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On the Interior Regularity of Weak Solutions to Nonlinear Elliptic Systems of Second Order

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Es wird die $C^{1,\alpha}$-Regularität der schwachen Lösung (mit dem Gradienten im BMO-Raum) eines nichtlinearen elliptischen Systems partieller Differentialgleichungen zweiter Ordnung untersucht. Das Problem ist unter der Voraussetzung lösbar, daß das System die verallgemeinerte Liouvillesche Bedingung im BMO-Raum statt wie gewöhnlich im L^∞-Raum erfüllt. Zum Schluß wird gezeigt, daß die Liouvillesche Bedingung im Fall des $\mathbf{R}^2$ gilt.

Исследуется $C^{1,\alpha}$-регулярность слабого решения (с градиентом в BMO-пространстве) нелинейной эллиптической системы дифференциальных уравнений второго порядка. Проблема положительно разрешима в предположении, что система удовлетворяет обобщенному условию Лиувилля в BMO-пространстве вместо как обычно в L^∞-пространстве. В конце доказано, что условие Лиувилля выпольнено в случае $\mathbf{R}^2$.

The interior $C^{1,\alpha}$-regularity for a weak solution (with gradient in the BMO-space) of a nonlinear second order elliptic system is investigated. The positive answer is obtained on the assumption that the elliptic system satisfy the generalized Liouville condition considered in the BMO-space instead of the usually used L^∞-space. Finally it is proved that the Liouville condition holds in the case of $\mathbf{R}^2$.

0. Introduction

In this paper, which is a modified version of the thesis [4], we prove regularity for a weak solution (with gradient in the BMO-space) of the following nonlinear elliptic system ($i = 1, \ldots, N$):

$$-D_\alpha a_i^\alpha(x, u, Du) + a_i(x, u, Du) = -D_\alpha f_i^\alpha(x) + f_i(x), \tag{0.1}$$

where x belongs to a bounded open set Ω of $\mathbf{R}^n$, $n \geqq 3$, $u: \Omega \to \mathbf{R}^N$, $N > 1$, $u(x) = \big(u^1(x), \ldots, u^N(x)\big)$ is a vector-valued function, $Du = (D_1u, \ldots, D_nu)$, $D_\alpha = \partial/\partial x_\alpha$; we will use the summation convention over repeated indices.

In [6–9, 12] the so-called Liouville condition (L) is formulated in terms of the space L^∞. On the other hand, the proof of L^∞-boundedness of the gradient of a weak solution for the system (0.1) has not yet been achieved in reasonably wide extent and the possibility of this proof is questionable.

The following definition is a generalized form of the Liouville property from [7, 8] and reads as follows.

Definition 0.1: The system (0.1) satisfies the *Liouville property* (L) if for every $x^0 \in \Omega$ and every $u \in \mathbf{R}^N$ the only solutions v in $\mathbf{R}^n$ to

$$-D_\alpha a_i^\alpha\big(x^0, u, Dv(x)\big) = 0, \qquad (i = 1, \ldots, N) \tag{0.2}$$

with $Dv \in \mathrm{BMO}(\mathbf{R}^n)$ are polynomials of at most first degree.

The main result of this paper is the fact that if system (0.1) has property (L), then Du is locally Hölder continuous in Ω. To this effect it represents a generalization of [7, 8]. Because it is easier to verify that the gradient of the solution is an element of the BMO-space ($L^\infty \subsetneqq \mathrm{BMO}$), the generalization reached in this paper has a fundamental meaning. The approach stated in this paper has been used in [15], which deals with quasilinear parabolic systems.

1. Notations and definitions

In the sequel Ω will be a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary $\partial\Omega$. The meaning of $\Omega_0 \subset\subset \Omega$ is that the closure of Ω_0 is contained in Ω, i.e. $\bar{\Omega}_0 \subset \Omega$. For the sake of simplification we denote by $|\cdot|$ and $(\cdot, \cdot)$ the norm and scalar product in $\mathbf{R}^n$ as well as in $\mathbf{R}^N$ and $\mathbf{R}^{nN}$. If $x \in \mathbf{R}^n$ and r is a positive real number, we set $B(x, r) = \{y \in \mathbf{R}^n : |y - x| < r\}$, $\Omega(x, r) = \Omega \cap B(x, r)$ and $Q(x, r)$ will be the cube in $\mathbf{R}^n$ with the center in the point x and length of the side r.

By $\mathcal{P}_k$, $k \geqq 0$ integer, we denote the set of all vector-valued polynomials $P = (P^1, \ldots, P^N)$ with real coefficients defined on $\mathbf{R}^n$ such that the degree of P^i is less than k for each $i = 1, \ldots, N$.

Beside the usually used Hölder and Sobolev spaces (for detailed information see, e.g., [3, 6, 12]) we will use the following ones.

Definition 1.1 (*Campanato-Morrey spaces*): Let $\lambda \in [0, n]$, $p \in [1, \infty)$. The space $L^{p,\lambda}(\Omega)$ is the subspace of such functions $f \in L^p(\Omega)$ for which

$$\|f\|_{L^{p,\lambda}(\Omega)} = \left\{\sup_{x\in\bar{\Omega}, r>0} r^{-\lambda} \int_{\Omega(x,r)} |f(y)|^p \, dy\right\}^{1/p} < \infty. \tag{1.1}$$

Let k be a non-negative integer and $\lambda \in [0, n + (k + 1)\, p]$. The space $\mathcal{L}_k^{p,\lambda}(\Omega)$ is the subspace of such functions $f \in L^p(\Omega)$ for which

$$\|f\|_{\mathcal{L}_k^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{\mathcal{L}_k^{p,\lambda}(\Omega)} < \infty, \tag{1.2}$$

where

$$[f]_{\mathcal{L}_k^{p,\lambda}(\Omega)} = \left\{\sup_{x\in\bar{\Omega}, r>0} \left[r^{-\lambda} \inf_{P\in\mathcal{P}_k} \int_{\Omega(x,r)} |f(y) - P(y)|^p \, dy\right]\right\}^{1/p}.$$

With the norms (1.1) and (1.2), $L^{p,\lambda}(\Omega)$ and $\mathcal{L}_k^{p,\lambda}(\Omega)$ are Banach spaces. We will work mainly with the spaces $L^{2,\lambda}$, $\mathcal{L}_0^{2,\lambda}$ and $\mathcal{L}_1^{2,\lambda}$; instead of $\mathcal{L}_0^{2,\lambda}$ we will usually write $\mathcal{L}^{2,\lambda}$.

In our considerations we make use of the fact that for each function $u \in \mathcal{L}_k^{2,\lambda}(\Omega)$, each $x^0 \in \Omega$, $0 < r \leqq \operatorname{diam} \Omega$, there exists one and only one polynomial $P \in \mathcal{P}_k$, $P(x) = P(x, x^0, r, u)$ such that

$$\inf_{P\in\mathcal{P}_k} \int_{\Omega(x^0,r)} |u(x) - P(x)|^2 \, dx = \int_{\Omega(x^0,r)} |u(x) - P(x, x^0, r, u)|^2 \, dx.$$

For $k = 1$ we will write this polynomial P in the form

$$\begin{aligned} P(x, x^0, r, u) &= b^0(x^0, r, u) + \sum_{\alpha=1}^{n} b^\alpha(x^0, r, u)\,(x_\alpha - x_\alpha^0) \\ &= b^0(x^0, r, u) + \big(b(x^0, r, u), (x - x^0)\big), \end{aligned} \tag{1.3}$$

and for $k = 0$ it equals the constant

$$u_{x^0,r} = \fint_{B(x^0,r)} u(y)\, dy = \big(\operatorname{meas} B(x^0, r)\big)^{-1} \int_{B(x^0,r)} u(y)\, dy,$$

where meas $B(x^0, r)$ means the n-dimensional Lebesgue measure. Denote further $U(x^0, r) = \fint_{B(x^0,r)} |u(y) - u_{x,r}|^2 \, dy$, and define $\mathrm{BMO}(\mathbf{R}^n)$ as the set of all measurable functions u on $\mathbf{R}^n$ for which the set $\mathcal{U} = \{U(x, r) : x \in \mathbf{R}^n, r > 0\}$ is bounded, setting $\|u\|_{\mathrm{BMO}(\mathbf{R}^n)} = \sup \mathcal{U}$.

At last, let $H^{1,(\lambda)}(\Omega)$, $\lambda \in [0, n]$ be the Banach space of all functions $u \in H^1(\Omega)$, $D_\alpha u \in \mathcal{L}^{2,\lambda}(\Omega)$ with norm

$$\|u\|_{H^{1,(\lambda)}(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{\alpha=1}^{n} \|D_\alpha u\|_{\mathcal{L}^{2,\lambda}(\Omega)}.$$

Proposition 1.1: *We have the following important properties of the spaces defined above:*

(a) $L^{2,\lambda}(\Omega) = \mathcal{L}^{2,\lambda}(\Omega)$, $\lambda \in [0, n)$,

(b) $\mathcal{L}^{2,\lambda}(\Omega) = \mathcal{L}_1^{2,\lambda}(\Omega)$, $\lambda \in [0, n + 2)$,

(c) $\mathcal{L}^{2,n}(\Omega) \subset L^{2,\lambda_1}(\Omega) \subset L^{2,\lambda_2}(\Omega)$, $0 \leq \lambda_2 < \lambda_1 < n$,

(d) $L^{2,n}(\Omega) = L^\infty(\Omega) \subsetneqq \mathcal{L}^{2,n}(\Omega)$,

(e) $\mathcal{L}^{p,n}(\Omega) = \mathcal{L}^{s,n}(\Omega) = \mathrm{BMO}(\Omega)$ *for all* $p, s \in [1, \infty)$, Ω *being a cube,*

(f) $H^{1,(n)}(\Omega) \subset C^{0,\gamma}(\Omega_0)$ *for each* $\Omega_0 \subset\subset \Omega$, $\gamma \in (0, 1)$ *and*

$$\|\cdot\|_{C^{0,\gamma}(\Omega_0)} \leq c(n, \gamma, \operatorname{diam} \Omega, \operatorname{dist}(\Omega_0, \partial\Omega)) \|\cdot\|_{H^{1,(n)}(\Omega)}.$$

For the proofs and more detailed information about the Campanato-Morrey spaces see, e.g., [1–3, 6, 12]. In the sequel we will denote all important constants by the symbol $\mathscr{C}$ and other ones by c.

A function $u \in H^1(\Omega)$ is called *weak solution* of (0.1) in Ω if

$$\int_\Omega a_i^\alpha(x, u, Du) \, D_\alpha\varphi^i(x) \, dx + \int_\Omega a_i(x, u, Du) \, \varphi^i(x) \, dx$$
$$= \int_\Omega f_i^\alpha(x) \, D_\alpha\varphi^i(x) \, dx + \int_\Omega f_i(x) \, \varphi^i(x) \, dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega), \tag{1.4}$$

where $a_i^\alpha, a_i, f_i^\alpha, f_i$ are functions fulfilling for each $(x, u, p) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}$ with $|u| \leq L$ the following conditions:

$$a_i^\alpha, a_i \in C^1(\Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}), \tag{1.5}$$

$$|a_i^\alpha(x, u, p)|, |a_i(x, u, p)| \leq \mathscr{C}_1(L) \, (1 + |p|), \tag{1.6}$$

$$|\partial a_i^\alpha(x, u, p)/\partial p_j^\beta|, |\partial a_i(x, u, p)/\partial p_j^\beta| \leq \mathscr{C}_1(L), \tag{1.7}$$

$$\left.\begin{array}{l} |\partial a_i^\alpha(x, u, p)/\partial u_k|, |\partial a_i^\alpha(x, u, p)/\partial x_s| \\ |\partial a_i(x, u, p)/\partial u_k|, |\partial a_i(x, u, p)/\partial x_s| \end{array}\right\} \leq \mathscr{C}_1(L) \, (1 + |p|), \tag{1.8}$$

$$\partial a_i^\alpha(x, u, p)/\partial p_j^\beta \text{ is uniformly continuous on } \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}, \tag{1.9}$$

$$\partial a_i^\alpha(x, u, p)/\partial p_j^\beta \to d_{ij}^{\alpha\beta}(x, u) \quad \text{as} \quad |p| \to \infty, \quad \text{for all } (x, u) \in \Omega \times \mathbf{R}^N \tag{1.10}$$

$$f_i^\alpha \in H^{1,q}(\Omega), \qquad f_i \in H^{1,q/2}(\Omega), \qquad q > n, \tag{1.11}$$

$$\sum \|f_i^\alpha\|_{H^{1,q}(\Omega)} + \sum \|f_i\|_{H^{1,q/2}(\Omega)} \leq \mathscr{C}_2, \tag{1.12}$$

$$\partial a_i^\alpha(x, u, p)/\partial p_j^\beta \, \eta_\alpha^i \eta_\beta^j \geq \nu(L) \, |\eta|^2$$
$$\text{for all } \eta \in \mathbf{R}^{nN}, (x, u, p) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}. \tag{1.13}$$

It is known that $u \in H^2_{\mathrm{loc}}(\Omega)$ if the function u fulfiles the conditions stated above (see, e.g., [3]).

2. The results

Principal result of this paper is the following theorem.

Theorem 2.1: *Let $u \in H^{1,(n)}(\Omega)$ be a weak solution of the system* (0.1) *and suppose that the conditions* (1.5)—(1.13) *hold. If the system* (0.1) *has the Liouville property* (L), *then $u \in C^{1,1-n/q}_{\text{loc}}(\Omega)$.*

There arise two natural questions:

1. Do there exist systems of the form (0.1) with weak solutions in the space $H^{1,(n)}(\Omega)$?
2. Under which assumptions has the system of the form (0.1) the Liouville property (L)?

A partial answer on the first question is given in [5]. The problem of the $H^{1,(\lambda)}$-regularity of weak solutions is studied in detail in [3]. The second question is positively answered in the case of $n = 2$ and $N > 1$ by the following

Proposition 2.2: *Let the system* (0.1) *satisfy conditions* (1.5)—(1.8), (1.11)—(1.13) *and let $n = 2$. Then it has property* (L).

In the case $n \geqq 3$, $N > 1$ some conditions under which linear elliptic systems with L^∞-coefficients, quasilinear or nonlinear systems, respectively, have property (L) are shown in [11], [13] and [10], respectively. From [14] it follows that there are nonlinear elliptic systems without property (L).

3. Lemmas

The following two lemmas concern the estimate of the coefficients of the polynomials from (1.5).

Lemma 3.1 [1: pp. 140—144]: *Let $P \in \mathcal{P}_k$, $s \in [1, \infty)$ and E be a measurable subset of the ball $B(x^0, r) \subset \mathbf{R}^n$ satisfying the condition* meas $E \geqq Ar^n$, *A a positive constant. Then there is a constant $c = c(n, k, s, A)$ such that for each multiindex α we have*

$$|[D_\alpha P(x)]_{x=x_0}|^s \leqq (c/r^{n+|\alpha|s}) \int_E |P(x)|^s \, dx.$$

Lemma 3.2 [1: pp. 146]: *Let $u \in \mathcal{L}_1^{2,n+2}(\Omega)$. Then there exists a constant $c = c(n)$ such that for every $x \in \Omega$ and for all r, r_0, $0 < r \leqq r_0 \leqq$* diam Ω, *we have*

$$|b^0(x, r_0) - b^0(x, r)| \leqq cr_0[u]_{\mathcal{L}_1^{2,n+2}(\Omega)},$$

$$|b^\alpha(x, r_0) - b^\alpha(x, r)| \leqq c\big(1 + \ln (r_0/r)\big) [u]_{\mathcal{L}_1^{2,n+2}(\Omega)}$$

for all $\alpha = 1, \ldots, n$, where b^0, b^α are defined in (1.3).

Another important result needed for the proof of Theorem 2.1 is the following

Proposition 3.3 [2: pp. 373]: *Let Ω be convex. Then there is a constant $c = c(n$,* diam Ω, meas $\Omega)$ *such that for each $\lambda \in [0, n+2]$ we have*

$$H^{1,(\lambda)}(\Omega) \subset \mathcal{L}_1^{2,\lambda+2}(\Omega),$$

$$\|u\|_{\mathcal{L}_1^{2,\lambda+2}(\Omega)} \leqq c \, \|u\|_{H^{1,(\lambda)}(\Omega)} \quad \text{for all } u \in H^{1,(\lambda)}(\Omega).$$

Now we present a fundamental result concerning the partial regularity of weak solutions to the quasilinear elliptic systems of the type

$$D_\alpha[A_{ij}^{\alpha\beta}(x, u)\, D_\beta u^j] + A_{ij}^\beta(x, u)\, D_\beta u^j = -D_\alpha g_i^\alpha + g_i. \tag{3.1}$$

Assume that the coefficients $A_{ij}^{\alpha\beta}$ are uniformly continuous, A_{ij}^β are continuous in $\Omega \times \mathbf{R}^N$, $g_i^\alpha \in L^q(\Omega)$, $g_i \in L^{q/2}(\Omega)$, $q > n$ and that ($c, \mu > 0$ constants)

$$\sum_{i,j,\alpha,\beta} |A_{ij}^{\alpha\beta}| + \sum_{i,j,\beta} |A_{ij}^\beta| + \sum_{i,\alpha} \|g_i^\alpha\|_{L^q} + \sum_i \|g_i\|_{L^{q/2}} \leqq c,$$

$$A_{ij}^{\alpha\beta}(x, u)\, \xi_\alpha^i \xi_\beta^j \geqq \mu\, |\xi|^2 \qquad \text{for all } (x, u) \in \Omega \times \mathbf{R}^N, \quad \xi \in \mathbf{R}^{nN}.$$

Consider the solutions to the system (3.1) belonging to the space $H^1 \cap \mathcal{L}^{2,n}(\Omega)$.

Proposition 3.4 [12: pp. 147—149]: *Let u be a weak solution of the system* (3.1). *Suppose that $U(x, r) \to 0$ as $r \to 0+$ uniformly in each compact set $K \subset \Omega$. Then $u \in C_{\mathrm{loc}}^{0,\alpha}(\Omega)$ with $\alpha = 1 - n/q$ and the a-priori estimate $\|u\|_{C^{0,\alpha}(K)} \leqq c_1(\mu, c, K, \operatorname{dist}(K, \partial\Omega))$ holds.*

4. Proof of the results

Let $\Omega_0 \subset\subset \Omega$, $x^0 \in \Omega_0$ be fixed, $R_0 = \min\{1, \operatorname{dist}(\Omega_0, \partial\Omega)\}$. For $R \in (0, R_0)$ and $u \in H^{1,(n)}(\Omega)$ (u is a weak solution of the system (0.1)) we define

$$y = y(x) = (x - x^0)/R, \tag{4.1}$$

$$u_R(y) = \big(u(x^0 + Ry) - b^0(x^0, R) - R(b(x^0, R), y)\big)/R, \tag{4.2}$$

where $b^0(x^0, R) = b^0(x^0, R, u) \in \mathbf{R}^N$ and $b(x^0, R) = b(x^0, R, u) \in \mathbf{R}^{nN}$ are the coefficients of the polynomial $P(x, x^0, R, u)$ from (1.3) since $u \in \mathcal{L}_1^{2,n+2}(B(x^0, R))$ for each $B(x^0, R) \subset \Omega$ due to Proposition 3.3. From (4.1) it can be seen that for each $a > 0$ there exists $R(a) \in (0, R_0]$ such that for all $R \in (0, R(a))$ we have $B(0, 2a\sqrt{n}) \subset O_R$ (O_R is the image of Ω through the transformation (4.1)). From (4.2) it follows that there exists a constant $c > 0$ such that for each $r > 0$, $y^0 \in \mathbf{R}^n$ and all $R \in (0, R(y^0))$ ($R(y^0) = R_0$ in the case $y^0 = 0$) we have

$$\int_{B(y^0,r)} |Du_R(y) - (Du_R)_{y^0,r}|^2\, dy \leqq c[Du]_{\mathcal{L}^{2,n}(\Omega)}\, r^n \tag{4.3}$$

and the equation (1.4) has the following form:

$$\int_{O_R} a_i^\alpha\big(x^0 + Ry, b^0(x^0, R) + Ru_R(y) + R(b(x^0, R), y), b(x^0, R) + Du_R(y)\big)\, D_\alpha\psi^i(y)\, dy$$

$$+ \int_{O_R} Ra_i\big(x^0 + Ry, b^0(x^0, R) + Ru_R(y) + R(b(x^0, R), y), b(x^0, R) + Du_R(y)\big)\, \psi^i(y)\, dy$$

$$= \int_{O_R} f_i^\alpha(x^0 + Ry)\, D_\alpha\psi^i(y)\, dy + \int_{O_R} Rf_i(x^0 + Ry)\, \psi^i(y)\, dy \qquad \text{for all } \psi \in C_0^\infty(O_R). \tag{4.4}$$

As previously said, $u \in H_{\mathrm{loc}}^2(\Omega)$ and with respect to (4.2) also $u_R \in H_{\mathrm{loc}}^2(O_R)$. Then it follows that $v_R = D_\gamma u_R$ satisfies the equation in variations

$$\int_{O_R} (\partial a_i^\alpha/\partial p_j^\beta D_\beta v_R^j + R\partial a_i^\alpha/\partial u^k (b_k^\gamma + v_R^k) + R\partial a_i^\alpha/\partial x_\gamma)\, D_\alpha\psi^i\, dy$$

$$+ \int_{O_R} (R\partial a_i/\partial p_j^\beta D_\beta v_R^j + R^2\partial a_i/\partial u^k\, (b_k^\gamma + v_R^k) + R^2\partial a_i/\partial x_\gamma)\, \psi^i\, dy$$

$$= \int_{O_R} (R\partial f_i/\partial x_\gamma D_\alpha\psi^i + R^2\partial f_i/\partial x_\gamma)\, \psi^i\, dy \qquad \text{for all } \psi \in C_0^\infty(O_R). \tag{4.5}$$

In what follows we are going to prove that for each $a > 0$ the set $\mathcal{M}_0 = \{u_R : 0 < R < R(a)\}$ is bounded in $H^2(B(0, a))$ by a constant depending only on a. For this reason it is enough to prove the boundedness of sets $\mathcal{M}_0$ and $\mathcal{M}_2 = \{D^2u_R : 0 < R < R(a)\}$ in $L^2(B(0, a))$. The set $\mathcal{M}_1 = \{Du_R : 0 < R < R(a)\}$ is then bounded according to the Gagliardo-Nirenberg Theorem (see, e.g., [3: pp. 25]).

First, let us prove the boundedness of $\mathcal{M}_2$. For $a > 0$ denote $B(a) = B(0, a)$. Further choose $\eta \in C_0^\infty(B(2a))$ such that $0 \leqq \eta \leqq 1$, $\eta = 1$ on $B(a)$ and $|D\eta| \leqq c/a$. Substituting for ψ in the equation (4.5) the function $\psi(y) = \eta^2[v_R(y) - (v_R)_{0,2a}]$, we have for each $\varepsilon > 0$ from the assumptions (1.7), (1.8), (1.11)–(1.13), Young's inequality, Proposition 1.1 and properties of the function η that

$$\begin{aligned}
&\nu(L) \int_{B(0,2a)} \eta^2 |Dv_R|^2 dy \\
&\leqq \varepsilon c(L) \int_{B(0,2a)} \eta^2 |Dv_R|^2 dy + c(\varepsilon, L)\, a^{-2} \int_{B(0,2a)} |v_R - (v_R)_{0,2a}|^2 dy \\
&\quad + c(\varepsilon, L) \Big\{ R^2(1 + |b(x^0, R)|^2) \int_{B(0,2a)} |Du_R|^2 dy + R^2 \int_{B(0,2a)} |Du_R|^4 dy \\
&\quad + R^2(1 + |b(x^0, R)|^2 + |b(x^0, R)|^4)\, a^n + R^2 \int_{B(0,2a)} |D\tilde{\tilde{f}}|^2 dy + R^4 \int_{B(0,2a)} |D\tilde{f}|^2 dy \Big\}, \qquad (4.6)
\end{aligned}$$

here $\tilde{\tilde{f}} = (f_i^\alpha)$, $\tilde{f} = (f_i)$, $L = L(\operatorname{dist}(\Omega_0, \partial\Omega), \operatorname{diam}\Omega, \|u\|_{H^{1,(n)}(\Omega)})$; in the case $q < 4$ it is necessary to replace the last integral in (4.6) by $R^q \int |Df|^{q/2} dy$. Choosing $\varepsilon > 0$ in (4.6) small enough, we obtain

$$\begin{aligned}
&\int_{B(0,a)} |Dv_R|^2 dy \\
&\leqq c(L) \Big\{ a^{-2} \int_{B(0,2a)} |v_R - (v_R)_{0,2a}|^2 dy + R^2(1 + |b(x^0, R)|^2) \int_{B(0,2a)} |Du_R|^2 dy \\
&\quad + R^2 \int_{B(0,2a)} |Du_R|^4 dy + R^2(1 + |b(x^0, R)|^2 + |b(x^0, R)|^4)\, a^n \\
&\quad + R^2 \int_{B(0,2a)} |D\tilde{\tilde{f}}|^2 dy + R^4 \int_{B(0,2a)} |D\tilde{f}|^2 dy \Big\} \\
&= c(L)\{A + B + C + D + E + F\}.
\end{aligned}$$

Estimate now the individual terms in brackets. Since $Du \in \mathcal{L}^{2,n}(\Omega)$, we have

$$A = a^{-2}R^{-n} \int_{B(x^0,2aR)} |\partial u/\partial x_\gamma - (\partial u/\partial x_\gamma)_{x^0,2aR}|^2 dx \leqq c[Du]_{\mathcal{L}^{2,n}(\Omega)}\, a^{n-2}.$$

Further from Lemma 3.1, Lemma 3.2 and the fact that $Du \in L^{2,\lambda}(\Omega)$ for each $\lambda \in [0, n)$ (according to Proposition 1.1/(c)) we obtain

$$\begin{aligned}
B &= (1 + |b(x^0, R)|^2)\, R^{-n+2} \int_{B(x^0,2aR)} |Du - b(x^0, R)|^2 dx \\
&\leqq c[R^{\lambda+2-n}(1 + |b(x^0, R)|^2)\, a^\lambda + R^2(|b(x^0, R)|^2 + |b(x^0, R)|^4)]\, a^n \\
&\leqq c(\lambda, R_0)\,(1 + \ln^4 R)\, R^{\lambda+2-n}(a^\lambda + a^n)\, \|u\|_{H^{1,(n)}(\Omega)} \\
&\leqq c(\lambda, R_0, \|u\|_{H^{1,(n)}(\Omega)})\,(a^\lambda + a^n),
\end{aligned}$$

where $\lambda \in (n-2, n)$ is arbitrary. In estimating the term C we use the fact that $Du \in L^{s,\mu}(Q)$ for each cube $Q \subset\subset \Omega$, $s \in [1, \infty)$, $\mu \in [0, n)$ (see Proposition 1.1/(c)) and we proceed analogously as in the estimation of term B and obtain $C \leqq c(\lambda, R_0, \|u\|_{H^{(1,n)}(\Omega)})\, a^\lambda$, where $\lambda \in (n-2, n)$ is arbitrary. From Lemma 3.2 it follows that $D \leqq c(R_0)\, a^n$ and from the assumptions (1.11), (1.12) we have $E \leqq c(R_0, \mathscr{C}_2)\, a^{n(1-2/q)}$, $F \leqq c(R_0, \mathscr{C}_2)\, a^{n(1-4/q)}$ in case $q > 4$ and $F \leqq c(R_0, \mathscr{C}_2)$ in case $q \leqq 4$. From these estimates it then follows

$$\int_{B(0,a)} |Dv_R|^2\, dy \leqq c(\mathscr{C}_2, R_0, \operatorname{diam} \Omega, \|u\|_{H^{1,(n)}(\Omega)}, a) \leqq c(a)$$

for each $R \in (0, R(a))$. Hence $\int_{B(0,a)} |D^2 u_R|^2\, dy \leqq c(a)$ for any $R \in (0, R(a))$ and the boundedness of the set $\mathscr{M}_2$ is proved.

Now we are going to prove the boundedness of $\mathscr{M}_0$. From Lemma 3.2, Proposition 3.3 and (4.1), (4.2) we have

$$\begin{aligned}
\int_{B(0,a)} |u_R(y)|^2\, dy &= R^{-n-2} \int_{B(x^0,aR)} \left|u(x) - b^0(x^0, R) - \big(b(x^0, R), (x - x^0)\big)\right|^2 dx \\
&\leqq 2a^{n+2}(aR)^{-n-2} \int_{B(x^0,aR)} \left|u(x) - b^0(x^0, aR) - \big(b(x^0, aR), (x - x^0)\big)\right|^2 dx \\
&\quad + 2R^{-n-2} \int_{B(x^0,aR)} |b^0(x^0, aR) - b^0(x^0, R) \\
&\quad + \big(\big(b(x^0, aR) - b(x^0, R)\big), (x - x^0)\big)\big|^2\, dx \\
&\leqq c[u]_{\mathscr{L}^{2,n+2}(B(x_0,aR))}\,(1 + \ln^2 a) \max\{a^n, a^{n+2}\} \leqq [Du]_{\mathscr{L}^{2,n}(\Omega)}\, c(a).
\end{aligned}$$

Hence $\int_{B(0,a)} |u_R(y)|^2\, dy \leqq c(a)$ for any $R \in (0, R(a))$ and the boundedness of $\mathscr{M}_0$ in $H^2(B(0, a))$ is proved.

Compactness of the imbedding of $H^2(B(0, a))$ into $H^1(B(0, a))$ allows us to choose a sequence $R_k \to 0$ such that $u_{R_k} \to z$ in $H^1(B(0, a))$. Using the diagonal process we get a subsequence (we use the same notation for it) such that

$$\lim_{k\to\infty} u_{R_k} = z \text{ in } H^1_{\mathrm{loc}}(\mathbf{R}^n), \qquad \lim_{k\to\infty} Du_{R_k} = Dz \quad \text{a.e. in } \mathbf{R}^n. \tag{4.7}$$

According to (4.3) we obtain that there exists a constant $c > 0$ such that for each $y^0 \in \mathbf{R}^n$, $r > 0$ there holds

$$\int_{B(y^0,r)} |Dz(y) - (Dz)_{y^0,r}|^2\, dy \leqq c[Du]_{\mathscr{L}^{2,n}(\Omega)}\, r^n. \tag{4.8}$$

Further we deduce from (4.4) the equation for the limit function z. For passing to the limit in equation (4.4) the behaviour of $\sup\{b(x^0, R_k);\ k = 1, 2, \ldots\}$ is important. Remember for the following considerations that $Rb(x^0, R) \to 0$, $b^0(x^0, R) \to B^0 \in \mathbf{R}^N$ as $R \to 0+$ exist due to Lemma 3.2 and from the definition of u_R follows boundedness of the set $\{u_R : R > 0\}$ by a constant independent of R.

(a) Let $\sup\{|b(x^0, R_k)| : k = 1, 2, \ldots\}$ be a finite number. In this case there exists a subsequence (we use the same notation for it) $\{b(x^0, R_k)\}$ such that $b(x^0, R_k) \to B \in \mathbf{R}^{nN}$ as $k \to \infty$. According to (1.6), (1.12), (4.7) and the Vitali Convergence Theorem we can pass to the limit with $k \to \infty$ in the equation (4.4) (for the fixed function ψ). We see that the second integral on the left-hand side and the integrals on the right-hand side in (4.4) tend to zero. Thus we obtain that $B + Dz(y)$ is a weak solution of the system

$$\int_{\mathbf{R}^n} a_i^\alpha(x^0, B^0, B + Dz)\, D_\alpha \psi^i\, dy = 0 \qquad \text{for all } \psi \in H_0^1(\mathbf{R}^n).$$

Now from the Liouville property of the system (1.4) it follows that z is a polynomial of at most first degree.

(b) Let $\sup\{|b(x^0, R_k)| : k = 1, 2, \ldots\}$ be infinite. In this case we can suppose $|b(x^0, R_k)| \to \infty$ as $k \to \infty$. Denoting in the sequel $b_k = b(x^0, R_k)$, $b_k{}^0 = b^0(x^0, R_k)$, $u_k(y) = u_{R_k}(y)$, $w_k(y) = R_k(u_{R_k}(y) + (b(x^0, R_k), y))$ we can rewrite equation (4.4) as follows:

$$\int_{\mathbf{R}^n} \big[a_i{}^\alpha\big(x^0 + R_k y, b_k{}^0 + w_k(y), b_k + Du_k(y)\big) - a_i{}^\alpha\big(x^0 + R_k y, b_k{}^0 + w_k(y), b_k\big)$$
$$+ a_i{}^\alpha\big(x^0 + R_k y, b_k{}^0 + w_k(y), b_k\big) - a_i{}^\alpha(x^0 + R_k y, b_k{}^0, b_k)$$
$$+ a_i{}^\alpha(x^0 + R_k y, b_k{}^0, b_k) - a_i{}^\alpha(x^0, b_k{}^0, b_k)\big]\, D_\alpha \psi^i(y)\, dy$$
$$+ R_k \int_{\mathbf{R}^n} a_i\big(x^0 + R_k y, b^0 + w_k(y), b_k + Du_k(y)\big)\, \psi^i(y)\, dy$$
$$= \int_{\mathbf{R}^n} f_i{}^\alpha(x^0 + R_k y)\, D_\alpha \psi^i(y)\, dy + R_k \int_{\mathbf{R}^n} f_i(x^0 + R_k y)\, \psi^i(y)\, dy \qquad \text{for all } \psi \in C_0{}^\infty(\mathbf{R}^n).$$

Using the theorem on the mean value in the integrals from the previous system we can rewrite this system in the following form:

$$\int_{\mathbf{R}^n} \int_0^1 \partial a_i{}^\alpha/\partial p_j{}^\beta \big(x^0 + R_k y, b_k{}^0 + w_k(y), b_k + tDu_k(y)\big)\, D_\beta u_k{}^j(y)\, D_\alpha \psi^i(y)\, dt\, dy$$
$$+ R_k \int_{\mathbf{R}^n} \int_0^1 \partial a_i{}^\alpha/\partial u^s \big(x^0 + R_k y, b_k{}^0 + tw_k(y), b_k\big)\, w_k{}^s(y)\, D_\alpha \psi^i(y)\, dt\, dy$$
$$+ R_k \int_{\mathbf{R}^n} \int_0^1 \partial a_i{}^\alpha/\partial x_\gamma\, (x^0 + tR_k y, b_k{}^0, b_k)\, y_\gamma D_\alpha \psi^i(y)\, dt\, dy$$
$$+ R_k \int_{\mathbf{R}^n} a_i\big(x^0 + R_k y, b_k{}^0 + w_k(y), b_k + Du_k(y)\big)\, \psi^i(y)\, dy$$
$$= \int_{\mathbf{R}^n} f_i{}^\alpha(x^0 + R_k y)\, D_\alpha \psi^i(y)\, dy + R_k \int_{\mathbf{R}^n} f_i(x^0 + R_k y)\, \psi^i(y)\, dy \qquad \text{for all } \psi \in C_0{}^\infty(\mathbf{R}^n).$$

Taking into account (1.7), (1.9), (1.10), (1.12), (4.7) we can pass in the previous equation to the limit with $k \to \infty$ (for the fixed function ψ) and we have that the second, third and fourth integral in the left-hand side and the integrals on the right-hand side tend to zero. Due to (1.10) and the assumption $|b(x^0, R_k)| \to \infty$ as $k \to \infty$, we obtain that the function z satisfies the equation

$$\int_{\mathbf{R}^n} d_{ij}^{\alpha\beta}(x^0, B^0)\, D_\beta z^j D_\alpha \psi^i\, dy = 0 \qquad \text{for all } \psi \in C_0{}^\infty(\mathbf{R}^n).$$

It is a linear elliptic system with the same constant of ellipticity and constant coefficients and by means of (4.8) we have that $Dz \in \mathrm{BMO}(\mathbf{R}^n)$. In this case z is a polynomial at most first degree again.

Returning to the x-coordinates, we prove that for each $x^0 \in \Omega_0$ there exists a sequence $R_k \to 0$ such that

$$\lim_{R_k \to 0} \fint_{B(x^0, R_k)} |Du(x) - (Du)_{x^0, R_k}|^2\, dx = 0. \tag{4.9}$$

We have

$$\fint_{B(x^0,R_k)} |Du(x) - (Du)_{x^0,R_k}|^2\,dx = \int_{B(0,1)} |Du_{R_k}(y) - (Du_{R_k})_{0,1}|^2\,dy$$
$$\leq \int_{B(0,1)} |Du_{R_k} - t|^2\,dy \qquad \text{for all } t \in \mathbf{R}^{nN}.$$

Now we put $t = Dz$ (Dz is a constant) and, passing to the limit, we see that (4.9) holds.

Now let us consider the equation in variations for the system (1.4) in Ω_0. If we denote by v_γ the derivative $D_\gamma u$, we get as before that

$$\int_{\Omega_0} (\partial a_i^\alpha/\partial p_j^\beta\, D_\beta v_\gamma^j + \partial a_i^\alpha/\partial u^k\, v_\gamma^k + \partial a_i^\alpha/\partial x_\gamma)\, D_\alpha \varphi^i\,dx$$
$$+ \int_{\Omega_0} (\partial a_i/\partial p_j^\beta\, D_\beta v_\gamma^j + \partial a_i/\partial u^k\, v_\gamma^k + \partial a_i/\partial x_\gamma)\,\varphi^i\,dx$$
$$= \int_{\Omega_0} (\partial f_i^\alpha/\partial x_\gamma\, D_\alpha\varphi^i + \partial f_i/\partial x_\gamma\, \varphi^i)\,dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega_0),\quad \gamma = 1, \ldots, n \tag{4.10}$$

Set

$$A_{ij}^{\alpha\beta}(x, v) = \partial a_i^\alpha/\partial p_j^\beta\,\big(x, u(x), v\big), \qquad A_{ij}^{\beta}(x, v) = \partial a_i/\partial p_j^\beta\,\big(x, u(x), v\big),$$
$$g_i^{\alpha\gamma}(x) = -\partial a_i^\alpha/\partial u^k\,\big(x, u(x), Du(x)\big)\, v_\gamma^k(x) - \partial a_i^\alpha/\partial x_\gamma\,\big(x, u(x), Du(x)\big) + \partial f_i^\alpha/\partial x_\gamma(x),$$
$$g_i^{\gamma}(x) = -\partial a_i/\partial u^k\,\big(x, u(x), Du(x)\big)\, v_\gamma^k(x) - \partial a_i/\partial x_\gamma\,\big(x, u(x), Du(x)\big) + \partial f_i/\partial x_\gamma(x).$$

From the assumption of the theorem it follows that $A_{ij}^{\alpha\beta}$ are uniformly continuous and bounded in $\Omega_0 \times \mathbf{R}^{nN}$, A_{ij}^{β} are continuous and bounded in $\Omega_0 \times \mathbf{R}^{nN}$, $g_i^{\alpha\gamma} \in L^q(\Omega_0)$ and $g_i^{\gamma} \in L^{q/2}(\Omega_0)$. Then the system (4.10) can be rewritten as

$$\int_{\Omega_0} \delta_{\theta\gamma}[A_{ij}^{\alpha\beta}(x, v)\, D_\beta v_\gamma^j\, D_\alpha \varphi_\theta^i + A_{ij}^{\beta}(x, v)\, D_\beta v_\gamma^j\, \varphi_\theta^i]\,dx$$
$$= \int_{\Omega_0} [g_i^{\alpha\theta}(x)\, D_\alpha\varphi_\theta^i + g_i^{\theta}(x)\,\varphi_\theta^i]\,dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega_0).$$

Thus v is a solution of a quasilinear system of the type (3.1) for which partial regularity (Proposition 3.4) holds ((4.9) guarantees that the assumption of Proposition 3.4 is satisfied) ∎

Proof of Proposition 2.2: Let $v \in H^1_{\text{loc}}(\mathbf{R}^2)$ with $Dv \in \text{BMO}(\mathbf{R}^2)$ be a weak solution in $\mathbf{R}^2$ of

$$\int_{\mathbf{R}^2} a_i^\alpha(x^0, u, Dv)\, D_\alpha\varphi^i(x)\,dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^2).$$

The equation in variations is

$$\int_{\mathbf{R}^2} \partial a_i^\alpha/\partial p_j^\beta\,(x^0, u, Dv)\, D_\beta v_\gamma^j\, D_\alpha\varphi^i\,dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^2), \tag{4.11}$$

where $v_\gamma = D_\gamma v$. Now we prove that $Dv_\gamma \in L^2(\mathbf{R}^2)$. Let $y^0 \in \mathbf{R}^2$, $T > 0$ be an arbitrary constant. Setting $\varphi^i = \eta^2(v_\gamma^i - (v_\gamma^i)_{y^0,2T})$, $\eta \in C_0^\infty\big(B(y^0, 2T)\big)$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B(y^0, T)$, $|D\eta| \leq c/T$ in equation (4.11), we get $\int_{B(y^0,T)} |Dv_\gamma|^2\,dx \leq c$ for $\gamma = 1, \ldots, n$, where c is independent of y^0 and T. It is known that a sequence $\{\varphi_k\} \subset C_0^\infty(\mathbf{R}^2)$ exists such that $D\varphi_k \to Dv_\gamma$ in $L^2(\mathbf{R}^2)$ and therefore from (4.11) we have

$$\int_{\mathbf{R}^2} \partial a_i^\alpha/\partial p_j^\beta(x^0, u, Dv)\, D_\beta v_\gamma^j D_\alpha v_\gamma^i\,dx = 0$$

and together with the condition of ellipticity (1.13) gives the result ∎

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