Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (6) 1990, S. 519–534

On a Lattice Problem for Geodesic Double Differential Forms in the n-Dimensional Hyperbolic Space

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Im n-dimensionalen hyperbolischen Raum $\mathbf{H}_n$ wird in Verallgemeinerung eines Gitterpunktproblems ein Gitterproblem für geodätische Doppeldifferentialformen gelöst. Dazu verwenden wir eine eigentlich diskontinuierliche Gruppe $\mathfrak{G}$ von Isometrien des $\mathbf{H}_n$ mit kompaktem Fundamentalbereich. Es wird der Zusammenhang von Gittersummen und dem p-Eigenwertspektrum des Laplace-Operators für Differentialformen auf der hyperbolischen Raumform $\mathbf{H}_n/\mathfrak{G}$ betrachtet. Zur Lösung des Gitterproblems werden Mittelwertoperatoren für Differentialformen, deren Kerndoppelformen und ein Landausches Differenzenverfahren verwendet.

Пусть $\mathbf{H}_n$ — гиперболическое n-мерное пространство, в котором определена вполне разрывная группа изометрий с компактной фундаментальной областью. В $\mathbf{H}_n$ рассматривается обобщенная сеточная задача для геодезических двойных дифференциальных форм. Изучается связь между суммами значений двойственной формы и собственным спектром оператора Лапласа для дифференциальных форм в пространстве гиперболических форм Клиффорда-Клейна $\mathbf{H}_n/\mathfrak{G}$. Для решения задачи используются операторы усреднения дифференциальной формы, их ядря, а также разностный метод Ландау.

Generalizing a lattice point problem we solve a lattice problem for geodesic double differential forms in the n-dimensional hyperbolic space. Thereby we use a properly discontinuous group $\mathfrak{G}$ of isometrics of $\mathbf{H}_n$ with compact fundamental domain. We study the relation between lattice sums and the eigenvalue spectrum of the Laplace operator for p-forms on the hyperbolic space form $\mathbf{H}_n/\mathfrak{G}$. Our approach essentially uses mean value operators for differential forms, their kernel double differential forms and a Landau difference method.

1. Introduction

Let $\mathfrak{G}$ be a properly discontinuous group of isometries of the n-dimensional hyperbolic space $\mathbf{H}_n$ of constant curvature -1 without fixed points (with the exception of the identity map id) with compact fundamental domain $\mathcal{F}$. We denote the corresponding Killing-Hopf space form $\mathbf{H}_n/\mathfrak{G}$ by V. Let d and δ be the differential and codifferential operator acting on differential p-forms on a Riemannian manifold. The p-eigenform spectrum S_p of the Laplace operator $\Delta = d\delta + \delta d$ plays an important role for various problems. A variety of results on the spectrum of the Laplace operator on a compact hyperbolic space form and its geometry as well as relations between them follow from the Selberg trace formula, cf. [12, 13, 22].

P. Günther [9, 11] has treated a lattice problem which generalizes the Landau ellipsoid problem and pointed out that a certain lattice sum determines the 0-spectrum of compact hyperbolic space forms. In order to determine the p-spectrum by lattice sums, we use the following geodesic double differential forms, introduced by P. GÜNTHER [6]:

$$\begin{aligned} \sigma_0(x, y) &= 1, & \tau_0(x, y) &= 0, \\ \sigma_1(x, y) &= \sinh r(x, y)\, d\hat{d}r(x, y), & \tau_1(x, y) &= dr(x, y)\, \hat{d}r(x, y), \\ \sigma_p &= (1/p)\, \sigma_{p-1} \wedge \hat{\wedge}\, \sigma_1, & \tau_p &= \sigma_{p-1} \wedge \hat{\wedge}\, \tau_1 \end{aligned} \tag{1}$$

with the geodesic distance $r(x, y)$ of the points $x, y \in \mathbf{H}_n$; d and $\hat{d}$ denote the differential with respect to x and y, respectively. Every isometry b of $\mathbf{H}_n$ induces a mapping b^* for differential forms. If we use b^* in connection with a double differential form it shall be taken with respect to the second variable. We are interested in

$$\mathcal{S}^\sigma(t, x, y) = \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} b^* \sigma_p(x, by), \qquad \mathcal{S}^\tau(t, x, y) = \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} b^* \tau_p(x, by). \tag{2}$$

This paper is a continuation of the treatment of lattice problems in [24], in that paper we have studied the Euclidean case. In Section 3 we show that $\mathcal{S}^\sigma(t, x, x,)$ and $\mathcal{S}^\tau(t, x, x)$ or even $\int_{\mathcal{F}} \operatorname{tr} \mathcal{S}^\sigma(t, x, x)\, dv$ and $\int_{\mathcal{F}} \operatorname{tr} \mathcal{S}^\tau(t, x, x)\, dv$ determine the spectrum S_p of Δ (for the definition of the traces cf. Section 2). $\mathcal{S}^\sigma$ and $\mathcal{S}^\tau$ also contain information about the eigenforms of Δ. We use Selberg transformations (cf. [25]) for the treatment of these problems. As usual in the context of lattice problems, we are interested in the asymptotic behaviour of $\mathcal{S}^\sigma(t, x, y)$ and $\mathcal{S}^\tau(t, x, y)$ for $t \to \infty$.

A p-form is called $\mathfrak{G}$-*automorphic*, if it is invariant under b^* for all $b \in \mathfrak{G}$. We identify the $\mathfrak{G}$-automorphic differential forms on $\mathbf{H}_n$ and the differential forms on V. In the space of quadratic integrable p-forms over V there exists a complete orthonormal system E_p of eigenforms ω of Δ which we can suppose to be closed ($d\omega = 0$) or coclosed ($\delta\omega = 0$). Thereby we have used the scalar product which we get by integration over V from a pointwise scalar product for p-forms defined in Section 2 by (4). We decompose E_p into $E_p = E_p{}^h \cup E_p{}^d \cup E_p{}^\delta$ with

$$\omega \in \begin{cases} E_p{}^h & \text{if } \Delta\omega = 0 \text{ (harmonic eigenforms)}, \\ E_p{}^d & \text{if } \delta\omega = 0,\ \delta\omega \neq 0, \\ E_p{}^\delta & \text{if } \delta\omega = 0,\ d\omega \neq 0. \end{cases}$$

The dimension of $E_p{}^h$ is the p-th Betty number B_p of V. The eigenvalue of an eigenform $\omega \in E_p$ we always denote by μ_ω: $\Delta\omega = \mu_\omega \omega$. We get the

Theorem: *Set $N = (n-1)/2$ and define*

$$\mathcal{H}^\sigma(t, x, y) = \begin{cases} \displaystyle\sum_{\substack{\omega \in E_0 \\ 0 \le \mu_\omega < n\left(\frac{n-1}{n+1}\right)^2}} c_0(\mu_\omega) \exp\left(\left(\sqrt{N^2 - \mu_\omega} + N\right) t\right) \omega(x)\, \omega(y) & \text{for } p = 0, \\ \displaystyle\sum_{\substack{\omega \in E^d_{n-1} \\ 0 < \mu_\omega < n\left(\frac{n-1}{n+1}\right)^2}} c_1(\mu_\omega) \exp\left(\left(\sqrt{N^2 - \mu_\omega} + N\right) t\right) \omega(x)\, \omega(y) & \text{for } p = n-1, \\ 0 & \text{for } p \neq 0, p \neq n-1. \end{cases}$$

Then we get for the lattice remainder $\mathcal{R}^\sigma = \mathcal{S}^\sigma - \mathcal{H}^\sigma$ the error estimate

$$\|\mathcal{R}^\sigma(t, x, y)\| = O\left(\exp\left((n - 2 + 2/(n+1))\, t\right)\right.$$

with the pointwise norm $\|\cdot\|$ *with respect to x and y given by* (6). c_0 *and* c_1 *are defined by*

$$c_0(\mu) = \pi^N \Gamma(c_2(\mu))/\Gamma(c_2(\mu) + N + 1),$$
$$c_1(\mu) = (-1)^n c_0(\mu) \mu(c_2(\mu) + N)^{-2} \tag{3}$$

with $c_2(\mu) = \sqrt{N^2 - \mu}$. *The same statement is true for* τ *instead of* σ *if we replace* $p = 0$, $p = n - 1$, E_0, E_{n-1}^d *by* $p = n$, $p = 1$, E_n, E_1^δ, *respectively.*

We want to give some conclusions and remarks. The special case $p = 0$ for $\mathcal{S}^\sigma$ is the known result for

$$A(t, x, y) = \sum_{\substack{b \in \mathfrak{G}, \\ r(x, by) < t}} 1$$

of [11]. The leading term of $\mathcal{S}^\sigma$ is determined by the eigenforms with eigenvalue 0 for $p = 0$. For $p = n - 1$ the order of magnitude of the leading term is determined by the smallest positive eigenvalue lesser $n((n-1)/(n+1))^2$ and $\mathcal{H}^\sigma(t, x, y)$ is closed, there is no harmonic part. This is in contrast to the situation we have seen in [24], in the Euclidean case the leading term of the lattice sum $\mathcal{S}^\sigma$ for $p = 0, \ldots, n - 1$ is always determined (for $\mathcal{H}^\sigma(t, x, y) \neq 0$ in the terminology of this paper) by the p-eigenforms of eigenvalue 0. The leading term of the lattice sum $\mathcal{S}^\sigma + \mathcal{S}^\tau$ is harmonic for $p = 0$ and $p = n$, is closed but not coclosed for $p = n - 1$ and is coclosed but not closed for $p = 1$ (if there exist eigenvalues $\mu \in E_{n-1}^d$ and $\mu \in E_1^\delta$ with $0 < \mu < n((n-1)/(n+1))^2$ and $n \neq 2$, respectively).

For the proof of the theorem we use mean value operators for differential forms. A more detailed treatment of these operators and relations between them one can find in [19, 23]. We apply a Landau difference method which also was essential for the treatment of a lattice problem for p-forms in the Euclidean space, cf. [24]. The solution $z(t, \lambda, \mu, n)$ of the Euler-Poisson-Darboux equation

$$\left(d^2/dt^2 + \lambda \coth t \, d/dt + \mu + 1/4(\lambda^2 - (n-1)^2)\right) z(t, \lambda, \mu, n) = 0,$$
$$z(0, \lambda, \mu, n) = 1, \qquad \left(dz(t, \lambda, \mu, n)/dt\right)\big|_{t=0} = 0$$

is essential for the eigenform expansion of the kernels (18) of mean value operators. The basic properties of $z(t, \lambda, \mu, n)$ are recalled in Section 2.

2. Double differential forms and spherical mean values

We define the double differential forms

$$\alpha_p(x, y) = \frac{1}{p!} \{d\hat{d} \cosh r(x, y)\}^p \qquad \text{(exterior power)},$$

$$\beta_p(x, y) = \frac{1}{p!} \cosh r(x, y) \left\{\cosh r(x, y) \, d\hat{d}\left(\ln \cosh r(x, y)\right)\right\}^p.$$

By (1) we get

$$\alpha_p = \sigma_p + \cosh r \, \tau_p; \qquad \beta_p = \cosh r \, \sigma_p + \tau_p.$$

According to [6, 23], we have

$$\Delta \alpha_p = -p(n - p + 1) \, \alpha_p, \qquad \Delta \beta_p = -(p + 1)(n - p) \, \beta_p,$$
$$d\alpha_p = 0, \qquad \delta \beta_p = 0.$$

The differential operators are taken with respect to the first variable. We use a global coordinate system $(x^1, \ldots, x^n)$ of $\mathbf{H}_n$. In order to recall the geometric background of the geodesic double differential forms σ_p, τ_p, we define the component of a p-form

$$\varphi = \varphi_{i_1 \cdots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

in the direction of a vector $v = (v^i)$ of the tangent space in $x \in \mathbf{H}_n$ with $v^i v_i = 1$ by

$$\varphi_{|v} = p v_{[i_1} v^i \varphi_{|i| i_2 \cdots i_p]} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

We adopt the convention of summing over repeated indices. $[\ldots |i| \ldots]$ shall denote the alternation without i. Further on we set $\varphi_{|v}^{\perp} = \varphi - \varphi_{|v}$. We define the pointwise scalar product of p-forms φ and ψ by

$$(\varphi \cdot \psi)(x) = p! \, \varphi_{i_1 \cdots i_p}(x) \, \psi^{i_1 \cdots i_p}(x). \tag{4}$$

Using the Hodge star operator $*$, we have

$$\delta = (-1)^{pn+n+1} * d * \text{ for } p\text{-forms and } (\varphi \cdot \psi) * 1 = \varphi \wedge * \psi. \tag{5}$$

The scalar product of double differential forms

$$\varphi(x, y) = \varphi_{i_1 \cdots i_p j_1 \cdots j_p}(x, y) \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \, dy^{j_1} \hat{\wedge} \cdots \hat{\wedge} \, dy^{j_p},$$

$$\psi(x, y) = \psi_{i_1 \cdots i_p j_1 \cdots j_p}(x, y) \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \, dy^{j_1} \hat{\wedge} \cdots \hat{\wedge} \, dy^{j_p}$$

is defined by

$$(\varphi \cdot \psi)(x, y) = p!^2 \, \varphi_{i_1 \cdots i_p j_1 \cdots j_p}(x, y) \, \psi^{i_1 \cdots i_p j_1 \cdots j_p}(x, y) \tag{6}$$

and the trace in (x, x) by

$$\operatorname{tr} \varphi(x, x) = p! \, \varphi_{i_1 \cdots i_p}{}^{i_1 \cdots i_p}(x, x).$$

Let $c(t)$ be a geodesic line with $c(0) = y$ and $c(1) = x$. If we do a parallel translation of the component $\varphi(y)_{|\dot{c}(0)/|\dot{c}(0)|}$ of the p-form $\varphi(y)$ along $c(t)$ from y to x, we get $(-1)^p \tau_p(x, y) \cdot \varphi(y)$ according to [8]. If we take $\varphi_{|v}^{\perp}$ instead of $\varphi_{|v}$, we get $(-1)^p \times \sigma_p(x, y) \cdot \varphi(y)$ and $(-1)^p (\sigma_p + \tau_p)$ is the transport form.

We introduce the spherical mean values

$$M^{\alpha}[\omega](t, x) = c_3 \sinh^{1-n} t \int_{S(x,t)} \alpha_p(x, y) \cdot \omega(y) \, do_y,$$

$$M^{\beta}[\omega](t, x) = c_3 \sinh^{1-n} t \int_{S(x,t)} \beta_p(x, y) \cdot \omega(y) \, do_y$$

with $c_3 = \frac{1}{2} (-1)^p \pi^{-n/2} \Gamma(n/2)$ for differential p-forms ω, $S(x, t)$ denotes the sphere around x with geodesic radius t, do denotes the surface element. In [7, 8, 23] the spherical mean values defined with σ_p, τ_p instead of α_p, β_p have been treated. Using these results, we get

$$M^{\alpha}[\omega](t, x) = \begin{cases} \big(-(n-p)/n \sinh^2 t \, y(t, n+1, \mu, n) \\ \quad + \cosh t \, y(t, n-1, \mu, n)\big) \, \omega(x) & \text{for } \Delta\omega = \mu\omega, \, d\omega = 0, \\ x(t, n-1, \mu, n) \, \omega(x) & \text{for } \Delta\omega = \mu\omega, \, \delta\omega = 0, \end{cases}$$

$$M^{\beta}[\omega](t, x) = \begin{cases} y(t, n-1, \mu, n) \, \omega(x) & \text{for } \Delta\omega = \mu\omega, \, d\omega = 0, \\ \big(-p/n \sinh^2 t \, x(t, n+1, \mu, n) \\ \quad + \cosh t \, x(t, n-1, \mu, n)\big) \, \omega(x) & \text{for } \Delta\omega = \mu\omega, \, \delta\omega = 0 \end{cases} \tag{7}$$

with the functions

$$
\begin{aligned}
x(t, \lambda, \mu, n) &= z\big(t, \lambda, \mu + (p+1)(n-p) - n, n\big), \\
y(t, \lambda, \mu, n) &= z\big(t, \lambda, \mu + p(n+1-p) - n, n\big), \\
z(t, \lambda, \mu, n) &= \left(\frac{1}{2}(\cosh t + 1)\right)^{(1-\lambda)/2} \\
&\quad \times F\left(\frac{1}{2} - \chi, \frac{1}{2} + \chi, \frac{1}{2}(\lambda+1), \frac{1}{2}(1-\cosh t)\right). \\
\chi &= \sqrt{N^2 - \mu}, \qquad N = (n-1)/2.
\end{aligned}
\tag{8}
$$

F thereby denotes the Gauss hypergeometric function. The function z satisfies the Euler-Poisson-Darboux equation

$$(d^2/dt^2 + \lambda \coth t\, d/dt + \mu + \lambda^2/4 - N^2)\, z(t, \lambda, \mu, n) = 0$$

with the initial conditions $z(0, \lambda, \mu, n) = 1$ and $\big(dz(t, \lambda, \mu, n)/dt\big)\big|_{t=0} = 0$. We have $z(t, \lambda, \mu, n) = \big(1/(\lambda+1) \sinh t\, d/dt + \cosh t\big)\, z(t, \lambda+2, \mu, n)$. For $\lambda_2 \geqq \lambda_1 + 2$ we get

$$
\begin{aligned}
z(t, \lambda_2, \mu, n) &= \left(2 \sinh^{1-\lambda_2} t \Big/ B\left(\frac{1}{2}(\lambda_1+1), \frac{1}{2}(\lambda_2 - \lambda_1)\right)\right) \\
&\quad \times \int_0^t \big(2(\cosh t - \cosh s)\big)^{(\lambda_2-\lambda_1-2)/2} \sinh^{\lambda_1} s\, z(s, \lambda_1, \mu, n)\, ds.
\end{aligned}
\tag{9}
$$

The estimation

$$|z(t, \lambda, \mu, n)| \leqq c_4 \sinh^{-\lambda/2} t (\mu - N^2)^{-\lambda/4} \tag{10}$$

for $\mu > \tilde{\mu} > N^2$, $N = (n-1)/2$, $t > 0$ is of importance for the convergence of eigenform developments of kernels of mean value operators. By setting

$$
\begin{aligned}
u(t, \lambda, \mu, n) &= -(\lambda+1)^{-1} \big((\lambda+1)/2 - p + n/2\big) \sinh^2 t\, y(t, \lambda+2, \mu, n) \\
&\quad + \cosh t\, y(t, \lambda, \mu, n), \\
v(t, \lambda, \mu, n) &= -(\lambda+1)^{-1} \big((\lambda+1)/2 + p - n/2\big) \sinh^2 t\, x(t, \lambda+2, \mu, n) \\
&\quad + \cosh t\, x(t, \lambda, \mu, n)
\end{aligned}
\tag{11}
$$

we obtain $u(t, \lambda, \mu, n) = \big((\lambda+1)^{-1} \sinh t\, d/dt + \cosh t\big)\, u(t, \lambda+2, \mu, n)$ and the same equation for v instead of u. Using the asymptotic behaviour of $F\left(a, b, c, \frac{1}{2} \times (1-\cosh t)\right)$ for $t \to \infty$ (cf. [14]) we have for $0 \leqq \mu < N^2$, $t \to \infty$

$$
\begin{aligned}
z(t, \lambda, \mu, n) &= \frac{\Gamma\left(\frac{1}{2}(\lambda+1)\right) \Gamma(\chi)\, 2^{\lambda-1}}{\Gamma(\chi + \lambda/2)\, \Gamma(1/2)} (2 \cosh t)^{\chi - \lambda/2} \\
&\quad + O\big((\cosh t)^{-\lambda/2+\chi-1}\big) + O\big((\cosh t)^{-\lambda/2-\chi}\big)
\end{aligned}
\tag{12}
$$

with $\chi = \sqrt{N^2 - \mu}$. Further on we get by a short calculation (cf. [19])

$$z(t, \lambda, N^2, n) = O\big(t \sinh^{-\lambda/2} t\big). \tag{13}$$

3. Lattice sums and spectra

The Laplace operator $\Delta = d\delta + \delta d$ for p-forms on a compact hyperbolic space form V has a discrete spectrum S_p. The eigenspace of an eigenvalue $\mu \in S_p$ has finite dimension $d_p(\mu)$. For $\mu > 0$ it is the orthogonal sum (with respect to the scalar product (4) integrated over V) of a subspace of closed ($d\omega = 0$) and coclosed ($\delta\omega = 0$) eigenforms ω with the dimensions $d_p{}^d(\mu)$ and $d_p{}^\delta(\mu)$, respectively. Put $S_p' = S_p \setminus \{0\}$. By the telescopage theorem of McKean and Singer we have $d_p{}^\delta(\mu) = d_{p+1}^d(\mu)$ for $\mu \in S_p'$ (cf. [1]).

To begin, we consider the relationship between lattice sums and the eigenvalue spectrum for $p = 0$. We define

$$K(x, y) = \sum_{b\in\mathfrak{G}} k\left(\sinh^2 \frac{1}{2} r(x, by)\right)$$

for a continuous function k on the real line with compact support. For every pair $(x, y) \in \mathbf{H}_n \times \mathbf{H}_n$ the sum is finite because $\mathfrak{G}$ is properly discontinuous. We have $K(x, y) = K(y, x)$. Using

$$A(t, x, y) = \sum_{\substack{b\in\mathfrak{G} \\ r(x,by)<t}} 1$$

we can write $K(x, y)$ as a Stieltjes integral:

$$K(x, y) = \int_0^\infty k\left(\sinh^2 \frac{1}{2} r\right) dA(\cdot, x, y)\,(r). \tag{14}$$

P. GÜNTHER [11] considered the asymptotic behaviour ($t \to \infty$) of $A(t, x, y)$. Following standard ideas for the proof of the Selberg trace formula, we use the integral operator

$$K: \varphi \to \int_{\mathcal{F}} K(x, y)\, \varphi(y)\, dv_y = \int_{\mathbf{H}_n} k\left(\sinh^2 \frac{1}{2} r(x, y)\right) \varphi(y)\, dv_y$$

for $\mathfrak{G}$-automorphic functions φ on $\mathbf{H}_n$, which are quadratically integrable over a fundamental domain $\mathcal{F}$ of $\mathfrak{G}$ in $\mathbf{H}_n$ (dv denotes the volume element). The last integral can also be written in the form

$$\left(2\pi^{n/2}/\Gamma(n/2)\right) \int_0^\infty k\left(\sinh^2 \frac{1}{2} r\right) \sinh^{n-1} r\, M^a[\varphi]\,(r, x)\, dr.$$

By virtue of (7) the eigenfunction expansion of the kernel $K(x, y)$ with respect to the eigenfunction system E_0 gives (as an L^2 equation over $\mathcal{F}$ with respect to y)

$$\sum_{b\in\mathfrak{G}} k\left(\sinh^2 \frac{1}{2} r(x, by)\right) = \sum_{\omega\in E_0} h\left(\sqrt{\mu_\omega - N^2}\right) \omega(x)\, \omega(y) \tag{15}$$

with

$$h\left(\sqrt{\mu_\omega - N^2}\right) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty k\left(\sinh^2 \frac{1}{2} r\right) \sinh^{n-1} r\, z(r, n-1, \mu_\omega, n)\, dr. \tag{16}$$

According to [25], the functions h and k are related by the Selberg transformations:

$$\left.\begin{aligned} k(t) &= \left((-1^m)\pi^m\right) Q^{(m)}(w)|_{w=t} \\ Q(w) &= \pi^m \int_w^\infty \int_{t_1}^\infty \dots \int_{t_{m-1}}^\infty k(t)\, dt \dots dt_2\, dt_1 \end{aligned}\right\} \text{ for } n = 2m+1,$$
$$\left.\begin{aligned} k(t) &= \left((-1)^m/\pi^m\right) \int_t^\infty Q^{(m)}(w)/\sqrt{w-t}\, dw \\ Q(w) &= \pi^{m-1} \int_w^\infty \int_{t_1}^\infty \dots \int_{t_{m-1}}^\infty k(t)/\sqrt{t-t_{m-1}}\, dt \dots dt_2\, dt_1 \end{aligned}\right\} \text{ for } n = 2m, \tag{17}$$

(for $n = 2$, $m = 1$ set $t_{m-1} = w$) and

$$Q(w) = Q(2\cosh u - 2) = g(u),$$

$$h(r) = \int_{-\infty}^{\infty} \exp(iru)\, g(u)\, du, \qquad g(u) = (1/2\pi) \int_{-\infty}^{\infty} \exp(-iru)\, h(r)\, dr.$$

We could deduce (16) directly from these transformation formulas. We want to allow for the possibility that k no longer has compact support. Thereby we use $A(t, x, y) = O\{\exp[(n-1)t]\}$ (cf. [11]) and the fact, that

$$\sum_{\omega \in E_0, \mu_\omega > 0} (\mu_\omega)^{-(n+\varepsilon)/2}\, \omega(x)\, \omega(y) \qquad (\varepsilon > 0)$$

is absolutely convergent in order to prove that the series of both sides of (15) are convergent. By a standard approximation argument (cf. [13, 25]) we see that (15) is still valid (even as a pointwise equation with respect to x and y), if we suppose the following condition:

(C) $h = h(r)$ is an even analytic function in the strip

$$|\mathrm{Im}\, r| < (n-1)/2 + \varepsilon,\ \varepsilon > 0 \text{ and } h(r) = O\left((1+|r|)^{-\max(3,n)}\right).$$

If we choose $h(r) = \exp(-r^2)$ and use (14) and (15), it follows easily that $A(t, x, y)$ or even $A(t) = \int_{\mathcal{F}} A(t, x, x)\, dv_x$ determines the 0-spectrum S_0. Notice further that $A(t, x, y)$ determines $\sum_{\omega \in E_0, \mu_\omega = \mu} \omega(x)\, \omega(y)$ for $\mu \in S_0$.

In order to get a relation between the p-spectrum and lattice sums, we define for $\lambda \geqq n+3$

$$\tilde{\mathcal{M}}^{\alpha}(t, \lambda, x, y) = c_5(\lambda) \left(2(\cosh t - \cosh r(x, y))\right)_+^{(\lambda-n-3)/2} \alpha_p(x, y)$$

with

$$(t)_+^{\nu} = \begin{cases} t^{\nu} & \text{if } t > 0, \\ 0 & \text{if } t \leqq 0, \end{cases} \quad c_5(\lambda) = \frac{(-1)^p}{\pi^{n/2}} \frac{\Gamma\left(\frac{1}{2}(\lambda-1)\right)}{\Gamma\left(\frac{1}{2}(\lambda-n-1)\right)}$$

and the analogous equation with β instead of α. It is convenient to set

$$\begin{aligned}
&\mathcal{M}^\alpha(t,\lambda,x,y) = \sum_{b\in\mathfrak{G}} b^*\tilde{\mathcal{M}}^\alpha(t,\lambda,x,by), \qquad \mathcal{M}^\beta(t,\lambda,x,y) = \sum_{b\in\mathfrak{G}} b^*\tilde{\mathcal{M}}^\beta(t,\lambda,x,by),\\
&\mathcal{A}(t,\lambda,x,y) = (\lambda-3)\cosh t\, \mathcal{M}^\beta(t,\lambda-2,x,y) - (\lambda-3)\,\mathcal{M}^\alpha(t,\lambda-2,x,y)\\
&\qquad - \big((\lambda-3)/2 - p + n/2\big)\,\mathcal{M}^\beta(t,\lambda,x,y),\\
&\mathcal{B}(t,\lambda,x,y) = (\lambda-3)\cosh t\, \mathcal{M}^\alpha(t,\lambda-2,x,y) - (\lambda-3)\,\mathcal{M}^\beta(t,\lambda-2,x,y)\\
&\qquad - \big((\lambda-3)/2 + p - n/2\big)\,\mathcal{M}^\beta(t,\lambda,x,y).
\end{aligned} \tag{18}$$

For $\mathfrak{G}$-automorphic p-forms ω, we have

$$\begin{aligned}
\int_{\mathcal{F}} \mathcal{M}^\alpha(t,\lambda,x,y)\,\omega(y)\,dv_y &= \int_{\mathbf{H}_n} \tilde{\mathcal{M}}^\alpha(t,\lambda,x,y)\,\omega(y)\,dv_y\\
&= \frac{c_5(\lambda)}{c_3}\int_0^t \big(2(\cosh t - \cosh r)\big)^{(\lambda-n-3)/2}\sinh^{n-1} r\, M^\alpha[\omega]\,(r,x)\,dr.
\end{aligned}$$

If we use the mean value formula (7) for $\mathcal{M}^\alpha$ and equation (9), we obtain the eigenform expansion of $\mathcal{M}^\alpha$; for more details cf. [23]. More precisely, we have the

Proposition 1: *For $x, y \in \mathbf{H}_n$ the following eigenform expansions are valid as L^2 equations over $\mathcal{F}$ with respect to y:*

(i) $$\mathcal{M}^\alpha(t,\lambda,x,y) = \sum_{\omega\in E_p^d} u(t,\lambda-2,\mu_\omega,n)\sinh^{\lambda-3} t\,\omega(x)\,\omega(y) + \sum_{\omega\in E_p^\delta\cup E_p^h} x(t,\lambda-2,\mu_\omega,n)\sinh^{\lambda-3} t\,\omega(x)\,\omega(y);$$

(ii) $$\mathcal{M}^\beta(t,\lambda,x,y) = \sum_{\omega\in E_p^d\cup E_p^h} y(t,\lambda-2,\mu_\omega,n)\sinh^{\lambda-3} t\,\omega(x)\,\omega(y) + \sum_{\omega\in E_p^\delta} v(t,\lambda-2,\mu_\omega,n)\sinh^{\lambda-3} t\,\omega(x)\,\omega(y);$$

(iii) $$\mathcal{A}(t,\lambda,x,y) = -\sum_{\omega\in E_p^\delta}\frac{\mu_\omega}{\lambda-1}\,x(t,\lambda,\mu_\omega,n)\sinh^{\lambda-1} t\,\omega(x)\,\omega(y); \tag{19}$$

(iv) $$\mathcal{B}(t,\lambda,x,y) = -\sum_{\omega\in E_p^d}\frac{\mu_\omega}{\lambda-1}\,y(t,\lambda,\mu_\omega,n)\sinh^{\lambda-1} t\,\omega(x)\,\omega(y). \tag{20}$$

The series are pointwise and absolute convergent in x and y for $\lambda > 2n+2$.

It remains to prove the pointwise convergence. But this is a consequence of the well-known fact, that $\sum_{\omega\in E_p^d\cup E_p^\delta}(\mu_\omega)^{-\lambda/2}\,\|\omega(x)\|\,\|\omega(y)\|$ is convergent and of the estimate (10) ∎

Taking the trace in (19) and (20), we obtain for $\lambda > 2n+2$ by integration over $\mathcal{F}$

$$\begin{aligned}
\int_{\mathcal{F}} \operatorname{tr}\mathcal{A}(t,\lambda,x,x)\,dv_x &= \sum_{\omega\in E_p^\delta}(-\mu_\omega)(\lambda-1)^{-1}\sinh^{\lambda-1} t\, x(t,\lambda,\mu_\omega,n),\\
\int_{\mathcal{F}} \operatorname{tr}\mathcal{B}(t,\lambda,x,x)\,dv_x &= \sum_{\omega\in E_p^d}(-\mu_\omega)(\lambda-1)^{-1}\sinh^{\lambda-1} t\, y(t,\lambda,\mu_\omega,n).
\end{aligned}$$

If we suppose the condition (C) for h with the dimension $\lambda + 1$, we deduce by integration ($\lambda > 2n + 2$, $\lambda \in \mathbf{Z}$)

$$\int_0^\infty k\left(\sinh^2 \frac{1}{2} r\right) \sinh^{-1} r \int_{\mathcal{F}} \operatorname{tr} \mathcal{A}(t, \lambda, x, x)\, dv_x\, dr$$

$$= \sum_{\omega \in E_p^\delta} (-\mu_\omega) (\lambda - 1)^{-1} h\left(\sqrt{\tilde{\mu}_\omega - ((\lambda - 1)/2)^2}\right)$$

with $\tilde{\mu}_\omega = \mu_\omega + (p + 1)(n - p) - n + \frac{1}{4}\left(\lambda^2 - (n - 1)^2\right)$. The functions h and k are thereby related by the Selberg transformations (16) and (17) with respect to the dimension $\lambda + 1$. We get the analogous equation for $\mathcal{B}$ instead of $\mathcal{A}$ if we replace E_p^δ by E_p^d and $\tilde{\mu}_\omega$ by $\tilde{\tilde{\mu}}_\omega = \mu_\omega + p(n + 1 - p) - n + \frac{1}{4}\left(\lambda^2 - (n - 1)^2\right)$. Again, we use $h(r) = \exp(-r^2)$. Then it follows that $\mathcal{A}(t, \lambda, x, y)$ and $\mathcal{B}(t, \lambda, x, y)$ or even the traces $\operatorname{tr} \mathcal{A}(t, \lambda, x, x)$ and $\operatorname{tr} \mathcal{B}(t, \lambda, x, x)$ determine $d_p^\delta(\mu)$, $d_p^d(\mu)$ for $\mu \in S_p'$ and S_p' itself.

4. Lattice estimates

In order to extract information about the asymptotic behaviour of $\mathcal{S}^\sigma(t, x, y)$ and $\mathcal{S}\cdot(t, x, y)$ for $t \to \infty$ we shall use

$$\mathcal{S}(t, x, y) = \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} \sinh^2 r(x, by)\, b^* \sigma_p(x, by).$$

Observe that

$$\mathcal{S}^\sigma(t, x, y) = \int_0^t \sinh^{-2} \tau\, d\mathcal{S}(\cdot, x, y)(\tau). \tag{21}$$

Partial integration gives

$$\mathcal{S}^\sigma(t, x, y) = 2 \int_0^t \cosh \tau \sinh^{-3} \tau\, \mathcal{S}(\tau, x, y)\, d\tau + \sinh^{-2} t\, \mathcal{S}(t, x, y).$$

In view of this we consider the asymptotic behaviour of $\mathcal{S}(t, x, y)$. We can write this double differential form in terms of the kernels defined in Section 3:

$$\begin{aligned} \mathcal{S}(t, x, y) &= c^*\big(-\mathcal{M}^\beta(t, n + 5, x, y) + (n + 2) \cosh t\, \mathcal{M}^\beta(t, n + 3, x, y) \\ &\qquad - (n + 2)\, \mathcal{M}^\alpha(t, n + 3, x, y)\big) \\ &= c^*\big((n - p)\, \mathcal{M}^\beta(t, n + 5, x, y) + \mathcal{A}(t, n + 5, x, y)\big) \end{aligned} \tag{22}$$

with $c^* = \frac{1}{2} (-1)^p \pi^{-n/2} / \Gamma(n/2 + 2)$.

We use the pointwise convergence for $\lambda > 2n + 2$ of the eigenform expansion of $\mathcal{M}^\alpha$, $\mathcal{M}^\beta$ stated in Proposition 1 and the following recursion equation which is an immediate consequence from the definition.

Lemma 2: *For $\lambda \geqq n+3$ we have*

$$\partial/\partial t\, \mathcal{M}^\alpha(t, \lambda+2, x, y) = (\lambda-1) \sinh t\, \mathcal{M}^\alpha(t, \lambda, x, y).$$

The same equation is valid with β instead of α.

In connection with its eigenform expansion we break $\mathcal{M}^\alpha$ in two parts

$$\begin{aligned}\mathcal{H}^\alpha(t, \lambda, x, y) = & \sum_{\substack{\omega \in E_p{}^d \\ \mu_\omega < (N+1-p)^2}} u(t, \lambda-2, \mu_\omega, n) \sinh^{\lambda-3} t\, \omega(x)\, \omega(y) \\ & + \sum_{\substack{\omega \in E_p{}^\delta \cap E_p{}^h \\ \mu_\omega < (N-p)^2}} x(t, \lambda-2, \mu_\omega, n) \sinh^{\lambda-3} t\, \omega(x)\, \omega(y)\end{aligned} \tag{23}$$

and

$$\mathcal{R}^\alpha(t, \lambda, x, y) = \mathcal{M}^\alpha(t, \lambda, x, y) - \mathcal{H}^\alpha(t, \lambda, x, y). \tag{24}$$

In the analogous equation for $\mathcal{M}^\beta$ we use y, v instead of u, x. We remark, that $u(t, \lambda, 0, n) = x(t, \lambda, 0, n)$ and $v(t, \lambda, 0, n) = y(t, \lambda, 0, n)$, therefore it is of no importance if we include $E_p{}^h$ in the first or second sum. Using (23), (24) and Lemma 2, we get

Lemma 3: *For $\lambda \geqq n+3$ we have*

$$\partial/\partial t\, \mathcal{H}^\alpha(t, \lambda+2, x, y) = (\lambda-1) \sinh t\, \mathcal{H}^\alpha(t, \lambda, x, y).$$

The same equation is valid for $\mathcal{R}^\alpha$ instead of $\mathcal{H}^\alpha$.

The following estimates for the eigenforms $\omega \in E_p$ are well known (cf. [5, 11]).

Lemma 4: *We have*

$$\begin{aligned}&\sum_{\omega \in E_p, 0<\mu_\omega<\xi} \|\omega(x)\|^2 \mu_\omega{}^{-\varrho} = O(\xi^{n/2-\varrho}) \quad \text{for } \varrho < n/2, \\ &\sum_{\omega \in E_p, \mu_\omega > \xi} \|\omega(x)\|^2 \mu_\omega{}^{-\varrho} = O(\xi^{n/2-\varrho}) \quad \text{for } \varrho > n/2.\end{aligned} \tag{25}$$

As a consequence of (10), (23) and (25) we deduce for $\lambda > 2n+2$ the inequality $\|\mathcal{M}^\alpha(t, \lambda, x, y) - \mathcal{H}^\alpha(t, \lambda, x, y)\| \leqq c_6 \sinh^{\lambda/2-1} t$ with a constant c_6 not depending on x, y and t. We now want to use the recursion equations of Lemma 1 and 2 and a Landau difference method (cf. [11, 15, 24]) to get information about the asymptotic behaviour of $\mathcal{M}^\alpha$ for $t \to \infty$ for smaller values of λ. We will need the values $\lambda = n+3$, $\lambda = n+5$.

Proposition 5: *The asymptotic behaviour of $\mathcal{M}^\alpha(t, n+3, x, y)$ for $t \to \infty$ is given by $\|\mathcal{R}^\alpha(t, n+3, x, y)\| = O\big((\cosh t)^{n-(n-1)/(n+1)}\big)$.*

Proof: It will turn out to be convenient to use the transformation $\xi = \cosh t$ which we indicate by $^-$, for instance $\bar{\mathcal{M}}^\alpha(\xi, \lambda, x, y) = \mathcal{M}^\alpha(t, \lambda, x, y)$. Then we can write Lemma 2 in the form $(\lambda-1)\, \bar{\mathcal{M}}^\alpha(\xi, \lambda, x, y) = \partial \bar{\mathcal{M}}^\alpha(\xi, \lambda+2, x, y)/\partial \xi$ and the same equation is true for $\bar{\mathcal{H}}^\alpha$, $\bar{\mathcal{R}}^\alpha$ instead of $\bar{\mathcal{M}}^\alpha$. We set $m = n/2$ for even n, $(n+1)/2$ for odd n and get

$$\bar{\mathcal{M}}^\alpha(\xi, \lambda+2m, x, y) = c_7(\lambda, m) \int_1^\xi \int_1^{\eta_m} \dots \int_1^{\eta_2} \bar{\mathcal{M}}^\alpha(\eta_1, \lambda, x, y)\, d\eta_1 \dots d\eta_m$$

with $c_7(\lambda, m) = 2^m \Gamma\big((\lambda - 1 + 2m)/2\big)/\Gamma\big((\lambda - 1)/2\big)$. For a function F from $\mathbf{R}$ into an arbitrary vector space we use the difference operator $\nabla_m{}^a F(\xi) = \sum_{\nu=0}^{m} (-1)^{m-\nu} \binom{m}{\nu} \times F(\xi + \nu\eta)$ with $\eta = \xi^a$, $a \in (0, 1)$. Applying $\nabla_m{}^a$ on $\bar{\mathcal{M}}^a$, we get

$$\nabla_m{}^a \bar{\mathcal{M}}^a(\xi, \lambda + 2m, x, y) = c_7(\lambda, m) \, I_m{}^a \bar{\mathcal{M}}^a(\xi, \lambda, x, y)$$

with

$$I_m{}^a F(\xi) = \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \dots \int_{\eta_2}^{\eta_2+\eta} F(\eta_1) \, d\eta_1 \dots, d\eta_m.$$

Thus

$$\begin{aligned} \bar{\mathcal{M}}^a(\xi, \lambda, x, y) &= c_7(\lambda, m)^{-1} \eta^{-m} \nabla_m{}^a \bar{\mathcal{M}}^a(\xi, \lambda + 2m, x, y) \\ &\quad + \eta^{-m} I_m{}^a\big(\bar{\mathcal{M}}^a(\xi, \lambda, x, y) - \bar{\mathcal{M}}^a(\cdot, \lambda, x, y)\big)(\xi). \end{aligned} \tag{26}$$

This notation means that the integral operator $I_m{}^a$ is taken with respect to the variable indicated by a point in formula (26) and the used argument is written at the end of the formula:

$$\begin{aligned} & I_m{}^a\big(\bar{\mathcal{M}}^a(\xi, \lambda, x, y) - \bar{\mathcal{M}}^a(\cdot, \lambda, x, y)\big)(\xi) \\ &= \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \int_{\eta_2}^{\eta_2+\eta} \big(\bar{\mathcal{M}}^a(\xi, \lambda, x, y) - \bar{\mathcal{M}}^a(\eta_1, \lambda, x, y)\big) \, d\eta_1 \dots d\eta_m. \end{aligned}$$

The equation (26) is also valid for $\bar{\mathcal{H}}^a$ and $\bar{\mathcal{R}}^a$ instead of $\bar{\mathcal{M}}^a$.

Combining these equations, we get

$$\begin{aligned} \bar{\mathcal{R}}^a(\xi, \lambda, x, y) &= c_7(\lambda, m)^{-1} \eta^{-m} \nabla_m{}^a \bar{\mathcal{R}}^a(\xi, \lambda + 2m, x, y) \\ &\quad - \eta^{-m} I_m{}^a\big(\bar{\mathcal{H}}^a(\xi, \lambda, x, y) - \bar{\mathcal{H}}^a(\cdot, \lambda, x, y)\big)(\xi) \\ &\quad + \eta^{-m} I_m{}^a\big(\bar{\mathcal{M}}^a(\xi, \lambda, x, y) - \bar{\mathcal{M}}^a(\cdot, \lambda, x, y)\big)(\xi). \end{aligned}$$

Now we are taking the pointwise norm $\|\cdot\|$ with respect to x and y. The paper [5] tells us how to get $\|\cdot\|$ under the integral sign. For $\lambda = n + 3$ we get

$$\|\bar{\mathcal{R}}^a(\xi, n + 3, x, y)\| \leqq c_8(\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3)$$

with

$$\begin{aligned} \mathcal{T}_1 &= \eta^{-m} \|\nabla_m{}^a \bar{\mathcal{R}}^a(\xi, n + 3 + 2m, x, y)\|, \\ \mathcal{T}_2 &= \eta^{-m} I_m{}^a \|\bar{\mathcal{H}}^a(\xi, n + 3, x, y) - \bar{\mathcal{H}}^a(\cdot, n + 3, x, y)\|(\xi), \\ \mathcal{T}_3 &= \eta^{-m} I_m{}^a \|\bar{\mathcal{M}}^a(\xi, n + 3, x, y) - \bar{\mathcal{M}}^a(\cdot, n + 3, x, y)\|(\xi). \end{aligned} \tag{27}$$

We will continue the proof by estimating $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Estimate of $\mathcal{T}_1$: We break the eigenform expansion of $\bar{\mathcal{R}}^a(\xi, n + 3 + 2m, x, y)$ in four parts with respect to $\omega \in E_p$ ($b > 0$, $N = (n - 1)/2$):

(i)	$\omega \in E_p{}^d$,	$(N + 1 - p)^2 < \mu_\omega \leqq \xi^b$,	(iii)	$\omega \in E_p{}^d$,	$\xi^b < \mu_\omega$,
(ii)	$\omega \in E_p{}^\delta$,	$(N - p)^2 < \mu_\omega \leqq \xi^b$,	(iv)	$\omega \in E_p{}^\delta$,	$\xi^b < \mu_\omega$.

As a consequence of

$$(\lambda - 1) \, u(t, \lambda - 2, \mu, n) \sinh^{\lambda-3} t = \partial\big(u(t, \lambda, \mu, n) \sinh^{\lambda-1} t\big)/\partial\xi,$$

$$(\lambda - 1) \, x(t, \lambda - 2, \mu, n) \sinh^{\lambda-3} t = \partial\big(x(t, \lambda, \mu, n) \sinh^{\lambda-1} t\big)/\partial\xi$$

with $\xi = \cosh t$ and by Lagrange's theorem of differential calculus, we obtain

$$\eta^{-m}\nabla_m{}^a\big(u(t, n+1+2m, \mu, n)\sinh^{n+2m} t\big) = c_9 u(t_\mu{}^*, n+1, \mu, n)\sinh^n t_\mu{}^*,$$

$$\eta^{-m}\nabla_m{}^a\big(x(t, n+1+2m, \mu, n)\sinh^{n+2m} t\big) = c_9 x(t_\mu{}^{**}, n+1, \mu, n)\sinh^n t_\mu{}^{**}$$

with $\xi_\mu{}^* = \cosh t_\mu{}^*$ and $\xi_\mu{}^{**} = \cosh t_\mu{}^{**}$, $\xi \leqq \xi_\mu{}^*$, $\xi_\mu{}^{**} \leqq \xi + m\eta$. Using the above decomposition of $\bar{\mathcal{R}}^a(\xi, n+3+2m, x, y)$, we get

$$\begin{aligned} & c_9{}'\eta^{-m}\,\|\nabla_m{}^a\bar{\mathcal{R}}^a(\xi, n+3+2m, x, y)\| \\ &\leqq \sum_{\substack{\omega\in E_p{}^d\\(N+1-p)^2<\mu_\omega\leqq\xi^b}} \eta^{-m}\left|\nabla_m{}^a\big(u(t, n+1+2m, \mu_\omega, n)\sinh^{n+2m} t\big)\right| \,\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^\delta\\(N-p)^2<\mu_\omega\leqq\xi^b}} \eta^{-m}\left|\nabla_m{}^a\big(x(t, n+1+2m, \mu_\omega, n)\sinh^{n+2m} t\big)\right| \,\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^d\\\xi^b<\mu_\omega}} |u(t^*_{\mu_\omega}, n+1, \mu_\omega, n)\sinh^n t^*_{\mu_\omega}|\,\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^\delta\\\xi^b<\mu_\omega}} |x(t^{**}_{\mu_\omega}, n+1, \mu_\omega, n)\sinh^n t^{**}_{\mu_\omega}|\,\|\omega(x)\|\,\|\omega(y)\|. \end{aligned}$$

We suppose ξ to be large enough, so that $(N-p)^2 < \xi^b$ and $(N+1-p)^2 < \xi^b$. Assuming (8), (10), (11) and Lemma 4, we will have

$$\begin{aligned} & c_{10}\eta^{-m}\,\|\nabla_m{}^a\bar{\mathcal{R}}^a(\xi, n+3+2m, x, y)\| \\ &\leqq \sum_{\substack{\omega\in E_p{}^d\\(N+1-p)^2<\mu_\omega\leqq\xi^b}} \left(\xi^{\frac{n+1}{2}+m(1-a)}\mu_\omega^{-\frac{n+3+2m}{4}} + \xi^{\frac{n+1}{2}+m(1-a)}\mu_\omega^{-\frac{n+1+2m}{4}}\right)\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^\delta\\(N-p)^2<\mu_\omega\leqq\xi^b}} \xi^{\frac{n-1}{2}+m(1-a)}\mu_\omega^{-\frac{n+1+2m}{4}}\,\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^d\\\xi^b<\mu_\omega}} \left(\xi^{\frac{n+1}{2}}\mu_\omega^{-\frac{n+3}{4}} + \xi^{\frac{n+1}{2}}\mu_\omega^{-\frac{n+1}{4}}\right)\|\omega(x)\|\,\|\omega(y)\| \\ &\quad + \sum_{\substack{\omega\in E_p{}^\delta\\\xi^b<\mu_\omega}} \xi^{\frac{n-1}{2}}\mu_\omega^{-\frac{n+1}{4}}\,\|\omega(x)\|\,\|\omega(y)\| \\ &= c_{11}\left(\xi^{\frac{n+1}{2}+m(1-a)+\left(\frac{n}{2}-\frac{n+3+2m}{4}\right)b} + \xi^{\frac{n+1}{2}+m(1-a)+\left(\frac{n}{2}-\frac{n+1+2m}{4}\right)b} + \xi^{\frac{n-1}{2}+\left(\frac{n}{2}-\frac{n+1}{4}\right)b}\right. \\ &\quad \left. + \left(\xi^{\frac{n+1}{2}+\left(\frac{n}{2}-\frac{n+3}{4}\right)b} + \xi^{\frac{n+1}{2}+\left(\frac{n}{2}-\frac{n+1}{4}\right)b}\right) + \xi^{\frac{n-1}{2}+\left(\frac{n}{2}-\frac{n+1}{4}\right)b}\right). \end{aligned}$$

Plugging in $b = 2(1-a)$ gives

$$\mathcal{T}_1 \leqq c_{12}\xi^{n-\frac{1}{2}(n-1)a}.$$

Estimate of $\mathcal{T}_2$: There is only a finite number of eigenforms $\omega \in E_p{}^d$, $E_p{}^\delta$ with eigenvalues μ_ω not exceeding $(N+1-p)^2$, $(N-p)^2$, respectively. These eigenforms are of class C^∞ and so their norms are bounded by a common constant. Using again Lagrange's theorem of differential calculus, we obtain

$$c_{13}\mathcal{T}_2 \leqq \eta \sum_{\substack{\omega\in E_p{}^a \\ \mu_\omega<(N+1-p)^2}} |u(t^*_{\mu_\omega}, n-1, \mu_\omega, n)| \sinh^{n-2} t^*_{\mu_\omega}$$

$$+ \eta \sum_{\substack{\omega\in E_p{}^\delta\cup E_p{}^h \\ \mu_\omega<(N-p)^2}} |x(t^{**}_{\mu_\omega}, n-1, \mu_\omega, n)| \sinh^{n-2} t^{**}_{\mu_\omega}.$$

By (12), (13) we get

$$c_{14}\mathcal{T}_2 \leqq \left(\xi^{n-\frac{n+1}{2}+\frac{n-1}{2}+a} + \xi^{n-1-\frac{n-1}{2}+\frac{n-1}{2}+a}\right) + \xi^{n-2-\frac{n-1}{2}+\frac{n-1}{2}+a} \leqq 3\xi^{n-1-a}.$$

Estimate of $\mathcal{T}_3$: If we use (5), the dualization formulas $**\hat{\sigma}_p = (-1)^n \tau_{n-p}$, $**\hat{\tau}_p = (-1)^n \sigma_{n-p}$ (cf. [6]) and the property $*b^* = b^**$, a short calculation gives $\|b^*\sigma_p(x, by)\| \leqq c_{15}$, $\|b^*\tau_p(x, by)\| \leqq c_{16}$. We have

$$\mathcal{M}^\alpha(t, n+3, x, y) = c_{17} \sum_{\substack{b\in\mathfrak{G} \\ r(x,by)<t}} b^*\big(\sigma_p(x, by) + \cosh r(x, by)\, \tau_p(x, by)\big)$$

with $c_{17} = (-1)^p \pi^{-n/2}\Gamma(n/2+1)$. Thus

$$\bar{\mathcal{M}}^\alpha(\eta_1, n+3, x, y) - \bar{\mathcal{M}}^\alpha(\xi, n+3, x, y)$$
$$= c_{17} \sum_{t_1\leqq r(x,by)<t_2} b^*\big(\sigma_p(x, by) + \cosh r(x, by)\, \tau_p(x, by)\big)$$

with $\xi = \cosh t_1$ and $\eta_1 = \cosh t_2$. Because of the property of $\mathfrak{G}$ to be properly discontinuous, we can estimate the number of lattice points with $t_1 \leqq r(x, by) < t_2$ by c_{18} times the difference of the volumes of $K(x, t_2)$ and $K(x, t_1)$. Consequently we get

$$\|\bar{\mathcal{M}}^\alpha(\xi, n+3, x, y) - \bar{\mathcal{M}}^\alpha(\eta_1, n+3, x, y)\| \leqq c_{19}\, \xi^{n-1+a}$$

for $\xi \leqq \eta_1 \leqq \xi + m\eta$.

This implies by (27)

$$\mathcal{T}_3 \leqq c_{20}\xi^{n-1+a}.$$

Comparing the estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ we see that the best result is reached with $a = 2/(n+1)$. We get

$$\|\bar{\mathcal{R}}^\alpha(\xi, n+3, x, y)\| \leqq c_{21}\, \xi^{n-(n-1)/(n+1)}. \tag{28}$$

Analogously we get

$$\|\bar{\mathcal{R}}^\beta(\xi, n+3, x, y)\| \leqq c_{22}\xi^{n-(n-1)/(n+1)}. \tag{29}$$

As a consequence of Lemma 3 we have

$$\|\bar{\mathcal{R}}^\alpha(\xi, n+5, x, y)\| \leqq c_{23}\xi^{n+1-(n-1)/(n+1)} \tag{30}$$

and the same result for β instead of α ∎

Following (22) we define

$$\mathcal{R}^{\mathcal{S}}(t, x, y) = c^*[-\mathcal{R}^\beta(t, n+5, x, y)$$
$$+ (n+2)\cosh t\, \mathcal{R}^\beta(t, n+3, x, y) - (n+2)\, \mathcal{R}^\alpha(t, n+3, x, y)].$$

with $c^* = \frac{1}{2}(-1)^p \pi^{-n/2}/\Gamma(n/2+2)$, and $\mathcal{H}^{\mathcal{G}} = \mathcal{G} - \mathcal{R}^{\mathcal{G}}$. The equations (28)–(30) lead to

$$\|\mathcal{R}^{\mathcal{G}}(t,x,y)\| \leqq c_{24}(\cosh t)^{n+1-\frac{n-1}{n+1}}, \quad \|\mathcal{R}^{\mathcal{G}}(t,x,y)\| = O\left(e^{\left(n+1-\frac{n-1}{n+1}\right)t}\right). \tag{31}$$

So we are able to estimate the remainder $\mathcal{R}^{\mathcal{G}}$ of $\mathcal{G}$. We are now interested in the order of magnitude of $\mathcal{H}^{\mathcal{G}}$. A direct calculation in accordance with (11) and (23) gives

$$\begin{aligned}(1/c_5)\,\mathcal{H}^{\mathcal{G}}(t,x,y) = \sum_{\substack{\omega\in E_p{}^\delta \\ \mu_\omega \leqq (N-p)^2}} \Big[-\frac{\mu_\omega + (n-p)(p+2)}{n+4} x(t,n+5,\mu_\omega,n)\sinh^2 t \\ + (n-p)\cosh t\, x(t,n+3,\mu_\omega,n)\Big]\sinh^{n+2} t\,\omega(x)\,\omega(y) \\ + \sum_{\substack{\omega\in E_p{}^d\cup E_p{}^h \\ \mu_\omega \leqq (N+1-p)^2}} (n-p)\, y(t,n+3,\mu_\omega,n)\sinh^{n+2} t\,\omega(x)\,\omega(y).\end{aligned} \tag{32}$$

By (12) we have for $t \to \infty$ ($\mu < (N-p)^2$)

$$x(t,\lambda,\mu,n) = \frac{\Gamma((\lambda+1)/2)\,\Gamma(\chi_x(\mu))\,2^{\lambda-1}}{\Gamma(\lambda/2+\chi_x(\mu))\,\Gamma(1/2)} \exp\left((-\lambda/2+\chi_x(\mu))t\right) + O\left(\exp(r_x t)\right) \tag{33}$$

with

$$\chi_x(\mu) = \sqrt{(N-p)^2-\mu}, \qquad r_x = \begin{cases} -\lambda/2 + \chi_x(\mu) - 1 & \text{for } \chi_x(\mu) \geqq 1/2, \\ -\lambda/2 - \chi_x(\mu) & \text{for } \chi_x(\mu) < 1/2 \end{cases}$$

and the same equation with y

$$\chi_y(\mu) = \sqrt{(N+1-p)^2-\mu}, \quad r_y = \begin{cases} -\lambda/2 + \chi_y(\mu) - 1 & \text{for } \chi_y(\mu) \geqq 1/2, \\ -\lambda/2 - \chi_y(\mu) & \text{for } \chi_y(\mu) < 1/2 \end{cases}$$

instead of x, $\chi_x(\mu)$ and r_x (valid for $\mu < (N+1-p)^2$). This gives

$$\|\mathcal{H}^{\mathcal{G}}(t,x,y)\| = O\left(\exp\left((n+1-(n-1)/(n+1))\,t\right)\right) \tag{34}$$

for $p \neq 0$ and $p \neq n-1$. This is the same order of magnitude as for $\mathcal{R}^{\mathcal{G}}$. Next we consider $p=0$ and $p=n-1$. We have $E_0{}^d = 0$. After short calculations (32) and (33) gives

$$\begin{aligned}&\mathcal{H}^{\mathcal{G}}(t,x,y) \\ &= \sum_{\substack{\omega\in E_p{}^\delta\cup E_p{}^h \\ \mu_\omega < n\left(\frac{n-1}{n+1}\right)^2}} c_{25}\exp\left(\left(\sqrt{N^2-\mu_\omega}+N+2\right)t\right)\omega(x)\,\omega(y) + \mathcal{H}_*{}^{\mathcal{G}}(t,x,y)\end{aligned} \tag{35}$$

with

$$\mathcal{H}_*{}^{\mathcal{G}}(t,x,y) = O\left(\exp\left((n+1-(n-1)/(n+1))\,t\right)\right),$$

$$c_{25} = \frac{\Gamma(N+5/2)\,\Gamma\left(\sqrt{N^2-\mu_\omega}\right)}{\Gamma\left(\sqrt{N^2-\mu_\omega}+N+3\right)\Gamma(1/2)}\, c_{26},$$

$$c_{26} = \begin{cases} \frac{1}{2}\left(-\mu_\omega + n\sqrt{N^2-\mu_\omega} + N\right) & \text{for } p = 0, \\ -\mu_\omega - N + \sqrt{N^2-\mu_\omega} & \text{for } p = n-1. \end{cases}$$

We set $\mathcal{H}_*^{\mathfrak{s}} = \mathcal{H}^{\mathfrak{s}}$ for $p \neq 0, n-1$ and $\mathcal{H}_0^{\mathfrak{s}} = \mathcal{H}^{*\mathfrak{s}} - \mathcal{H}^{\mathfrak{s}}$, $\mathcal{R}_0^{\mathfrak{s}} = \mathcal{H}^{*\mathfrak{s}} + \mathcal{R}^{\mathfrak{s}}$ for $p = 0, \ldots, n$. We split S^σ in

$$\mathcal{H}^\sigma(t, x, y) = 2 \int_0^t \cosh \tau \sinh^{-3} \tau\, \mathcal{H}_0^{\mathfrak{s}}(\tau, x, y)\, d\tau + \sinh^{-2} t\, \mathcal{H}_0^{\mathfrak{s}}(t, x, y)$$

and

$$\mathcal{R}^\sigma(t, x, y) = 2 \int_0^t \cosh \tau \sinh^{-3} \tau\, \mathcal{R}_0^{\mathfrak{s}}(\tau, x, y)\, d\tau + \sinh^{-2} t\, \mathcal{R}_0^{\mathfrak{s}}(t, x, y).$$

Now the theorem follows from (21), (31), (34) and (35) after short calculations ∎

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Manuskripteingang: 20. 09. 1988; in revidierter Fassung 11. 07. 1989

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