\,\mu$ and $\mu X < +\infty$ for all $X \in \mathfrak{x}$ is called a *μ-partition*. Denote by $\Omega(\mu)$ or just by Ω the class of all μ-partitions of F. The set $\Omega(\mu)$ is non-empty if and only if the measure μ is σ-finite. Since we are interested only in the case $\Omega(\mu) \neq \emptyset$, we assume for the sequel that μ be σ-finite. (Remark: There is a finite μ-partition of F if and only if $\mu F < +\infty$.)

Definition 4: (a) For all $\mathfrak{x}, \mathfrak{y} \in \Omega(\mu)$, let $\mathfrak{x} \leq \mathfrak{y}$ hold if for each $Y \in \mathfrak{y}$ there exists an $X \in \mathfrak{x}$ such that $Y \subseteq X$; furthermore, let $\mathfrak{x} \vee \mathfrak{y} = \{X \cap Y \mid X \in \mathfrak{x}$ and $Y \in \mathfrak{y}$ and

$X \cap Y \neq \emptyset\}$. (Remark: $(\Omega(\mu), \leqq)$ is a directed set, and $\mathfrak{x} \vee \mathfrak{y}$ is the $\leqq$-supremum of $\{\mathfrak{x}, \mathfrak{y}\}$.)

(b) Let $\mathfrak{X} \subseteqq \Omega(\mu)$. The set $\mathfrak{X}$ is called a *μ-partition system* if $(\mathfrak{X}, \leqq | \mathfrak{X})$ is a directed set. $\mathfrak{X}$ is called to be *$\vee$-closed* if $\mathfrak{x} \vee \mathfrak{y} \in \mathfrak{X}$ holds for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$.

For (c)—(f), let $\mathfrak{X}$ be a μ-partition system.

(c) The *filter of perfinality on $\mathfrak{X}$ w.r. to* $\leqq | \mathfrak{X}$ is denoted by $\mathcal{F}\mathfrak{X}$ (in accord with the notation introduced in § 0).

(d) For all $(\mathfrak{x}, \varphi), (\mathfrak{y}, \psi) \in \mathrm{S}_{\mathfrak{z} \in \Omega(\mu)} \mathcal{P}\mathfrak{z}$, let $(\mathfrak{x}, \varphi) \leqq^{\#} (\mathfrak{y}, \psi)$ hold if $\mathfrak{x} \leqq \mathfrak{y}$. We put $\mathrm{S}_{\mathfrak{x} \in \mathfrak{X}} \mathcal{P}\mathfrak{x} = \mathfrak{X}^{\#}$. (Remark: $(\mathfrak{X}^{\#}, \leqq^{\#} | \mathfrak{X}^{\#})$, in particular $(\Omega(\mu)^{\#}, \leqq^{\#})$, is a directed set; for the notations, see § 0.)

(e) The *filter of perfinality on $\mathfrak{X}^{\#}$ w.r. to* $\leqq^{\#} | \mathfrak{X}^{\#}$ is denoted by $\mathcal{F}(\mathfrak{X}^{\#})$ or $\mathcal{F}\mathfrak{X}^{\#}$ (in accord with the notation introduced in §0).

(f) If $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$ with $\mathfrak{x} \leqq \mathfrak{y}$, and $X \in \mathfrak{x}$, then the set $\{Y \in \mathfrak{y} \mid Y \subseteqq X\}$ is denoted by $\mathfrak{y}_X$.

Remarks: 3. In (d), one has $\mathcal{F}\mathfrak{X}^{\#} = {}^{\mathfrak{a}}\mathrm{S}\varphi$ with $\mathfrak{a} = \mathcal{F}\mathfrak{X}$ and $\varphi(\mathfrak{x}) = \{\mathcal{P}\mathfrak{x}\}$ for all $\mathfrak{x} \in \mathfrak{X}$, where ${}^{\mathfrak{a}}\mathrm{S}\varphi$ denotes the $\mathfrak{a}$-filtered sum of φ (see § 0). 4. Clearly, if a subset $\mathfrak{X}$ of $\Omega(\mu)$ is non-empty and $\vee$-closed, then it is also a μ-partition system. In [12, p. 76], where a "μ-Zerlegung" is not subjected to any finiteness condition w.r. to μ, the analogue to our μ-partition system (the "μ-Zerlegungssystem") is required to be $\vee$-closed. In this paper, we require the $\vee$-closedness only in parts of § 5. In fact, the most natural examples of μ-partition systems, namely $\Omega(\mu)$ and, if $\mu F < +\infty$, $\{\mathfrak{x} \in \Omega(\mu) \mid \mathfrak{x}$ is finite$\}$, have this property.

Proposition 16: *If $\mathfrak{X}$ is a μ-partition system and $\mathfrak{Y} \in \mathcal{F}\mathfrak{X}$, then $\mathfrak{Y}$ is a μ-partition system, and one has $\mathcal{F}\mathfrak{Y} = (\mathcal{F}\mathfrak{X})_{\mathfrak{Y}}$ and $\mathcal{F}\mathfrak{Y}^{\#} = (\mathcal{F}\mathfrak{X}^{\#})_{\mathfrak{Y}^{\#}}$.*

Proof: In view of (0.2), it suffices to show that $\mathfrak{Y}^{\#} \in \mathcal{F}\mathfrak{X}^{\#}$. Since $\mathfrak{Y} \in \mathcal{F}\mathfrak{X}$, there is an $\mathfrak{x}_0 \in \mathfrak{X}$ such that $\mathfrak{Z} := \{\mathfrak{x} \in \mathfrak{X} \mid \mathfrak{x}_0 \leqq \mathfrak{x}\} \subseteqq \mathfrak{Y}$. Since $\mathfrak{Z} \in \mathcal{F}\mathfrak{X}$, $\mathfrak{Z}$ is a μ-partition system (by (0.2)), and $\mathfrak{Z}^{\#} \subseteqq \mathfrak{Y}^{\#} \subseteqq \mathfrak{X}^{\#}$ (because $\mathfrak{Z} \subseteqq \mathfrak{Y} \subseteqq \mathfrak{X}$). Choose $\varphi_0 \in \mathcal{P}\mathfrak{x}_0$. Then, $\mathfrak{Z}^{\#} = \{(\mathfrak{x}, \varphi) \mid \mathfrak{x} \in \mathfrak{Z}$ and $\varphi \in \mathcal{P}\mathfrak{x}\} = \{(\mathfrak{x}, \varphi) \in \mathfrak{X}^{\#} \mid (\mathfrak{x}_0, \varphi_0) \leqq^{*} (\mathfrak{x}, \varphi)\} \in \mathcal{F}\mathfrak{X}^{\#}$, therefore $\mathfrak{Y}^{\#} \in \mathcal{F}\mathfrak{X}^{\#}$ ∎

For the remainder of this paper, let $\mathfrak{X}$ be a μ-partition system of F and f (except for § 8) be a mapping on F into $\mathfrak{E}$. For all $X \subseteqq F$, $f[X]$ denotes the set $\bigcup\{f(x) \mid x \in X\}$ ($= \{ef(x) \mid x \in X$ and $f(x) \neq \emptyset\}$) (for the definition of $\mathfrak{E}$ and e, see § 0/a)). We call f to be *singleton-valued* if $f(u) \neq \emptyset$ for all $u \in F$.

We have now made all preparations to define an integral of f.

Definition 5: (a) For each $\mathfrak{x} \in \Omega$ and $\varphi \in \mathcal{P}\mathfrak{x}$, the set $R(f, \mathfrak{x}, \varphi) = \sum_{X \in \mathfrak{x}} f(\varphi(X))\, \mu X$ is called the *Riemann sum of f belonging to μ, $\mathfrak{x}$, and φ*. Most often, we write $R(\mathfrak{x}, \varphi)$ instead of $R(f, \mathfrak{x}, \varphi)$.

(b) The set ${}^{\mathfrak{X}}\!\int f\, d\mu = {}^{\mathcal{F}\mathfrak{X}^{\#}}\!\lim_{(\mathfrak{x}, \varphi) \in \mathfrak{X}^{\#}} R(f, \mathfrak{x}, \varphi)$ is called *$(\mu, \mathfrak{X})$-integral of f*. The mapping assigning to each $g: F \to \mathfrak{E}$ its $(\mu, \mathfrak{X})$-integral ${}^{\mathfrak{X}}\!\int g\, d\mu$ is denoted by ${}^{\mathfrak{X}}\!\int \cdot\, d\mu$ and called *$(\mu, \mathfrak{X})$-integral*. Occasionally, we write ${}^{\mathfrak{X}}\!\int_{u \in F} f(u)\, d\mu$ instead of ${}^{\mathfrak{X}}\!\int f\, d\mu$ (see § 8).

(c) Let $\mathfrak{x} \in \Omega$. The set $B(f, \mathfrak{x}) = \sum_{X \in \mathfrak{x}} f[X]\, \mu X$ [resp. $C(f, \mathfrak{x}) = \sum_{\mathfrak{x}}{}^{*}\big((f[X]\, \mu X)^{X \in \mathfrak{x}}\big)$] is called the *Birkhoff sum* [resp. *weak Birkhoff sum*] *of f belonging to μ, and $\mathfrak{x}$*. Most often, we write $B(\mathfrak{x})$ [resp. $C(\mathfrak{x})$] instead of $B(f, \mathfrak{x})$ [resp. $C(f, \mathfrak{x})$]. (For the definition of $\sum_{\mathfrak{x}}{}^{*}$, see Remark 1.)

Remarks: 5. Riemann sums with countably many summands have been introduced and used to represent by means of them measure integrals of functions ranging in the extended real line already by HAHN and ROSENTHAL [14, p. 183–184]. 6. For the explicit introduction of the sums $B(f, \mathfrak{x})$ (occuring in Definition 5/c), see [5, p. 53] (called there "Riemann-Summen"). BIRKHOFF [2, p. 367] has, implicitly, introduced the sums $B(f, \mathfrak{x})$ and has used them for the introduction of an integral of mappings on F into E. Birkhoff's integral of a mapping $g: F \to E$ coincides essentially with our (μ, Ω)-integral of the mapping $g^{\sim}$ defined by $g^{\sim}(u) = \{g(u)\}$ for all $u \in F$ (for a more precise statement, see Remark 7).

The simple relationship between Riemann sums, Birkhoff sums, and weak Birkhoff sums is described in the following

Proposition 17: *Let $\mathfrak{x} \in \Omega$. Then the five statements* (a), (e), (d) $\Rightarrow$ (c) $\wedge$ (f), (b) $\wedge$ (f) $\Rightarrow$ (d), (c) $\Rightarrow$ (b) *hold, where* (a)–(f) *are defined next:*

(a) $B(\mathfrak{x}) \subseteq C(\mathfrak{x})$; (b) $B(\mathfrak{x}) = C(\mathfrak{x})$;
(c) $B(\mathfrak{x}) \neq \emptyset$; (d) $R(\mathfrak{x}, \varphi) \neq \emptyset$ *for all* $\varphi \in \mathcal{P}\mathfrak{x}$;
(e) $C(\mathfrak{x}) = \bigcup_{\varphi \in \mathcal{P}\mathfrak{x}} R(\mathfrak{x}, \varphi)$. (f) *$f$ is singleton-valued.*

Proof: One uses the following remarks: For each $\psi \in \mathop{\mathsf{P}}_{X \in \mathfrak{x}} f[X]\,\mu X$ there is a $\varphi \in \mathcal{P}\mathfrak{x}$ such that $\psi(X) \in f(\varphi(X))\,\mu X$ holds for all $X \in \mathfrak{x}$. If (f) holds, then, for each $\varphi \in \mathcal{P}\mathfrak{x}$, the mapping $(ef(\varphi(X))\,\mu X)_{X \in \mathfrak{x}}$ is a member of $\mathop{\mathsf{P}}_{X \in \mathfrak{x}} f[X]\,\mu X$ ∎

We recall that for all non-empty countable sets I and all $\varphi: I \to \mathfrak{E}$, $\sum_{i \in I} \varphi(i) \neq \emptyset$ implies that $\varphi(i) \neq \emptyset$ for all $i \in I$ (see Theorem 17/b in [13] or Proposition 8). Similarly, one has for the $(\mu, \mathfrak{X})$-integral

Proposition 18: *If ${}^{\mathfrak{X}}\!\!\int f\,d\mu \neq \emptyset$, then f is singleton-valued.*

Proof: Assume that the premise be true, but the conclusion be false. Choose $x \in {}^{\mathfrak{X}}\!\!\int f\,d\mu$ and $u \in F$ such that $f(u) = \emptyset$. Let $U \in \mathfrak{B}_r x$. Then, there exists an $\mathfrak{x}_0 \in \mathfrak{X}$ such that $\emptyset \neq R(f, \mathfrak{x}, \varphi) \subseteq U$ holds for all $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_0 \leq \mathfrak{x}$ and all $\varphi \in \mathcal{P}\mathfrak{x}$. Choose an $\mathfrak{x}$ with $\mathfrak{x}_0 \leq \mathfrak{x}$. Then, there is an $X \in \mathfrak{x}$ such that $u \in X$. Choose a $\varphi \in \mathcal{P}\mathfrak{x}$ with $\varphi(X) = u$. Then, one has $R(f, \mathfrak{x}, \varphi) = \emptyset$. This is a contradiction ∎

Corollary: $({}^{\mathfrak{X}}\!\!\int \cdot\, d\mu)^{\vee}$ *is an F-ary partial operation in E, and one has* $\big(({}^{\mathfrak{X}}\!\!\int \cdot\, d\mu)^{\vee}\big)^{\wedge} = {}^{\mathfrak{X}}\!\!\int \cdot\, d\mu$ (for the terminology, see § 0/a)).

Proof: Theorem 5 in [13] ∎

As immediate consequence of Proposition 16 and (0.1), we obtain

Proposition 19: *If $\mathfrak{Y} \in \mathcal{F}\mathfrak{X}$, then $\mathfrak{Y}$ is a μ-partition system, and one has ${}^{\mathfrak{Y}}\!\!\int f\,d\mu = {}^{\mathfrak{X}}\!\!\int f\,d\mu$.*

This is a first answer to the question, how our integral behaves under the change of $\mathfrak{X}$. How does it depend on $\mathfrak{X}$ in general?

Our next aim is to show that the mapping assigning to each μ-partition system $\mathfrak{Y}$ the set ${}^{\mathfrak{Y}}\!\!\int f\,d\mu$ is monotone w.r. to $\subseteq$. For preparation, we prove the following two propositions. (The sum occuring in Proposition 20/b is of course understood to be taken w.r. to the summation $\sum_{\mathfrak{y}_X}$ in the Banach space $\mathbb{R}$.)

Proposition 20: *Let $\mathfrak{x}, \mathfrak{y} \in \Omega$ with $\mathfrak{x} \leq \mathfrak{y}$ and $X \in \mathfrak{x}$. Then, one has* (a)–(c):

(a) $\{\mathfrak{y}_X \mid X \in \mathfrak{x}\}$ *is a partition of* $\mathfrak{y}$.

(b) $\mathfrak{y}_X$ *is a partition of* X *and (thus)* $\sum_{Y \in \mathfrak{y}_X} \mu Y = \mu X$.

(c) $\sum_{Y \in \mathfrak{y}_X} f[Y]\,\mu Y \subseteqq \tau\mathrm{co}\,(f[X]\,\mu X)$.

Proof: (a) and (b) are clear. *Ad* (c). In view of (2.1), we may assume that $\mu Y \neq 0$ holds for all $Y \in \mathfrak{y}_X$. Let $z \in \sum_{Y \in \mathfrak{y}_X} f[Y]\,\mu Y$. Then, there is a $\varphi \in \mathsf{P}_{Y \in \mathfrak{y}_X} f[Y]$ such that $z \in \sum_{Y \in \mathfrak{y}_X} \{\varphi(Y)\}\,\mu Y$. For all $K \in \mathfrak{e}(\mathfrak{y}_X)$, one has $\left(\sum_{Y \in K} \varphi(Y)\,\mu Y\right)\left(\sum_{Y \in K} \mu Y\right)^{-1} \in \mathrm{co}\,f[X]$, therefore, by (b), (0.3), (1.8) and (1.9), $z \in \tau\mathrm{co}\,(f[X]\,\mu X)$. ∎

Proposition 21 (cf. [5, 4.3.1]): *Let* f *be singleton-valued,* $\mathfrak{x}, \mathfrak{y} \in \Omega$ *and* $B(f, \mathfrak{x})$ *be bounded and non-empty. Then,* $\mathfrak{x} \leqq \mathfrak{y}$ *implies* $\emptyset \neq B(f, \mathfrak{y}) \subseteqq \tau\mathrm{co}\,B(f, \mathfrak{x})$.

Proof: Put, for abbreviation, $S(L) = \sum_{Y \in L} f[Y]\,\mu Y$ for all L with $\emptyset \neq L \subseteqq \mathfrak{y}$. Since f is singleton-valued, one has $f[Z] \neq \emptyset$ for all Z with $\emptyset \neq Z \subseteqq F$. We recall the terminological agreements in § 0/c. — Assume $\mathfrak{x} \leqq \mathfrak{y}$. Using this, we shall apply Proposition 20/a, b tacitly in the following.

1. We show first that $S(\mathfrak{y}) \neq \emptyset$. This holds trivially, if $\mathfrak{y}$ is finite; assume $\mathfrak{y}$ to be infinite. Let $\varepsilon > 0$. By Proposition 15/b, the set $\sum_{X \in \mathfrak{x}} \tau\mathrm{aco}\,(f[X]\,\mu X)$ is non-empty; therefore (by Proposition 13), if $\mathfrak{x}$ is infinite, there exists a set $K \in \mathfrak{e}\mathfrak{x}$ (which we fix for the following) such that

$$(1)\quad \emptyset \neq \sum_{X \in L} \tau\mathrm{aco}\,(f[X]\,\mu X) \subseteqq B_{\varepsilon/2}(0) \text{ for all } L \in \mathfrak{e}(\mathfrak{x} \setminus K).$$

If $\mathfrak{x}$ is finite, let $K = \mathfrak{x}$ for the following.

Let $X \in K$. In view of Proposition 14, the boundedness of the non-empty set $B(f, \mathfrak{x})$ implies the boundedness of $f[X]\,\mu X$, and this (in the Banach space $\mathbb{R}$) $\sum_{Y \in \mathfrak{y}_X} \|f[Y]\,\mu Y\| \neq \emptyset$. Thus (by Proposition 9), one has $S(\mathfrak{y}_X) \neq \emptyset$. Therefore (by Proposition 13), if $\mathfrak{y}_X$ is infinite, there exists a set $P(X) \in \mathfrak{e}\mathfrak{y}_X$ (which we fix for the following) such that

$$(2)\quad \emptyset \neq \|S(L)\| < (\varepsilon/2)\,(\mathrm{card}\,K)^{-1} \text{ for all } L \in \mathfrak{e}\big(\mathfrak{y}_X \setminus P(X)\big).$$

If $\mathfrak{y}_X$ is finite, let $P(X) = \mathfrak{y}_X$ for the following. — Proceeding so for all $X \in \mathfrak{x}$, we have defined a family $\big(P(X)\big)_{X \in K}$.

Let $N = \bigcup \{P(X) \mid X \in K\}$; then $N \in \mathfrak{e}\mathfrak{y}$. Let $Q \in \mathfrak{e}(\mathfrak{y} \setminus N)$. In order to show that $S(\mathfrak{y}) \neq \emptyset$, it suffices (by Proposition 13) to prove that $\|S(Q)\| < \varepsilon$ holds. For abbreviation, we put, for each $X \in \mathfrak{x}$, $Q_X = \{Y \in Q \mid Y \subseteqq X\}$, furthermore $L_0 = \{X \in \mathfrak{x} \mid Q_X \neq \emptyset\}$ and $K_0 = K \cap L_0$. Then, L_0 is finite and non-empty, one has $Q = \bigcup \{Q_X \mid X \in L_0\}$ and $(L_0 \setminus K_0) \subseteqq (\mathfrak{x} \setminus K)$. Clearly, $Q_X \subseteqq \mathfrak{y}_X$ holds for all $X \in L_0$. By Proposition 11, one has $\|S(Q)\| = \left\|\sum_{X \in L_0} S(Q_X)\right\|$. Now, we consider the following three cases:

Case a). Let $K_0 \neq \emptyset$. (This holds, e.g., if $K_0 = L_0$.) Then, one has [since $Q_X \in \mathfrak{e}\big(\mathfrak{y}_X \setminus P(X)\big)$ holds for all $X \in K_0$] by (2) the inequality

$$\left\|\sum_{X \in K_0} S(Q_X)\right\| < (\mathrm{card}\,K)\,(\varepsilon/2)\,(\mathrm{card}\,K)^{-1} = \varepsilon/2.$$

Case b). Let $L_1 = L_0 \setminus K_0 \neq \emptyset$. (This holds, e.g., if $K_0 = \emptyset$.) Then, one has [in view of (1.7), (1.8) reformulated for "aco" instead of "co", $L_1 \in \mathfrak{e}(\mathfrak{x} \setminus K)$ and (1)] the chain

$$\left\|\sum_{X \in L_1} S(Q_X)\right\| \leqq_0 \left\|\sum_{X \in L_1} \tau\mathrm{aco}\,(f[X]\,\mu X)\right\| < \varepsilon/2.$$

Case c). Let $K_0 \neq \emptyset$ and $L_0 \setminus K_0 \neq \emptyset$. Then, one obtains [by Proposition 11, the triangle inequality, the combination of the cases a and b, and the transitivity of the relation $\leqq_0$] the chain

$$\left\| \sum_{X \in L_0} S(Q_X) \right\| \leqq_0 \left\| \sum_{X \in K_0} S(Q_X) \right\| + \left\| \sum_{X \in L_0 \setminus K_0} S(Q_X) \right\| < \varepsilon .$$

Summarizing, we have shown $\|S(Q)\| < \varepsilon$ in the general situation. (Remark: In view of Proposition 7/b, one is allowed to replace in this Part 1 of the proof "τaco" by "aco" everywhere.)

2. We show now that $B(f, \mathfrak{y}) \subseteqq \tau\mathrm{co}\, B(f, \mathfrak{x})$. If one uses, in this order: Proposition 20/a, Part 1 of this proof and Proposition 11; Proposition 20/c, $B(f, \mathfrak{x}) \neq \emptyset$ and Proposition 15; Proposition 15, one obtains the chain

$$B(f, \mathfrak{y}) \subseteqq \sum_{X \in \mathfrak{x}} S(\mathfrak{y}_X) \subseteqq \sum_{X \in \mathfrak{x}} \tau\mathrm{co}\, (f[X]\, \mu X) \subseteqq \tau\mathrm{co}\, B(f, \mathfrak{x})\ ∎$$

Corollary: *Assume f to be singleton-valued. Define g by $g(\mathfrak{x}) = B(f, \mathfrak{x})$ for all $\mathfrak{x} \in \mathfrak{X}$ and K by $K = \{\mathfrak{x} \in \mathfrak{X} \mid g(\mathfrak{x})$ non-empty and bounded$\}$. If $K \neq \emptyset$, then $(g \mid K, K, \leqq \mid K)$ is a* τco-*nested net, and one has* $\mathrm{LIM}\,(g, \mathfrak{X}, \leqq) = \mathrm{LIM}\,(g \mid K, K, \leqq \mid K)$.

Proof: Use that for each $(h, J, \mathfrak{b}) \in \Phi(\mathfrak{P}E)$, $\mathrm{LIM}\,(h \mid B, B, \mathfrak{b}_B) = \mathrm{LIM}\,(h, J, \mathfrak{b})$ for all $B \in \mathfrak{b}$ (for the notation, see § 0/b)) ∎

§ 4 The $(\mu, \mathfrak{X})$-integral in terms of Birkhoff sums

If I is a non-empty set and Θ is an I-ary partial operation in E, we define, for each family $(Z_i)_{i \in I}$ in $\mathfrak{P}E$, $\Theta_{i \in I} Z_i$ by letting, for all $x \in E$, $x \in \Theta_{i \in I} Z_i$ if $\mathbf{P}_{i \in I} Z_i \subseteqq \mathrm{Dmn}\,\Theta$ and $x = \Theta\varphi$ for some $\varphi \in \mathbf{P}_{i \in I} Z_i$. Then, if $Z_i \in \mathfrak{E}$ for all $i \in I$, $\Theta^\wedge\big((Z_i)_{i \in I}\big) = \Theta_{i \in I} Z_i \in \mathfrak{E}$ (see § 0/a) or [13, p. 121]).

Proposition 22: *Let $(\Theta_\mathfrak{x})_{\mathfrak{x} \in \mathfrak{X}}$ be a family of $\mathfrak{x}$-ary partial operations $\Theta_\mathfrak{x}$ in E and $\mathfrak{a}$ a filter on $\mathfrak{X}$. Put, for abbreviation, $K = \mathbf{S}_{\mathfrak{x} \in \mathfrak{X}} \mathscr{P}\mathfrak{x}$ and $\mathfrak{b} = {}^{\mathfrak{a}}\mathbf{S}_{\mathfrak{x} \in \mathfrak{X}} \{\mathscr{P}\mathfrak{x}\}$ and define the mappings S and T by letting, for all $\mathfrak{x} \in \mathfrak{X}$ and all $(\mathfrak{x}, \varphi) \in K$, $S(\mathfrak{x}) = \Theta_{\mathfrak{x}\, X \in \mathfrak{x}} f[X]\, \mu X$ and $T(\mathfrak{x}, \varphi) = \Theta_{\mathfrak{x}\, X \in \mathfrak{x}} f\big(\varphi(X)\big)\, \mu X$. Then, one has* (a) *and, if f is singleton-valued,* (b):

(a) $\displaystyle {}^{\mathfrak{b}}\lim_{(\mathfrak{x}, \varphi) \in K} T(\mathfrak{x}, \varphi) \subseteqq {}^{\mathfrak{a}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x})$, (b) $\displaystyle {}^{\mathfrak{a}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x}) \subseteqq {}^{\mathfrak{b}}\lim_{(\mathfrak{x}, \varphi) \in K} T(\mathfrak{x}, \varphi)$.

Proof: 1. Let x be a member of the left side of (a). Let $U \in \mathfrak{B}_\tau x$. Then, there is an $A \in \mathfrak{a}$ such that

(1) $\emptyset \neq T(\mathfrak{x}, \varphi) \subseteqq U$ for all $\mathfrak{x} \in A$ and all $\varphi \in \mathscr{P}\mathfrak{x}$.

Let $\mathfrak{x} \in A$ and $\psi \in \mathbf{P}_{X \in \mathfrak{x}} f[X]\, \mu X$. Then, there is a $\varphi \in \mathscr{P}\mathfrak{x}$ such that $f\big(\varphi(X)\big)\, \mu X = \{\psi(X)\}$ for all $X \in \mathfrak{x}$. Thus, by (1), $\emptyset \neq \Theta_{\mathfrak{x}\, X \in \mathfrak{x}} \{\psi(X)\} = T(\mathfrak{x}, \varphi) \subseteqq U$, therefore $\emptyset \neq S(\mathfrak{x}) \subseteqq U$. By the choice of U, we have shown that $x \in {}^{\mathfrak{a}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x})$. 2. Assume f to be singleton-valued. Let $x \in {}^{\mathfrak{a}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x})$ and $U \in \mathfrak{B}_\tau x$. Then, there is an $A \in \mathfrak{a}$ such that

(2) $\emptyset \neq S(\mathfrak{x}) \subseteqq U$ holds for all $\mathfrak{x} \in A$.

Let $\mathfrak{x} \in A$ and $\varphi \in \mathcal{P}\mathfrak{x}$. By the assumption on f, $f(\varphi(X))\mu X$ is non-empty for all $X \in \mathfrak{x}$; let $\psi(X) := e(f(\varphi(X))\mu X)$ for all $X \in \mathfrak{x}$. Then $\psi \in \mathbf{P}_{X\in\mathfrak{x}} f[X]\,\mu X$, therefore (by (2)) $\emptyset \neq \Theta_{\mathfrak{x}}\{\psi(X)\}_{X\in\mathfrak{x}} = T(\mathfrak{x},\varphi) \subseteq U$, and so, by the choice of U, x is element of the right-hand side of (b) ∎

Corollary: *One has* (a) *and, if f is singleton-valued,* (b):

(a) ${}^{\mathfrak{X}}\!\!\int f\,d\mu \subseteq {}^{\mathcal{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x}\in\mathfrak{X}}\, B(f,\mathfrak{x})$, (b) ${}^{\mathfrak{X}}\!\!\int f\,d\mu = {}^{\mathcal{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x}\in\mathfrak{X}}\, B(f,\mathfrak{x})$.

First proof: In Proposition 22, define $\mathfrak{a} = \mathcal{F}\mathfrak{X}$ and, for all $\mathfrak{x} \in \mathfrak{X}$, $\Theta_{\mathfrak{x}} = \sum_{\mathfrak{x}}$. Second proof: Combine Definition 2 and Definition 5 with Proposition 17 ∎

Theorem 1 (cf. [5, 4.4.3]): *Let $x \in E$. Then, the statement* (a) *implies the statement* (b); *if f is singleton-valued, the statements* (a) *and* (b) *are equivalent*:

(a) $x \in {}^{\mathfrak{X}}\!\!\int f\,d\mu$.

(b) *For each $U \in \mathfrak{B}_\tau x$, there is an $\mathfrak{x} \in \mathfrak{X}$ such that $\emptyset \neq B(f,\mathfrak{x}) \subseteq U$.*

Proof: If (a) holds, then f is singleton-valued by Proposition 18. Therefore, we may (and do) assume that f be singleton-valued. We show that (a) is equivalent to (b). We refer to the definitions of g and K in the Corollary to Proposition 21. Define (c) to be the statement $x \in \mathrm{LIM}\,(g, \mathfrak{X}, \leqq |\mathfrak{X})$. If (b) or (c) holds, then K is non-empty, therefore, by the mentioned corollary, the net $(g, K, \leqq |K)$ is τco-nested, thus, by Proposition 4, (b) and (c) are equivalent. Furthermore, one applies the Corollary to Proposition 22 ∎

Theorem 2 (cf. [5, 4.4.4]): *The statement* (a) *implies the statement* (b); *if f is singleton-valued, the statements* (a) *and* (b) *are equivalent*:

(a) ${}^{\mathfrak{X}}\!\!\int f\,d\mu \neq \emptyset$.

(b) *For each $\varepsilon > 0$, $\emptyset \neq \|B(f,\mathfrak{x}) - B(f,\mathfrak{x})\| < \varepsilon$ holds for some $\mathfrak{x} \in \mathfrak{X}$.*

Proof: Copy the preceding proof, except for defining (c), now, to be the statement $\mathrm{LIM}\,(g, \mathfrak{X}, \leqq | \mathfrak{X}) \neq \emptyset$ and referring, now, to Proposition 3 instead of Proposition 4 ∎

Theorem 3: *If ${}^{\mathfrak{X}}\!\!\int f\,d\mu$ is non-empty, then ${}^{\mathfrak{X}}\!\!\int f\,d\mu = \bigcap\,\{\tau\mathrm{co}\,B(f,\mathfrak{x}) \mid \mathfrak{x} \in \mathfrak{X}$ and $B(f,\mathfrak{x})$ is non-empty and bounded$\}$.*

Proof: Use the proof of Theorem 2 partly and refer, now, to Proposition 6 instead of Proposition 3 ∎

Remark 7: Let $\mathfrak{X} = \Omega$ and $g\colon F \to E$; assume that $f(u) = \{g(u)\}$ for all $u \in F$. By Theorem 13 in Birkhoff's paper [2, p. 367], g is "integrable" (in the sense of Birkhoff) if and only if (b) in our Theorem 2 holds; and in this case, the (single) element of the right side of the equation in our Theorem 3 is called, by Birkhoff (see Definition 4 and Theorem 12, both in [2, p. 367]), the "integral" of g. In this sense, the Corollary to Proposition 22 gives a representation of Birkhoff's integral of g by means of Riemann sums.

By the preceding remark, it is justified to call, from now on,

$${}^{\Omega}\!\!\int \cdot\, d\mu \text{ also the } \textit{Birkhoff integral (belonging to } \mu).$$

Now we can formulate a result, being indicated as a goal before Proposition 20.

Theorem 4 (cf. [5, 4.8.4]): *Let $\mathfrak{Y}$ be (beside $\mathfrak{X}$) a μ-partition system. Then,*

$$\mathfrak{X} \subseteq \mathfrak{Y} \text{ implies } {}^{\mathfrak{X}}\!\!\int f\,d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f\,d\mu;$$

Proof: Use Proposition 18 and Theorem 2, the latter applied first for $\mathfrak{X}$ and then for $\mathfrak{Y}$ ∎

Example 1: Let $\mu F < +\infty$. The classical examples for $\mathfrak{X}$ and $\mathfrak{Y}$ being in the relationship $\mathfrak{X} \sqsubseteq \mathfrak{Y}$ are $\mathfrak{X} = \{\mathfrak{x} \mid \mathfrak{x}$ is a finite μ-partition of $F\}$ and $\mathfrak{Y} = \Omega$.

For an illustration of Theorem 4, see Remark 10 in § 6.

Definition 6: f is called an *$\mathfrak{X}$-step function* if there is an $\mathfrak{x} \in \mathfrak{X}$ and a mapping χ: $\mathfrak{x} \to \mathfrak{E}$ such that $f(x) = \chi(X)$ for all $(X, x) \in \mathscr{S}\mathfrak{x}$. If f is in such a relationship with $\mathfrak{x}$ and χ, we say that f is the *$\mathfrak{X}$-step function determined by* $(\mathfrak{x}, \chi)$. (For the definition of $\mathscr{S}$, see § 0/e).)

The next statement (4.1) follows immediately from the definitions:

If f is the $\mathfrak{X}$-step function determined by $(\mathfrak{x}, \chi)$, then, for each $\mathfrak{y} \in \mathfrak{X}$ with $\mathfrak{x} \leqq \mathfrak{y}$, there is a λ: $\mathfrak{y} \to \mathfrak{E}$ such that f is the $\mathfrak{X}$-step function determined by $(\mathfrak{y}, \lambda)$. (4.1)

In many approaches to integration the definition of an integral is based on an "elementary integral" defined for step functions (mostly having finitely many steps) via a procedure of "completion" (whatever this word might mean in a particular theory). In the present approach, the "elementary integral" is already contained in the general case in the sense of the next theorem (which, roughly speaking, says that the $(\mathfrak{X}, \mu)$-integration extends unconditional summation).

Theorem 5 (cf. [5, 4.9.2]): *Let $\mathfrak{x} \in \mathfrak{X}$, $\chi: \mathfrak{x} \to \mathfrak{E}$, and f be the $\mathfrak{X}$-step function determined by $(\mathfrak{x}, \chi)$. Then, one has*

$$^{\mathfrak{X}}\!\int f\, d\mu = \sum_{X \in \mathfrak{x}} \chi(X)\, \mu X.$$

Proof: 1. One has $^{\mathfrak{X}}\!\int f\, d\mu \sqsubseteq \sum_{X \in \mathfrak{x}} \chi(X)\, \mu X$, since for all $\mathfrak{y} \in \mathfrak{X}$ with $\mathfrak{x} \leqq \mathfrak{y}$ and all $\varphi \in \mathscr{P}\mathfrak{y}$ the following chain [where one uses in this order: Proposition 20/a and Theorem 19 in [13] or Proposition 11; the supposition on f; a distributive law; Proposition 20/b] holds:

$$R(f, \mathfrak{y}, \varphi) \sqsubseteq \sum_{X \in \mathfrak{x}} \sum_{Y \in \mathfrak{y}_X} f(\varphi(Y))\, \mu Y = \sum_{X \in \mathfrak{x}} \sum_{Y \in \mathfrak{y}_X} \chi(X)\, \mu Y$$
$$= \sum_{X \in \mathfrak{x}} \chi(X) \sum_{Y \in \mathfrak{y}_X} \mu Y = \sum_{X \in \mathfrak{x}} \chi(X)\, \mu X.$$

2. For abbreviation, put $\{\mathfrak{x}\} = \mathfrak{Y}$. Then, one has $^{\mathfrak{Y}}\!\int f\, d\mu \sqsubseteq {}^{\mathfrak{X}}\!\int f\, d\mu$ by Theorem 4, while $^{\mathfrak{Y}}\!\int f\, d\mu = \sum_{X \in \mathfrak{x}} \chi(X)\, \mu X$ holds, since $R(f, \mathfrak{x}, \varphi) = \sum_{X \in \mathfrak{x}} \chi(X)\, \mu X$ is true for all $\varphi \in \mathscr{P}\mathfrak{x}$ ∎

Next, we answer the question to which extent the $(\mu, \mathfrak{X})$-integral of f can be represented by means of the weak Birkhoff sums $C(\mathfrak{x})$ belonging to $\mathfrak{x} \in \mathfrak{X}$ (see Definition 5).

Theorem 6: *One has* (a) *and* (b):

(a) *If $^{\mathfrak{X}}\!\int f\, d\mu \neq \emptyset$, then $^{\mathfrak{X}}\!\int f\, d\mu = {}^{\mathscr{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}}\, C(\mathfrak{x})$.*

(b) *If f is singleton-valued and there exists an $\mathfrak{x}_0 \in \mathfrak{X}$ such that $B(\mathfrak{x}_0)$ is non-empty and bounded, then $^{\mathscr{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}}\, C(\mathfrak{x}) \sqsubseteq {}^{\mathfrak{X}}\!\int f\, d\mu$.*

Proof: *Ad* (a). Assume $^{\mathfrak{X}}\!\int f\, d\mu \neq \emptyset$. Then, by Proposition 1 and the Corollary to Proposition 22, one has $^{\mathfrak{X}}\!\int f\, d\mu = {}^{\mathscr{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}}\, B(\mathfrak{x})$. Let $x \in {}^{\mathfrak{X}}\!\int f\, d\mu$ and $U \in \mathfrak{V}_{\tau}\, x$. Then, there is an $\mathfrak{x}_0 \in \mathfrak{X}$ such that

(1) $\emptyset \neq B(\mathfrak{x}) \subseteqq U$ holds for all $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_0 \leqq \mathfrak{x}$.

Let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_0 \leqq \mathfrak{x}$. Then, by (1) and Proposition 17, $B(\mathfrak{x}) = C(\mathfrak{x})$, therefore, by (1) and the choice of U, $x \in {}^{\mathcal{F}\mathfrak{X}}\underset{\mathfrak{y}\in\mathfrak{X}}{\mathrm{LIM}}\, C(\mathfrak{y})$. Furthermore, use Proposition 1.

Ad (b). Assume the premise within (b) and choose $\mathfrak{x}_0$ as there. Let $x \in {}^{\mathcal{F}\mathfrak{X}}\underset{\mathfrak{x}\in\mathfrak{X}}{\mathrm{LIM}}\, C(\mathfrak{x})$ and $U \in \mathfrak{B}_\tau x$. Then, there is an $\mathfrak{x}_1 \in \mathfrak{X}$ with $\mathfrak{x}_0 \leqq \mathfrak{x}_1$ such that

(2) $\emptyset \neq C(\mathfrak{x}) \subseteqq U$ holds for all $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_1 \leqq \mathfrak{x}$.

Let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_1 \leqq \mathfrak{x}$. By Proposition 21, one has $\emptyset \neq B(\mathfrak{x})$, since $\mathfrak{x}_0 \leqq \mathfrak{x}$, thus (by Proposition 17) we have $B(\mathfrak{x}) = C(\mathfrak{x})$, therefore (in view of (2) and the choice of U) $x \in {}^{\mathcal{F}\mathfrak{X}}\underset{\mathfrak{y}\in\mathfrak{X}}{\mathrm{LIM}}\, B(\mathfrak{y})$, and so (by the premise within (b) and the Corollary to Proposition 22) $x \in {}^{\mathfrak{X}}\!\int f\, d\mu$ ∎

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[2]) On p. 94, line 11, replace "$C \in \mathfrak{R}$" by "$C \in \mathfrak{R} \setminus \{\emptyset\}$".

[3]) In Example 4 on p. 119 replace "Then $\sup_I$ is ..." by "Then $\sup_I$ is neither an α-summation-like nor a β-summation-like I-ary partial operation on $\mathbb{R}$."

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