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The Asymptotics of the Solutions of Linear Elliptic Variational Problems in Domains with Edges

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Es sei G ein beschränktes Gebiet im $\mathbf{R}^n$ mit stückweise glattem Rand $\partial G = \bigcup_{j=1}^{T} \Gamma_j \cap \bigcup_{j=1}^{T-1} \mathcal{M}_j$, wobei Γ_j und $\mathcal{M}_j$ glatte zusammenhängende $(n-1)$- bzw. $(n-2)$-dimensionale Mannigfaltigkeiten sind. Ferner seien L ein elliptischer Differentialoperator der Ordnung $2m$, $a(u, v)$ eine zugehörige Dirichlet-Form und $\mathcal{V}^m(G)$ der Raum derjenigen Funktionen aus $W_2^m(G)$, die den stabilen Randbedingungen $B_k^{(j)}u = 0$ auf Γ_j und $N_k^{(j)}u = 0$ auf $\mathcal{M}_j$ (ord $B_k^{(j)} \leqq m-1$ und ord $N_k^{(j)} \leqq m-2$) genügen. Es wird gezeigt, daß die Lösung $u \in \mathcal{V}^m(G)$ des Variationsproblems $a(u, v) = \langle f, v \rangle$ für alle $v \in \mathcal{V}^m(G)$ unter entsprechenden Voraussetzungen an f Summe von gewissen singulären Termen und einem regulärem Term $u_1 \in W_2^{m+\beta}(G)$ $(\beta \in (0, 1])$ ist.

Пусть G ограниченная область в $\mathbf{R}^n$ с кусочно-гладкой границей $\partial G = \bigcup_{j=1}^{T} \Gamma_j \cap \bigcup_{j=1}^{T-1} \mathcal{M}_j$, где Γ_j и $\mathcal{M}_j$ являются гладкими связяными $(n-1)$- и $(n-2)$-мерными многообразиями, соответственно. Пусть, далее, L эллиптический дифференциальный оператор порядка $2m$, $a(u, v)$ соответствующая форма Дирихле и $\mathcal{V}^m(G)$ пространство всех функций из $W_2^m(G)$ удовлетворяющих устойчивым граничным условиям $B_k^{(j)}u = 0$ на Γ_j и $N_k^{(j)}u = 0$ на $\mathcal{M}_j$ (ord $B_k^{(j)} \leqq m-1$ и ord $N_k^{(j)} \leqq m-2$). Доказывается, что решение $u \in \mathcal{V}^m(G)$ вариационной задачи $a(u, v) = \langle f, v \rangle$ для всех $v \in \mathcal{V}^m(G)$ в соответствующих условиях на f является суммой некоторых сингулярных функций и регулярной функции $u_1 \in W_2^{m+\beta}(G)$ $(\beta \in (0, 1])$.

Let G be a bounded domain in $\mathbf{R}^n$ with piecewise smooth boundary $\partial G = \bigcup_{j=1}^{T} \Gamma_j \cap \bigcup_{j=1}^{T-1} \mathcal{M}_j$, where Γ_j and $\mathcal{M}_j$ are smooth connected $(n-1)$- and $(n-2)$-dimensional manifolds, respectively. Furthermore, let L be an elliptic differential operator of order $2m$, $a(u, v)$ a Dirichlet form corresponding to L and $\mathcal{V}^m(G)$ the space of all functions from $W_2^m(G)$ satisfying the stable boundary conditions $B_k^{(j)}u = 0$ on Γ_j and $N_k^{(j)}u = 0$ on $\mathcal{M}_j$ (ord $B_k^{(j)} \leqq m-1$ and ord $N_k^{(j)} \leqq m-2$). It is proved that the solution $u \in \mathcal{V}^m(G)$ of the variational problem $a(u, v) = \langle f, v \rangle$ for all $v \in \mathcal{V}^m(G)$ under certain conditions on f can be written as a sum of some singular terms and a regular term $u_1 \in W_2^{m+\beta}(G)$ $(\beta \in (0, 1])$.

1. Introduction

Let G be a bounded domain in $\mathbf{R}^n$ with boundary ∂G consisting of $(n-1)$-dimensional smooth connected manifolds $\Gamma_1, \ldots, \Gamma_T$ and $(n-2)$-dimensional smooth connected manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{T-1}$. We suppose that G in a neighbourhood of each point $\zeta \in \mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{T-1}$ is diffeomorphic to a dihedral angle $\mathcal{D} = K \times \mathbf{R}^{n-2}$, where K is a plane cone. Elliptic boundary value problems of the form

$$
\begin{aligned}
&L(x, D)\, u = f \quad \text{in } G \\
&B_k^{(j)}(x, D)\, u = g_k^{(j)} \quad \text{on } \Gamma_j \quad (j = 1, \ldots, T; \quad k = 1, \ldots, m)
\end{aligned}
\tag{1}
$$

are considered in [10, 12] in the class of the weighted Sobolev spaces $V^l_{p,\beta}(G)$ with the norm

$$\|u\|_{V^l_{p,\beta}(G)} = \Big(\int\limits_G \sum_{|\alpha| \le l} r^{p(\beta - l + |\alpha|)} |D^\alpha u|^p \, dx\Big)^{1/p} \tag{2}$$

$(1 < p < \infty,\ \beta \in \mathbf{R},\ r = r(x) = \operatorname{dist}(x, \mathcal{M}),\ D^\alpha = D^{\alpha_1}_{x_1} \dots D^{\alpha_n}_{x_n},$
$D_{x_j} = -\mathrm{i}\partial/\partial x_j)$.

A disadvantage of these results is that the verification of the conditions there (triviality of the kernel and cokernel of some model operators) is not easy in the general case. However, in the case of the Dirichlet problem, these results can be applied to the variational solution $u \in \mathring{W}_2{}^m(G)$ (see, for instance, [2, 3, 5, 6, 12]) because this solution is contained in the weighted Sobolev space $V^m_{2,0}(G)$. Unfortunately, the theory in the spaces $V^l_{p,\beta}(G)$ is not immediately applicable to the solution of arbitrary variational problems. Nevertheless similar results are obtained in [16] for the Neumann problem for elliptic differential equations of second order.

The regularity of the solution of the problem (1) is determined by the eigenvalues of some parameter-depending elliptic operators, which will be described now. Let ζ be an arbitrary point on $\mathcal{M}$ and let $\Gamma_{i_+(\zeta)}, \Gamma_{i_-(\zeta)}$ $\big(i_\pm(\zeta) \in \{1, \dots, T\}\big)$ be the sides of ∂G adjacent to ζ. The tangential half-planes $\Gamma_\zeta{}^\pm$ to $\Gamma_{i_\pm(\zeta)}$ at the point ζ determine a dihedral angle $\mathcal{D}_\zeta$ with the edge M_ζ. We consider the problem

$$\begin{aligned} &L_0(\zeta; D)\, u = \varphi(x) \quad \text{in } \mathcal{D}_\zeta, \\ &B^\pm_{k0}(\zeta; D)\, u = \varphi_k{}^\pm(x) \quad \text{on } \Gamma_\zeta{}^\pm \ (k = 1, \dots, m), \end{aligned} \tag{3}$$

where $L_0(\zeta; D)$ and $B^\pm_{k0}(\zeta; D)$ are the principal parts of $L(x, D)$ and $B_k{}^\pm(x, D)$ with coefficients frozen in ζ. Applying to (3) the transformation $x \to (y, z)$, where $(y, z) = (y_1, y_2, z_1, \dots, z_{n-2})$ are local Cartesian coordinates in a neighbourhood of ζ and z are coordinates in the direction of M_ζ, we obtain the problem

$$\begin{aligned} &\tilde{L}_0(\zeta; D_y, D_z)\, u = \sum_{|\alpha'| + |\alpha''| = 2m} \tilde{a}_{\alpha'\alpha''}(\zeta)\, D_y{}^{\alpha'} D_z{}^{\alpha''} u = \varphi \quad \text{in } \mathcal{D}_\zeta \\ &\tilde{B}^\pm_{k0}(\zeta; D_y, D_z)\, u = \sum_{|\alpha'| + |\alpha''| = m_k{}^\pm} \tilde{b}^\pm_{k;\alpha'\alpha''}(\zeta)\, D_y{}^{\alpha'} D_z{}^{\alpha''} u = \varphi_k{}^\pm \quad \text{on } \Gamma_\zeta{}^\pm \end{aligned} \tag{4}$$

$(k = 1, \dots, m)$. $\mathcal{D}_\zeta$ has the representation $\mathcal{D}_\zeta = K_\zeta \times \mathbf{R}^{n-2}$ in the coordinates (y, z), where K_ζ is a plane cone. If we denote by r, ω the polar coordinates in the y-plane, K_ζ can be written in the form

$$K_\zeta = \big\{y \in \mathbf{R}^2 \colon 0 < r < \infty,\ \omega \in \Omega_\zeta = \big(-\omega_0(\zeta)/2, +\omega_0(\zeta)/2\big)\big\}.$$

Furthermore, we can write the operators $\tilde{L}_0(\zeta; D_y, 0)$, $\tilde{B}^\pm_{k0}(\zeta; D_y, 0)$ as follows:

$$\begin{aligned} &\tilde{L}_0(\zeta; D_y, 0) = r^{-2m} \mathcal{L}(\zeta; \omega, D_\omega, rD_r), \\ &B^\pm_{k0}(\zeta; D_y, 0) = r^{-m_k{}^\pm} \mathcal{B}_k{}^\pm(\zeta; \omega, D_\omega, rD_r). \end{aligned}$$

We denote by $\mathfrak{U}(\zeta; \lambda)$ $(\lambda \in \mathbf{C})$ the operator of the parameter-depending problem

$$\begin{aligned} &\mathcal{L}(\zeta; \omega, D_\omega, \lambda)\, v(\omega) = \psi(\omega), \quad \omega \in \Omega_\zeta \\ &\mathcal{B}_k{}^\pm(\zeta; \omega, D_\omega, \lambda)\, v(\omega) = \psi_k{}^\pm, \quad \omega = \pm\, \omega_0(\zeta)/2 \quad (k = 1, \dots, m). \end{aligned} \tag{5}$$

The operator $\mathfrak{U}(\zeta; \lambda)$ is elliptic in the sense of Agranovič and Višik [1]. Therefore $\mathfrak{U}(\zeta; \lambda)$ is a homeomorphism $W_2{}^{2m}(\Omega_\zeta) \to L_2(\Omega_\zeta) \times \mathbf{C}^m \times \mathbf{C}^m$ for all $\lambda \in \mathbf{C}$ except a countable number of isolated points, the eigenvalues of $\mathfrak{U}(\zeta; \lambda)$. There lies only a finite number of eigenvalues in every strip $h_1 < \operatorname{Im} \lambda < h_2$ of the complex plane.

In the following we denote by $V^l_{2,\beta}(G)$ the closure of $C_0^\infty(\overline{G} \setminus \mathcal{M})$ with respect to the norm (2) and by $W^l_{2,\beta}(G)$ $(\beta > -1)$ the closure of $C_0^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{W^l_{2,\beta}(G)} = \Big(\int_G r^{2\beta} \sum_{|\alpha| \leqq l} |D^\alpha u|^p \, dx\Big)^{1/p}. \tag{6}$$

For simplicity we also denote by $L_{2,\beta}(G)$ the space $V^0_{2,\beta}(G) = W^0_{2,\beta}(G)$. Furthermore let $V^{l-1/2}_{2,\beta}(\Gamma_j)$ and $W^{l-1/2}_{2,\beta}(\Gamma_j)$ be the spaces of traces of functions from $V^l_{2,\beta}(G)$ and $W^l_{2,\beta}(G)$ $(l \geqq 1)$ on Γ_j. The following imbeddings directly follow from the definition of the spaces or can be easily proved by means of the Hardy inequality:

$$V^l_{2,\beta}(G) \subset V^{l-k}_{2,\beta'}(G) \quad \text{for} \quad k = 1, \ldots, l \quad \text{if } \beta' \geqq \beta - k,$$

$$W^l_{2,\beta}(G) \subset W^{l-k}_{2,\beta'}(G) \quad \text{for } k = 1, \ldots, l \quad \text{if } \beta' \geqq \beta - k, -1;$$

in particular we have $V^l_{2,\beta}(G) = W^l_{2,\beta}(G)$ if $\beta > 1 - l$. Moreover, for arbitrary real $\beta > 0$, the imbedding

$$W^{[\beta]+1}_{2,[\beta]-\beta+1}(G) \subset W_2{}^\beta(G) \subset W^{[\beta]}_{2,[\beta]-\beta}(G)$$

is valid, where $W_2{}^\beta(G)$ denotes the usual Sobolev-Slobodezkij space and $[\beta]$ is the largest integer number with $[\beta] \leqq \beta$ (see [14]).

2. Formulation of the variational problem

For simplicity we assume that the set $\mathcal{M}$ of the edges of the domain G is connected, that means $T = 2$. Then we write Γ_+ and Γ_- instead of Γ_1 and Γ_2. Let $L(x, D)$ be an elliptic differential operator in G with smooth coefficients, which is given in the variational form

$$L(x, D)\, u = \sum_{|\alpha|,|\gamma| \leqq m} D^\gamma (a_{\alpha\gamma} D^\alpha u), \tag{7}$$

and let

$$a(u, v) = \int_G \sum_{|\alpha|,|\gamma| \leqq m} a_{\alpha\gamma} D^\alpha u \overline{D^\gamma v} \, dx \tag{8}$$

be the corresponding Dirichlet form. For functions from $C_0^\infty(\overline{G} \setminus \mathcal{M})$ or in the class of the spaces $V^l_{p,\beta}(G)$ Green's formulas are not essential different from those in a domain with smooth boundary (see [8, 10, 17]). However, in general, for functions from $C_0^\infty(\overline{G})$ also integrals over $\mathcal{M}$ occur in Green's formulas additional to integrals over Γ_+ and Γ_-. In the following a system of differential operators

$$S_k{}^{(i)}(x, D) = \sum_{|\alpha| \leqq i} a^{(i)}_{k;\alpha}(x)\, D^\alpha \qquad (i = 0, \ldots, l - 1;\ k = 0, \ldots, n_i)$$

is said to be a *normal system on* $\mathcal{M}$ if the following conditions are satisfied:

(i) For all $x_0 \in \mathcal{M}$ there exists a vector ξ with direction being orthogonal to $\mathcal{M}$ in x_0 such that $S^{(i)}_{k0}(x_0, \xi) \neq 0$. Here $S^{(i)}_{k0}(x_0, D)$ denotes the principal part of $S_k{}^{(i)}(x, D)$ with coefficients frozen in x_0. That means, if we transform the operators $S^{(i)}_{k0}(x_0, D)$ into the operators $\tilde{S}^{(i)}_{k0}(x_0, D_{y_1}, D_{y_2}, D_z)$, where D_{y_1}, D_{y_2} are derivatives in directions being orthogonal to $\mathcal{M}$, then there exists at least one $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ such that $\tilde{S}^{(i)}_{k0}(x_0, \xi_1, \xi_2, 0) \neq 0$.

(ii) The polynomials $\tilde{S}^{(i)}_{k0}(x_0, \xi_1, \xi_2, 0)$ $(k = 0, \ldots, n_i)$ are linearly independent for every $x_0 \in \mathcal{M}$ and every i.

The system $\{S_k^{(i)}\}$ is said to be a *Dirichlet system of order l on $\mathcal{M}$* if furthermore the following condition is satisfied:

(iii) The polynomials $\tilde{S}_{k0}^{(i)}(x_0, \xi_1, \xi_2, 0)$ $(k = 0, \ldots, n_i)$ form a basis in the linear space of homogeneous polynomials $P(\xi_1, \xi_2)$ of degree i for $i = 0, 1, \ldots, l-1, x_0 \in \mathcal{M}$.

Integration by parts similarly to [8] yields the following Green's formula.

Lemma 1: *Let $\{F_j^{\pm}\}_{j=0,\ldots,m-1}$ be given Dirichlet systems of order m on $\Gamma_\pm$ and $\{S_k^{(i)}\}_{i=0,\ldots,m-2;k=0,\ldots,i}$ a Dirichlet system of order $m-1$ on $\mathcal{M}$, where the coefficients of $F_j^{\pm}$ and $S_k^{(i)}$ are smooth. Then there exist normal systems $\{G_j^{\pm}\}_{j=0,\ldots,m-1}$ of boundary operators on $\Gamma_\pm$ with $\operatorname{ord} G_j^{\pm} + \operatorname{ord} F_j^{\pm} = 2m-1$ and boundary operators $T_k^{(i)}$ on $\mathcal{M}$ $(i = 0, \ldots, m-2;\ k = 0, \ldots, i)$ with $\operatorname{ord} T_k^{(i)} + \operatorname{ord} S_k^{(i)} \leqq 2m-2$ (the system of the operators $T_k^{(i)}$ is not normal in general, some of them can be equal to zero), such that, for all $u, v \in C^\infty(\overline{G})$,*

$$a(u, v) = \int_G Lu\bar{v}\,dx + \sum_{\pm} \sum_{j=1}^{m} \int_{\Gamma_\pm} G_j^{\pm}u\overline{F_j^{\pm}v}\,d\sigma_\pm + \sum_{i=0}^{m-2} \sum_{k=0}^{i} \int_{\mathcal{M}} T_k^{(i)}u\overline{S_k^{(i)}v}\,d\mu.$$

Here $\sigma_\pm$ and μ are measures on $\Gamma_\pm$ and $\mathcal{M}$, respectively.

In the following let $\{B_k^{\pm}\}_{k=1,\ldots,m}$ be normal systems of boundary operators on $\Gamma_\pm$ with smooth coefficients and $\operatorname{ord} B_k^{\pm} \leqq m-1$ for $k = 1, \ldots, p^{\pm}$, $m \leqq \operatorname{ord} B_k^{\pm} \leqq 2m-1$ for $k = p^{\pm}+1, \ldots, m$. Furthermore, let $\{N_k\}_{k=1,\ldots,m'}$ be a normal system of boundary operators on $\mathcal{M}$ with smooth coefficients and $\operatorname{ord} N_k \leqq m-2$. We assume that Green's formula

$$a(u, v) = \int_G Lu\bar{v}\,dx + \sum_{\pm} \sum_{k=1}^{p^{\pm}} \int_{\Gamma_\pm} \Phi_k^{\pm}u\overline{B_k^{\pm}v}\,d\sigma_\pm + \sum_{\pm} \sum_{k=p^{\pm}+1}^{m} \int_{\Gamma_\pm} B_k^{\pm}u\overline{\Phi_k^{\pm}v}\,d\sigma_\pm + \sum_{k=1}^{m'} \int_{\mathcal{M}} \Psi_k u\overline{N_k v}\,d\mu \tag{9}$$

is valid for all $u, v \in C^\infty(\overline{G})$, where $\Phi_k^{\pm}$, Ψ_k are boundary operators on $\Gamma^{\pm}$ and $\mathcal{M}$, respectively, such that $\operatorname{ord} B_k^{\pm} + \operatorname{ord} \Phi_k^{\pm} = 2m-1$, $\operatorname{ord} N_k + \operatorname{ord} \Psi_k \leqq 2m-2$. We denote

$$\mathcal{V}^m(G) = \{u \in W_2^m(G):\quad B_k^{\pm}u = 0 \text{ on } \Gamma^{\pm} \text{ for } k = 1, \ldots, p^{\pm};\quad N_k u = 0 \text{ on } \mathcal{M} \text{ for } k = 1, \ldots, m_0'\}$$

$(m_0' \leqq m')$. Then we consider the following variational problem:

For given f from the dual space $(\mathcal{V}^m(G))'$, find a solution $u \in \mathcal{V}^m(G)$ of the equation

$$a(u, v) = \langle f, v\rangle \quad \text{for all } v \in \mathcal{V}^m(G). \tag{10}$$

We suppose that the Dirichlet form $a(\cdot, \cdot)$ is $\mathcal{V}^m(G)$-coercive, i.e.

(i) $|a(u, v)| \leqq c_1 \|u\|_{W_2^m(G)} \|v\|_{W_2^m(G)}$

(ii) $\operatorname{Re} a(u, u) + c_2\|u\|^2_{L_2(G)} \geqq c_3\|u\|^2_{W_2^m(G)}$ for all $u, v \in \mathcal{V}^m(G)$,

where c_1, c_2, c_3 are non-negative constants. Furthermore, we suppose that the operators $B_k^{\pm}$ $(k = 1, \ldots, m)$ cover the operator L on $\Gamma_\pm$. The $\mathcal{V}^m(G)$-coercivity ensures the local regularity of the solution outside of a neighbourhood of the edge $\mathcal{M}$, i.e., if $u \in \mathcal{V}^m(G)$ is a solution of (10) with $f \in L_{2,\mathrm{loc}}(\overline{G} \setminus \mathcal{M})$, then $u \in W^{2m}_{2,\mathrm{loc}}(\overline{G} \setminus \mathcal{M})$ and u satisfies the equations

$$Lu = f \text{ in } G, \qquad B_k^{\pm}u = 0 \text{ on } \Gamma_\pm \quad (k = 1, \ldots, m). \tag{11}$$

From this we obtain the following regularity assertion.

Lemma 2: *If* $u \in \mathcal{V}^m(G)$ *is a solution of* (10) *with the right-hand side* $f \in (\mathcal{V}^m(G))' \cap L_{2,m}(G)$, *then* $u \in W^{2m}_{2,m}(G)$.

Proof: Every function $u \in W_2{}^m(G)$ can be written in the form $u = u_1 + u_2$, where $u_1 \in W^{2m}_{2,m}(G)$ and $u_2 \in V^m_{2,0}(G)$ (see [14]). Since $f \in L_{2,m}(G)$, we have $u \in W^{2m}_{2,\mathrm{loc}}(\overline{G} \setminus \mathcal{M})$. Consequently, u is a solution of the equations (11) and u_2 is a solution of the problem

$$Lu_2 = f - Lu_1 \in V^0_{2,m}(G) \quad \text{in } G,$$

$$B_k{}^{\pm} u_2|_{\Gamma_\pm} = -B_k{}^{\pm} u_1|_{\Gamma_\pm} \in V^{2m-m_k{}^{\pm}-1/2}_{2,m}(\Gamma_\pm) \qquad (k = 1, \ldots, m).$$

Using [10: Theorem 10.2] we obtain $u_2 \in V^{2m}_{2,m}(G)$ and therefore $u \in W^{2m}_{2,m}(G)$ ∎

3. Asymptotics of the solution

We will investigate now the behaviour of the solution $u \in \mathcal{V}^m(G)$ of (10) in a neighbourhood of a point $\zeta \in \mathcal{M}$ if the right-hand side f belongs to the dual space $W_2{}^{-m+\beta}(G)$ of $W_2{}^{m-\beta}(G)$. For simplicity we assume that the domain G coincides in a neighbourhood $\mathcal{U}$ of ζ with a dihedral angle

$$\mathcal{D} = \{x = (y, z) \in \mathbf{R}^n : y = (y_1, y_2) \in K,\ z = (z_1, \ldots, z_{n-2}) \in \mathbf{R}^{n-2}\}.$$

Furthermore, we restrict ourselves to the case that the coefficients of the boundary operators $B_k{}^{\pm}$ $(k = 1, \ldots, p^{\pm})$ and N_k $(k = 1, \ldots, m_0')$ defining the subspace $\mathcal{V}^m(G)$ are independent of the variable z in this neighbourhood.

Lemma 3: *Let* $\varphi \in C^1(K \times \mathbf{R}^{n-2})$. *Then, for* $0 \leqq \beta < 1$,

$$\int_{\mathbf{R}^{n-2}} \int_K |y|^{-2\beta} |\varphi(y, \eta)|^2 \, dy \, d\eta \leqq c_\beta \int_{\mathbf{R}^{n-2}} \int_K |\eta|^{2\beta-2} (|\eta|^2 |\varphi|^2 + |\nabla_y \varphi|^2) \, dy \, d\eta.$$

Proof: Let $\chi \in C_0{}^\infty(\overline{\mathbf{R}}_+)$ be a cut-off function with $\operatorname{supp} \chi \subset [0, 2)$ and $\chi(r) = 1$ for $r < 1$. Then

$$\int_{\substack{\mathbf{R}^{n-2} \\ |\eta||y|>1}} \int_K |y|^{-2\beta} |\varphi(y, \eta)|^2 \, dy \, d\eta \leqq \int_{\mathbf{R}^{n-2}} \int_K |\eta|^{2\beta} |\varphi(y, \eta)|^2 \, dy \, d\eta$$

and, by the Hardy inequality,

$$\begin{aligned}
&\int_{\substack{\mathbf{R}^{n-2} \\ |\eta||y|<1}} \int_K |y|^{-2\beta} |\varphi(y, \eta)|^2 \, dy \, d\eta \\
&\leqq c \int_{\mathbf{R}^{n-2}} \int_K |y|^{-2\beta} |\chi(|\eta|\,|y|)\, \varphi(y, \eta)|^2 \, dy \, d\eta \\
&\leqq c \int_{\mathbf{R}^{n-2}} \int_K |y|^{-2\beta+2} \left|\nabla_y \big(\chi(|\eta|\,|y|)\, \varphi(y, \eta)\big)\right|^2 dy \, d\eta \\
&\leqq c \int_{\mathbf{R}^{n-2}} \int_K |\eta|^{2\beta-2} \left(|\eta|^2 |\varphi(y, \eta)|^2 + |\nabla_y \varphi(y, \eta)|^2\right) dy \, d\eta.
\end{aligned}$$

Hence, the assertion is true ∎

In the following let $\chi \in C^\infty(\overline{G})$ be a cut-off function being equal to zero outside of $\mathcal{U}$ and to one in a neigbourhood $\mathcal{U}' \subset\subset \mathcal{U}$ of ζ. Furthermore, if ζ is a function on $\mathcal{D}$ and $h \in \mathbf{R}$, ζ_h denotes the function defined by

$$h\varphi_h(y, z) = \varphi(y, z_1, \ldots, z_{n-3}, z_{n-2} + h) - \varphi(y, z_1, \ldots, z_{n-2}).$$

If f is a distribution, then the distribution f_h will be defined by the equation $\langle f_h, \varphi\rangle = \langle f, \varphi_{-h}\rangle$ for all φ.

Lemma 4: *For $0 < \beta < 1$, $f \in W_{2,1-\beta}^{-m+1}(\mathcal{D})$, $\operatorname{supp} f \subset \mathcal{U}'$, we have*

$$\int_0^\delta h^{1-2\beta} \|f_h\|_{W_{2,0}^{-m}(\mathcal{D})} \, dh \leqq c \, \|f\|^2_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})}.$$

Proof: For simplicity we restrict ourselves to the case $n = 3$. By the Riesz theorem there exists a function $w \in W_{2,0}^m(\mathcal{D})$ such that

$$(w, v)_{W_2{}^m(\mathcal{D})} := \int_{\mathcal{D}} \sum_{|\alpha|+j\leqq m} D_y{}^\alpha D_z{}^j w \overline{D_y{}^\alpha D_z{}^j v} \, dy \, dz = \langle f, v\rangle$$

for all $v \in W_2{}^m(\mathcal{D})$ and $\|w\|_{W_2{}^m(\mathcal{D})} = \|f\|_{W_2{}^{-m}(\mathcal{D})}$. Moreover, we have

$$(w_h, v)_{W_2{}^m(\mathcal{D})} = (w, v_{-h})_{W_2{}^m(\mathcal{D})} = \langle f, v_{-h}\rangle = \langle f_h, v\rangle$$

for all $v \in W_2{}^m(\mathcal{D})$ and $\|w_h\|_{W_2{}^m(\mathcal{D})} = \|f_h\|_{W_2{}^{-m}(\mathcal{D})}$. Let $\mathcal{F}_{z\to\eta}$ be the Fourier transformation with respect to z and $\tilde{v} = \mathcal{F}_{z\to\eta} v$. Then we denote by A the functional

$$A(\tilde{v}) = \int_{\mathbf{R}} \int_K \sum_{|\alpha|+j\leqq m} |\eta|^{2j} D_y{}^\alpha \tilde{w}(y, \eta) \overline{D_y{}^\alpha \tilde{v}(y, \eta)} \, dy \, d\eta.$$

Using Lemma 3 we then obtain the inequality

$$\begin{aligned}|A(\tilde{v})| &= |(w, v)_{W_2{}^m(\mathcal{D})}| = |\langle f, v\rangle| \leqq \|f\|_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})} \|v\|_{W_{2,\beta-1}^{m-1}(\mathcal{D})} \\ &= \|f\|_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})} \left(\int_{\mathbf{R}} \int_K r^{2\beta-2} \sum_{|\alpha|+j\leqq m-1} |\eta|^{2j} |D_y{}^\alpha \tilde{v}|^2 \, dy \, d\eta\right)^{1/2} \\ &\leqq c\|f\|_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})} \left(\int_{\mathbf{R}} \int_K |\eta|^{-2\beta} \sum_{\alpha+|j|\leqq m} |\eta|^{2j} |D_y{}^\alpha \tilde{v}|^2 \, dy \, d\eta\right)^{1/2}.\end{aligned}$$

Thus we get

$$\int_{\mathbf{R}} \int_K |\eta|^{2\beta} \sum_{|\alpha|+j\leqq m} |\eta|^{2j} |D_y{}^\alpha \tilde{w}|^2 \, dy \, d\eta \leqq c\|f\|^2_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})}$$

and

$$\begin{aligned}\int_0^\delta h^{-2\beta+1} \|f_h\|^2_{W_2{}^{-m}(\mathcal{D})} \, dh &= \int_0^\delta h^{-2\beta+1} \|w_h\|^2_{W_2{}^m(\mathcal{D})} \, dh \\ &= \int_0^\delta h^{-2\beta+1} \int_{\mathbf{R}} \int_K \sum_{|\alpha|+j\leqq m} |\eta|^{2j} |D_y{}^\alpha \tilde{w}|^2 \left|\frac{e^{i\eta h} - 1}{h}\right|^2 dy \, d\eta \, dh \\ &\leqq c \int_{\mathbf{R}} \int_K |\eta|^{2\beta} \sum_{|\alpha|+j\leqq m} |\eta|^{2j} |D_y{}^\alpha \tilde{w}|^2 \, dy \, d\eta \leqq c\|f\|_{W_{2,-\beta+1}^{-m+1}(\mathcal{D})}. \blacksquare\end{aligned}$$

Theorem 1: *Let $u \in \mathcal{V}^m(G)$ be a solution of the variational problem* (10) *with the right-hand side $f \in W_{2,-\beta+1}^{-m+1}(G)$. Then*

$$\chi \, \partial u/\partial z_i \in W_{2,-\beta}^{m-1}(G) \text{ for } 0 \leqq \beta < 1 \text{ and } \chi \, \partial u/\partial z_i \in W_2{}^m(G) \text{ for } \beta = 1$$

$(i = 1, \ldots, n - 2)$. *If furthermore $f \in L_{2,m-\beta}(G)$ $(0 \leqq \beta \leqq 1)$, then*

$$\chi \, \partial u/\partial z_i \in W_{2,m-\beta}^{2m-1}(G) \qquad (i = 1, \ldots, n - 2).$$

Proof: In order to simplify the notations we restrict ourselves to the case $n = 3$. The proof for $n > 3$ is completely analogous.

At first we prove the assertion for the case that the support of u lies in a neighbourhood $\mathcal{U}'$ of ζ. Then, for small $|h|$, the support of $u_h \in \mathcal{V}^m(G)$ is contained in a neighbourhood $\mathcal{U}$ of ζ and we obtain

$$\begin{aligned} a(u_h, v) &= -a(u, v_{-h}) + k(u, v) \\ &= -\langle f, v_{-h}\rangle + k(u, v) = \langle f_h, v\rangle + k(u, v), \end{aligned}$$

where $k(u, v) = \int\limits_G \sum\limits_{|\alpha|,|\gamma| \leqq m} a_{\alpha\gamma,-h}(x)\, D^\alpha u \overline{D^\gamma v}\, dx$. Since $a(\cdot, \cdot)$ is coercive we get

$$\begin{aligned} \|u_h\|^2_{W_2^m(G)} &\leqq c\big(|a(u_h, u_h)| + \|u_h\|^2_{L_2(G)}\big) \\ &\leqq c\big(|\langle f_h, u_h\rangle| + |k(u, u_h)| + \|u_h\|^2_{L_2(G)}\big) \\ &\leqq c(\|f_h\|_{W_2^{-m}(G)} + \|u\|_{W_2^m(G)})\, \|u_h\|_{W_2^m(G)} \end{aligned}$$

and therefore

$$\|u_h\|^2_{W_2^m(G)} \leqq c(\|f_h\|^2_{W_2^{-m}(G)} + \|u\|^2_{W_2^m(G)}). \tag{12}$$

We have, by Lemma 4,

$$\int\limits_0^\delta h^{-2\beta+1}(\|f_h\|^2_{W_2^{-m}(G)} + \|u\|^2_{W_2^m(G)})\, dh \leqq c(\|f\|^2_{W^{-m+1}_{2,-\beta+1}(G)} + \|u\|^2_{W_2^m(G)})$$

if $0 < \beta < 1$. Using Lemma 3 we further obtain

$$\begin{aligned} \|\partial u/\partial z\|^2_{W^{m-1}_{2,-\beta}(G)} &= \int\limits_{\mathbf{R}} \int\limits_K r^{-2\beta} \sum_{|\alpha|+j \leqq m-1} |\eta|^{2j}\, |D_y{}^\alpha \tilde{u}(y, \eta)|^2\, dy\, d\eta \\ &\leqq c \int\limits_{\mathbf{R}} \int\limits_K |\eta|^{2\beta} \sum_{|\alpha|+j \leqq m} |\eta|^{2j}\, |D_y{}^\alpha \tilde{u}(y, \eta)|^2\, dy\, d\eta, \end{aligned}$$

where

$$\int\limits_{|\eta| \leqq 1} \int\limits_K |\eta|^{2\beta} \sum_{|\alpha|+j \leqq m} |\eta|^{2j}\, |D_y{}^\alpha \tilde{u}(y, \eta)|^2\, dy\, d\eta \leqq c\, \|u\|^2_{W_2^m(G)}$$

and

$$\begin{aligned} \int\limits_{|\eta| \geqq 1} \int\limits_K \dots &= c \int\limits_{|\eta| \geqq 1} \int\limits_K \left(\int\limits_0^{\delta/|\eta|} h^{-2\beta-1}\, |e^{i\eta h} - 1|^2\, dh \right) \sum_{|\alpha|+j \leqq m} |\eta|^{2j}\, |D_y{}^\alpha \tilde{u}(y, \eta)|^2\, dy\, d\eta \\ &\leqq c \int\limits_0^\delta h^{-2\beta+1} \int\limits_{\mathbf{R}} \int\limits_K \sum_{|\alpha|+j \leqq m} |\eta|^{2j}\, |D_y{}^\alpha \tilde{u}(y, \eta)|^2 \left| \frac{e^{i\eta h} - 1}{h} \right|^2 dy\, d\eta\, dh \\ &= c \int\limits_0^\delta h^{-2\beta+1}\, \|u_h\|^2_{W_2^m(G)}\, dh. \end{aligned}$$

Hence, for $0 < \beta < 1$, we get

$$\|\partial u/\partial z\|_{W^{m-1}_{2,-\beta}(G)} \leqq c(\|f\|_{W^{-m+1}_{2,-\beta+1}(G)} + \|u\|_{W_2^m(G)}).$$

Now we can prove the assertion $\partial u/\partial z \in W^{2m-1}_{2,m-\beta}(G)$ similarly to Lemma 1. By [14] we can write $\partial u/\partial z = v_1 + v_2$, where $v_1 \in W^{2m-1}_{2,m-\beta}(G) \cap W^{2m}_{2,m-\beta+1}(G)$ and $v_2 \in V^{m-1}_{2,-\beta}(G)$.

Since $u \in W^{2m}_{2,m}(G)$ we get

$$Lv_2 = L\frac{\partial u}{\partial z} - Lv_1 = \frac{\partial f}{\partial z} + \left[L, \frac{\partial}{\partial z}\right] u - Lv_1 \in V^{-1}_{2,m-\beta}(G).$$

Here $[L, \partial/\partial z]$ denotes the commutator of L and $\partial/\partial z$, and $V^{-1}_{2,m-\beta}(G)$ denotes the dual space to $V^1_{2,\beta-m}(G)$. Analogously we obtain

$$B_k^{\pm} v_2|_{\Gamma_\pm} = -B_k^{\pm} v_1|_{\Gamma_\pm} + [B_k^{\pm}, \partial/\partial z]\, u|_{\Gamma_\pm} \in V^{2m-m_k^{\pm}-1/2}_{2,m-\beta+1}(\Gamma_\pm).$$

From this and from [12: Lemma 3.1] it follows that $v_2 \in V^{2m-1}_{2,m-\beta}(G)$ and $\partial u/\partial z \in W^{2m-1}_{2,m-\beta}(G)$.

We now consider the case $\beta = 1$. It is easily seen that $\|f_h\|_{W_2^{-m}(G)} \leqq \|f\|_{W_2^{-m+1}(G)}$ for $|h| < \delta$. Consequently, we have by (12)

$$\|u_h\|_{W_2^m(G)} \leqq c(\|f\|_{W_2^{-m+1}(G)} + \|u\|_{W_2^m(G)}) \quad \text{for } |h| < \delta,$$

where the constant c does not depend on h. Consequently $\partial u/\partial z \in W_2^m(G)$ and $\|\partial u/\partial z\|_{W_2^m(G)} \leqq c(\|f\|_{W_2^{-m+1}(G)} + \|u\|_{W_2^m(G)})$. Analogously to the first part of the proof we get $\partial u/\partial z \in W^{2m-1}_{2,m-1}(G)$ if $f \in W_2^{-m+1}(G) \cap L_{2,m-1}(G)$.

Finally we give some remarks for the case that the solution u has an arbitrary support in G. Since $u \in W^{2m}_{2,m}(G)$ if $f \in L_{2,m}(G)$ (see Lemma 2) we have $B_k^{\pm}(\chi u)|_{\Gamma_\pm} = [B_k^{\pm}, \chi]\, u|_{\Gamma_\pm} \in W^{2m-m_k^{\pm}+1/2}_{2,m}(\Gamma_\pm)$ for $k = 1, \ldots, p^{\pm}$ and $N_k(\chi u)|_{\mathcal{M}} = [N_k, \chi]\, u|_{\mathcal{M}} \in W_2^{m-r_k}(\mathcal{M})$ for $k = 1, \ldots, m_0'$. It can be proved (see [14]) that there exists a function $v \in W^{2m}_{2,m}(G)$ with the support in $\mathcal{U}$ such that $\chi u - v \in \mathcal{V}^m(G)$ and $\|\partial v/\partial z\|_{W^{2m}_{2,m}(G)} \leqq c\, \|u\|_{W^{2m}_{2,m}(G)}$. Then we can consider the function $w = \chi u - v$ instead of u and obtain the assertion for $\partial w/\partial z$ ∎

Lemma 5: *The space* $W_2^{-m+\beta}(G) \cap L_{2,m-\beta}(G)$ $(0 \leqq \beta \leqq 1)$ *is imbedded in* $W^{-m+1}_{2,1-\beta}(G)$.

Proof: Let $f \in W_2^{-m+\beta}(G) \cap L_{2,m-\beta}(G)$. Then

$$|\langle f, v\rangle| \leqq \begin{cases} c\|v\|_{L_{2,-m+\beta}(G)} & \text{for all } v \in L_{2,-m+\beta}(G), \\ c\|v\|_{W_2^{m-\beta}(G)} & \text{for all } v \in W_2^{m-\beta}(G). \end{cases}$$

If v is arbitrarily from $W^{m-1}_{2,\beta-1}(G)$, then we can write $v = v_1 + v_2$, where

$$v_1 \in V^{m-1}_{2,\beta-1}(G) \subset L_{2,\beta-m}(G), \quad v_2 \in W^{m}_{2,\beta}(G) \subset W_2^{m-\beta}(G),$$

and

$$\|v_1\|_{L_{2,\beta-m}(G)} + \|v_2\|_{W_2^{m-\beta}(G)} \leqq c\, \|v\|_{W^{m-1}_{2,\beta-1}(G)}.$$

Consequently, we get

$$|\langle f, v\rangle| \leqq |\langle f, v_1\rangle| + |\langle f, v_2\rangle|$$
$$\leqq c(\|v_1\|_{L_{2,\beta-m}(G)} + \|v_2\|_{W_2^{m-\beta}(G)}) \leqq c\, \|v\|_{W^{m-1}_{2,\beta-1}(G)}$$ ∎

Corollary: *If* $u \in \mathcal{V}^m(G)$ *is a solution of the variational problem* (10) *with the right-hand side* $f \in W_2^{-m+\beta}G) \cap L_{2,m-\beta}(G)$ $(0 \leqq \beta \leqq 1)$, *then* $\chi\, \partial u/\partial z_i \in W_2^{m-1+\beta}(G)$ $i = 1, \ldots, n-2)$.

This assertion follows from Theorem 1, Lemma 5 and the imbedding $W^{2m-1}_{2,m-\beta}(G) \subset W_2^{m-1+\beta}(G)$ ∎

We now can apply the formulas for the asymptotics of the solution of elliptic boundary value problems near conical points (see [4, 9]) to describe the behaviour of the solution of the variational problem (10) near the edge. We write the operators

$L(x, D)$, $B_k^\pm(x, D)$ in the neighbourhood $\mathcal{U}$ of ζ in the form

$$L(x, D) = L(y, z, D_y, D_z) = \sum_{|\alpha'|+|\alpha''|\leqq 2m} a_{\alpha'\alpha''}(y, z)\, D_y^{\alpha'} D_z^{\alpha''},$$

$$B_k^\pm(x, D) = B_k^\pm(y, z, D_y, D_z) = \sum_{|\alpha'|+|\alpha''|\leqq m_k\pm} b^\pm_{k;\alpha'\alpha''}(y, z)\, D_y^{\alpha'} D_z^{\alpha''}.$$

Then, for fixed z,

$$L_0(0, z, D_y, 0) = \sum_{|\alpha'|=2m} a_{\alpha'0}(0, z)\, D_y^{\alpha'}$$

and

$$B^\pm_{k0}(0, z, D_y, 0) = \sum_{|\alpha'|=m_k\pm} b^\pm_{k;\alpha'0}(0, z)\, D_y^{\alpha'}$$

are operators on the cone K and the sides $\gamma^\pm$ of K, respectively, where $L_0(0, z, D_y, 0)$ is elliptic in K and the boundary operators $B^\pm_{k0}(0, z, D_y, 0)$ cover $L_0(0, z, D_y, 0)$ on $\gamma^\pm$. We define $W^l_{2,\beta}(K)$ ($l \geqq 0$ integer, $\beta > -1$) analogously to the space $W^l_{2,\beta}(G)$ as the closure of $C_0^\infty(\bar{K})$ with respect to the norm $\|u\|_{W^l_{2,\beta}(K)} = \left(\int_K r^{2\beta} \sum_{|\alpha|\leqq l} |D_y^\alpha u(y)|^2\, dy\right)^{1/2}$ ($r = |y|$) and $W^{l-1/2}_{2,\beta}(\gamma^\pm)$ ($l \geqq 1$) as the space of the traces of functions from $W^l_{2,\beta}(K)$ on $\gamma^\pm$.

Lemma 6: *Let $u \in W^{2m}_{2,m+\varepsilon}(K)$ $(0 < \varepsilon < 1)$ be a solution of the elliptic boundary value problem*

$$L_0(0, z, D_y, 0)\, u = f \quad \text{in } K,$$

$$B^\pm_{k0}(0, z, D_y, 0)\, u = g_k^\pm \quad \text{on } \gamma^\pm \ (k = 1, \ldots, m)$$

with $f \in W^0_{2,m-\beta}(K)$ and $g_k^\pm \in W^{2m-m_k\pm-1/2}_{2,m-\beta}(\gamma^\pm)$ $(0 \leqq \beta \leqq 1)$. We suppose that supp u *is compact and that no eigenvalue of $\mathfrak{U}(z; \lambda)$ lies on the lines* $\operatorname{Im} \lambda = 1 - m - \beta$ *and* $\operatorname{Im} \lambda = 1 - m + \varepsilon$. *Then u has the representation*

$$u = \sum_{j=1}^{I'} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k} r^{i\lambda_j} \sum_{s=0}^{k} \frac{1}{s!} (\mathrm{i} \log r)^s \varphi_j^{(k-s,\sigma)}(\omega)$$

$$+ \psi(r) \sum_{s=0}^{S} c_s r^{m-1} \log^s r \varphi_s(\omega) + u_0,$$

where $u_0 \in W^{2m}_{2,m-\beta}(K)$. Here λ_j are the eigenvalues of $\mathfrak{U}(z; \lambda)$ in the strip $1 - m - \beta < \operatorname{Im} \lambda < 1 - m + \varepsilon$, $\varphi_j^{(k,\sigma)}$ are the eigenfunctions $(k = 0)$ and associated functions $(k > 0)$ of $\mathfrak{U}(z; \lambda)$ with respect to λ_j, and ψ is a smooth cut-off function with support in a neighbourhood of $r = 0$.

Proof: It follows from [4: Lemma 4.11] that $u \in W^{2m}_{2,m+\varepsilon}(K)$ can be written in the form $u = \sum_{i+j=0}^{m-2} u_{ij} y_1^i y_2^j / i!j! + u'$, where $u' \in V^{2m}_{2,m+\varepsilon}(K)$, $u_{ij} \in \mathbb{C}$. Furthermore, we have $g_k^\pm = \sum_{j=0}^{m-1-m_k\pm} g^\pm_{kj} r^j + g_k'^\pm$ if $m_k^\pm \leqq m - 1$, where $g_k'^\pm \in V^{2m-m_k\pm-1/2}_{2,m-\beta}(\gamma^\pm)$ and $g^\pm_{kj} \in \mathbb{C}$, whereas $g_k^\pm \in V^{2m-m_k\pm-1/2}_{2,m-\beta}(\gamma^\pm)$ if $m_k^\pm \geqq m$. Consequently, we get

$$L_0(0, z, D_y, 0)\, u' = f \in V^0_{2,m-\beta}(K),$$

$$B^\pm_{k0}(0, z, D_y, 0)\, u'|_{\gamma^\pm} = \begin{cases} g_k^\pm \in V^{2m-m_k\pm-1/2}_{2,m-\beta}(\gamma^\pm) & \text{if } m_k^\pm \geqq m, \\ g^\pm_{k,m-1-m_k\pm} r^{m-1-m_k\pm} + g_k'^\pm & \text{if } m_k^\pm \leqq m - 1. \end{cases}$$

There exists a function $u'' = \sum_{s=0}^{S'} c_s r^{m-1} \log^s r \varphi_s(\omega)$ (see [4: Theorem 1.3]) such that

$$L_0(0, z, D_y, 0)\, u'' = 0 \quad \text{in } K,$$

$$B_{k0}^{\pm}(0, z, D_y, 0)\, u'' = \begin{cases} 0 & \text{on } \gamma^{\pm} \quad (k = p^{\pm} + 1, \ldots, m), \\ g^{\pm}_{k,m-1-m_k^{\pm}} r^{m-1-m_k^{\pm}} & \text{on } \gamma^{\pm} \quad (k = 1, \ldots, p^{\pm}). \end{cases}$$

Since $u' - \psi u'' \in V^{2m}_{2,m+\varepsilon}(K)$ we obtain

$$u' - \psi u'' = \sum_{j,\sigma,k} c_{j\sigma k} r^{i\lambda_j} \sum_{s=0}^{k} \frac{1}{s!} (i \log r)^s \varphi_j^{(k-s,\sigma)}(\omega) + u_0,$$

where $u_0 \in V^{2m}_{2,m-\beta}(K)$ (see [4: Theorem 1.2], [9: Theorem 3.2]) ∎

For applying this lemma to the solution of the variational problem (10) we introduce the extension operator $\mathfrak{K}$,

$$(\mathfrak{K}f)\,(r, z) = \psi(r) \int_{\mathbf{R}^{n-2}} f(z + tr) \prod_{i=1}^{n-2} K(t_i)\, dt,$$

$t = (t_1, \ldots, t_{n-2})$, where $r = |y|$, ψ is a cut-off function being equal to one in a neighbourhood of $r = 0$, and K is a smooth function with compact support in $(-1, +1)$ and $\int_{\mathbf{R}} K(t_i)\, dt_i = 1$. It is known (see, e.g., [12: Lemma 1.2]) that $\mathfrak{K}$ is a continuous map

$$W_2^{\gamma}(\mathbf{R}^{n-2}) \to W^1_{2,-\gamma}(\mathcal{D}) \cap W^2_{2,-\gamma+1}(\mathcal{D}) \cap \ldots \cap W^{2m}_{2,-\gamma+2m-1}(\mathcal{D}) \quad (0 < \gamma < 1).$$

Theorem 2: *We suppose that the Dirichlet form $a(\cdot, \cdot)$ is $\mathcal{V}^m(G)$-coercive and that no eigenvalue of $\mathfrak{U}(z; \lambda)$ $(z \in \mathcal{U} \cap \mathcal{M})$ lies on the line $\operatorname{Im} \lambda = 1 - m - \beta$. If $\lambda_j(z)$ is an eigenvalue of $\mathfrak{U}(z; \lambda)$ with $\operatorname{Im} \lambda_j(z) = 1 - m$ for some $z \in \mathcal{U} \cap \mathcal{M}$, then let $\operatorname{Im} \lambda_j(z) = 1 - m$ for all $z \in \mathcal{U} \cap \mathcal{M}$. Furthermore, we suppose that the eigenvalue functions $\lambda_j (j = 1, \ldots, I)$ from the strip $1 - m - \beta < \operatorname{Im} \lambda < 1 - m$ do not intersect and change their multiplicity on $\mathcal{U} \cap \mathcal{M}$. Then the solution $u \in \mathcal{V}^m(G)$ of the variational problem* (10) *with the right-hand side $f \in W_2^{-m+\beta}(G) \cap L_{2,m-\beta}(G)$ $(0 \leqq \beta \leqq 1)$ has the form*

$$\chi u = \sum_{j=1}^{I} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} \hat{c}_{j\sigma k}(r, z)\, r^{i\lambda_j(z)} \sum_{s=0}^{k} \frac{1}{s!} (i \log r)^s\, \varphi_j^{(k-s,\sigma)}(\omega, z) + u_1,$$

where $u_1 \in W_2^{m+\beta-\varepsilon}(G)$ with an arbitrary positive real number ε and $\hat{c}_{j\sigma k} = \mathfrak{K} c_{j\sigma k}$ are the extensions of $c_{j\sigma k} \in W_2^{\operatorname{Im}\lambda_j(\cdot)+m+\beta-1-\varepsilon}(\mathcal{M})$.

Proof: By Theorem 1 we have $\partial(\chi u)/\partial z_i \in W^{2m-1}_{2,m-\beta}(G)$ $(i = 1, \ldots, n-2)$. For fixed $z \in \mathcal{U} \cap \mathcal{M}$ we then obtain

$$L_0(0, z, D_y, 0)\, (\chi u)\, (\cdot, z) \in W^0_{2,m-\beta}(K),$$

$$B_{k0}^{\pm}(0, z, D_y, 0)\, (\chi u)\, (\cdot, z)|_{\gamma^{\pm}} \in W^{2m-m_k-1/2}_{2,m-\beta}(\gamma^{\pm}).$$

Let $\varepsilon > 0$ be such that no eigenvalue of $\mathfrak{U}(z; \lambda)$ lies in the strip $1 - m < \operatorname{Im} \lambda < 1 - m + \varepsilon$. Since $(\chi u)\, (., z) \in W^{2m}_{2,m+\varepsilon}(K)$ Lemma 6 implies that

$$(\chi u)\, (y, z) = \sum_{j=1}^{I'} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k}(z)\, r^{i\lambda_j(z)} \sum_{s=0}^{k} \frac{1}{s!} (i \log r)^s\, \varphi_j^{(k-s,\sigma)}(\omega, z)$$

$$+ \psi(r) \sum_{s=0}^{S} c_s r^{m-1} \log^s r \varphi_s(\omega, z) + u_0(y, z), \tag{13}$$

where $1 - m + \beta < \operatorname{Im} \lambda_j(z) < 1 - m$ for $j = 1, \ldots, I$ and $\operatorname{Im} \lambda_j(z) = 1 - m$ for $j = I + 1, \ldots, I'$. Summing up in the expression (13) for $\chi u - u_0$ from $j = I + 1$ to $j = I'$ only, we denote this new function by w. Then we obtain

$$D_y^\alpha(\chi u) = D_y^\alpha w + D_y^\alpha u_0 + D_y^\alpha \sum_{j=1}^{I} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k}(z)\, r^{i\lambda_j(z)} \sum_{s=0}^{k} \frac{1}{s!} (i \log r)^s \varphi_j^{(k-s,\sigma)}(\omega, z).$$

Since $D_y^\alpha(\chi u)(\cdot, z) \in L_2(K)$ for $|\alpha| = m$, $(D_y^\alpha w)(\cdot, z)$ must also be a function from $L_2(K)$ for $|\alpha| = m$. However $D_y^\alpha w$ has the form

$$D_y^\alpha w = r^{-1} P(\omega, z, \log r) + \sum_{j=I+1}^{I'} r^{i\lambda_j(z)-m} Q_j(\omega, z, \log r),$$

where $\operatorname{Re}\big(i\lambda_j(z) - m\big) = -1$ and P, Q_j are polynomials of $\log r$. Consequently, $D_y^\alpha w$ must be equal to zero for $|\alpha| = m$ and (13) can be rewritten in the form

$$(\chi u)(y, z) = \sum_{j=1}^{I} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k}(z)\, r^{i\lambda_j(z)} \sum_{s=0}^{k} \frac{1}{s!} (i \log r)^s \varphi_j^{(k-s,\sigma)}(\omega, z) + u_0'(y, z), \tag{14}$$

where $u_0'(\cdot, z) = u_0(\cdot, z) + w(\cdot, z) \in W^{2m}_{2,m-\beta}(K)$ and

$$\|u_0'(\cdot, z)\|^2_{W^{2m}_{2,m-\beta}(K)} \leqq c \left(\|(\chi u)(\cdot, z)\|^2_{W^{2m}_{2,m}(K)} + \sum_{i=1}^{n-2} \left\| \frac{\partial(\chi u)}{\partial z_i} (\cdot, z) \right\|^2_{W^{2m-1}_{2,m-\beta}(K)} + \|f(\cdot, z)\|^2_{W^0_{2,m-\beta}(K)} \right) \tag{15}$$

(the constant c is independent of z). From (14) we get

$$D_r^m(\chi u)(y, z) = \sum_{j=1}^{I} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k}(z)\, r^{i\lambda_j(z)-m} \times \sum_{s=0}^{k} \frac{(i \log r)^s}{s!} \Phi_j^{(k-s,\sigma)}(\omega, z) + D_r^m u_0'(y, z). \tag{16}$$

Here

$$\Phi_j^{(0,\sigma)}(\omega, z) = \lambda_j(\lambda_j + i) \ldots (\lambda_j + m\,i - i)\, \varphi_j^{(0,\sigma)}(\omega, z)$$

and, for $s < k$,

$$\Phi_j^{(k-s,\sigma)}(\omega, z) = \sum_{\nu=s}^{k} \alpha_{jks\nu}(z)\, \varphi_j^{(k-\nu,\sigma)}(\omega, z) \quad (\alpha_{jks\nu} \text{ smooth}).$$

Let v be the function $v(r, \omega, z) = D_r^m(\chi u)(y, z) - D_r^m u_0'(y, z)$. Then

$$v(tr, \omega, z) = \sum_{j=1}^{I} \sum_{\sigma=1}^{\chi_j} \sum_{k=0}^{\varkappa_{\sigma j}-1} c_{j\sigma k}(z)\, (tr)^{i\lambda_j - m} \sum_{s=0}^{k} \frac{(i \log tr)^s}{s!} \Phi_j^{(k-s,\sigma)}(\omega, z). \tag{17}$$

Since the eigenfunctions $\varphi_j^{(0,\sigma)}$ $(\sigma = 1, \ldots, \chi_j)$ corresponding to the same eigenvalue λ_j are linearly independent for every j there exists a system of functions $\psi_j^{(\sigma)}$ such that

$$\big(\varphi_j^{(0,\sigma)}(\cdot, z), \psi_j^{(\sigma')}(\cdot, z)\big)_\Omega = \int_\Omega \varphi_j^{(0,\sigma)}(\omega, z)\, \overline{\psi_j^{(\sigma')}(\omega, z)}\, d\omega = \delta_{\sigma,\sigma'}.$$

From (17) we now obtain

$$\left(v(t_{j'\sigma'k'}r,\cdot,z),\psi_{j'}^{(\sigma')}(\cdot,z)\right)_\Omega=\sum_{j,\sigma,k}(t_{j'\sigma'k'}r)^{i\lambda_j(z)-m}\,c_{j\sigma k}(z)\times\sum_{s=0}^{k}\frac{(i\log t_{j'\sigma'k'}r)^s}{s!}\,a_{j,j',k-s}^{(\sigma,\sigma')}(z)\tag{18}$$

$$(j'=1,\ldots,I;\ \sigma'=1,\ldots,\chi_{j'};\ k'=0,1,\ldots,\varkappa_{\sigma'j'}-1),$$

where $a_{j,j',k-s}^{(\sigma,\sigma')}=(\varphi_j^{(k-s,\sigma)},\psi_{j'}^{(\sigma')})$ are smooth functions on $\mathcal{M}$ and $t_{j'\sigma'k'}$ are arbitrary real positive numbers. (18) can be considered as a linear algebraic system for the functions $r^{i\lambda_j-m}c_{j\sigma k}$. We can choose the numbers $t_{j'\sigma'k'}$ such that the coefficient determinant of this system is independent of r and has no zeros in a neighbourhood of ζ (see [11: Lemma 3.5]). Consequently, every function $r^{i\lambda_j-m}c_{j\sigma k}$ in (17) can be written in the form

$$r^{i\lambda_j(z)-m}c_{j\sigma k}(z)=\sum_{\nu=1}^{N}\left(v(t_\nu r,\cdot,z),\psi_\nu(\cdot,z)\right)_\Omega\,P_\nu(z;\log r),$$

where $\psi_\nu\in\{\psi_j^{(\sigma)}\}$ are smooth functions on $\Omega\times\mathcal{M}$ and P_ν are polynomials of $\log r$ with smooth coefficients. If we denote by v_0 and v_1 the functions

$$v_0(r,z)=r^{m-i\lambda_j(z)}\sum_{\nu=1}^{N}\int_\Omega\left(D_r^m(\chi u)\right)(t_\nu r,\omega,z)\,\overline{\psi_\nu(\omega,z)}\,d\omega P_\nu(z;\log r),$$

$$v_1(r,z)=r^{m-i\lambda_j(z)}\sum_{\nu=1}^{N}\int_\Omega(D_r^m u_0')(t_\nu r,\omega,z)\,\overline{\psi_\nu(\omega,z)}\,d\omega P_\nu(z;\log r),$$

we obtain $v_0(r,z)=c_{j\sigma k}(z)+v_1(r,z)$. Using (15) we get

$$\int_G r^{-2\beta-2\operatorname{Im}\lambda_j-2m+2\varepsilon}\,|v_0(r,z)-c_{j\sigma k}(z)|^2\,dy\,dz=\int_G r^{-2\beta-2\operatorname{Im}\lambda_j-2m+2\varepsilon}\,|v_1(r,z)|^2\,dy\,dz<\infty$$

and

$$\int_G r^{-2\beta-2\operatorname{Im}\lambda_j-2m+2+2\varepsilon}\,|D_r v_0(r,z)|^2\,dy\,dz=\int_G r^{-2\beta-2\operatorname{Im}\lambda_j-2m+2+2\varepsilon}\,|D_r v_1(r,z)|^2\,dy\,dz<\infty.$$

Furthermore, by Theorem 1 $\int_G r^{-2\beta-2\operatorname{Im}\lambda_j-2m+2+2\varepsilon}\,|D_z v_0(r,z)|^2\,dy\,dz<\infty$, i.e., $v_0\in W^1_{2,-\beta-\operatorname{Im}\lambda_j-m+1+\varepsilon}(G)$. Since $-\beta-\operatorname{Im}\lambda_j-m<-1$ we further obtain $v_0|_{\mathcal{M}}=c_{j\sigma k}$. Consequently the function $c_{j\sigma k}$ must belong to the space $W_2^{\operatorname{Im}\lambda_j(\cdot)+m+\beta-1-\varepsilon}(\mathcal{M})$.

We now write (14) in the form

$$u=\sum_{j=1}^{I}\sum_{\sigma=1}^{\chi_j}\sum_{k=0}^{\varkappa_{\sigma j}-1}(\Re c_{j\sigma k})\,r^{i\lambda_j}\sum_{s=0}^{k}\frac{(i\log r)^s}{s!}\,\varphi_j^{(k-s,\sigma)}+u_1,$$

where

$$u_1=u_0'+\sum_{j,\sigma,k}(c_{j\sigma k}-\Re c_{j\sigma k})\,r^{i\lambda_j}\sum_{s=0}^{k}\frac{(i\log r)^s}{s!}\,\varphi_j^{(k-s,\sigma)}.$$

Using (15) and the properties of the extension operator $\mathfrak{R}$ we obtain $D_y^\alpha u_1 \in L_{2,m-\beta-\varepsilon}(G)$ if $|\alpha| = 2m$ and applying Theorem 1 we can prove that

$$D_y^\alpha D_z^\gamma u_1 = D_y^\alpha D_z^\gamma(\chi u) - D_y^\alpha D_z^\gamma \sum_{j,\sigma,k} (\mathfrak{R}c_{j\sigma k})\, r^{i\lambda_j} \sum_{s=0}^{k} \frac{(\mathrm{i}\log r)^s}{s!}\, \varphi_j^{(k-s,\sigma)}$$

belongs to $L_{2,m-\beta+\varepsilon}(G)$ if $|\alpha| + |\gamma| = 2m$, $|\gamma| \geqq 1$ (cf. [12: Theorem 4.1]), i. e., $u_1 \in W^{2m}_{2,m-\beta+\varepsilon}(G) \subset W_2^{m+\beta-\varepsilon}(G)$ ∎

We remark that the number ε can be omitted in Theorem 2 if no logarithmic terms occur in the asymptotic.

Example: Let $u \in W_2^1(G)$ be a solution of the Neumann problem for the Laplace operator

$$-\Delta u = f \quad \text{in } G, \qquad \partial u/\partial \nu = 0 \quad \text{on } \Gamma_\pm. \tag{19}$$

Then u is also a solution of the variational problem

$$a(u, v) = \int_G |\Delta u|\, \Delta v\, dx = \langle f, v\rangle \quad \text{for all} \quad v \in W_2^1(G). \tag{20}$$

Obviously, $a(\cdot, \cdot)$ is $W_2^1(G)$-coercive and we can apply Theorem 2 to the solution of (19). For simplicity we suppose that the domain G coincides with the dihedral angle

$$\mathcal{D} = \{x = (y, z) \in \mathbf{R}^n : y = (y_1, y_2) \in K, \quad z = (z_1, \ldots, z_{n-2}) \in \mathbf{R}^{n-2}\}$$

in a neighbourhood $\mathcal{U}$ of a fixed point $x_0 \in \mathcal{M}$, where

$$K = \{y \in \mathbf{R}^2 : 0 < r = |y| < \infty,\ \omega \in \Omega = (-\omega_0/2, +\omega_0/2)\}$$

denotes a plane cone. In order to describe the behaviour of the solution in the neighbourhood of $\mathcal{M}$ we have to calculate the eigenvalues of the operator $\mathfrak{U}(z; \lambda)$ of the parameter-depending problem

$$\partial^2\tilde{u}/\partial\omega^2 - \lambda^2\tilde{u} = 0 \quad \text{in } \Omega, \qquad \partial\tilde{u}/\partial\omega = 0 \quad \text{for} \quad \omega = \pm\omega_0/2$$

from the strip $-\beta < \operatorname{Im} \lambda < 0$. The eigenvalues of $\mathfrak{U}(z; \lambda)$ are $k\pi i/\omega_0$ ($k \in \mathbb{Z}$) and the eigenfunctions corresponding to $k\pi i/\omega_0$ ($k \neq 0$) are $\cos(k\pi(\omega/\omega_0 + 1/2))$. Let χ be an arbitrary smooth cut-off function with $\operatorname{supp} \chi \subset \mathcal{U}$. Then we obtain the following assertion if $f \in L_{2,1-\beta}(G)$ ($0 < \beta \leqq 1$):

a) $\chi u \in W_2^{1+\beta}(G)$ if $\beta < \pi/\omega_0$,

b) $\chi u = \sum_{l=1}^{k} c_l(r, z)\, r^{l\pi/\omega_0} \cos(l\pi(\omega/\omega_0 + 1/2)) + u_1$ if $k\pi/\omega_0 < \beta < (k+1)\,\pi/\omega_0$, where $u_1 \in W_2^{1+\beta}(G)$, and c_l are extensions of functions from $W_2^{\beta - l\pi/\omega_0}(\mathcal{M})$.

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