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Integration by Means of Riemann Sums in Banach Spaces II[1])

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Es wird die Diskussion des im Teil I eingeführten $(\mu, \mathfrak{X})$-Integrals fortgesetzt: Wir untersuchen die Beziehungen dieses Integrals zum Haupt-Aumann-Pauc-Integral, zum Pettis-Integral und zum Bochner-Integral. Ferner behandeln wir die Darstellung iterierter Integrale als Limites gefilterter Familien von iterierten Riemann-Summen.

Продолжается дискуссия о $(\mu, \mathfrak{X})$-интеграле введённом в части I: Мы исследуем отношения этого интеграла к интегралу Гаупт-Ауман-Паука, к интегралу Петтиса и к интегралу Бохнера. Далее мы занимаемся представлением итерированных интегралов в виде пределов фильтрующихся семейств итерированных сумм Римана.

The discussion of the $(\mu, \mathfrak{X})$-integral introduced in Part I is continued: We investigate the relationship of this integral to the Haupt-Aumann-Pauc integral, the Pettis integral and the Bochner integral. Furthermore, we deal with the representation of iterated integrals as limits of filtered families of iterated Riemann sums.

This Part II of the paper "Integration of Riemann sums in Banach spaces" is an immediate continuation of its Part I (see [6]). All numbered references to definitions, statements, etc., of Part I are made just by the numbers used in Part I without adding an extra hint to [6].

§ 5 contains a description of the behaviour of $\int_{\mathfrak{X}} \cdot\, d\mu$ under the transition of F to a subset A of F and the corresponding transition of $\int_{\mathfrak{X}} \cdot\, d\mu$ to the "subintegral" $\int_{\mathfrak{X}_A} \cdot\, d(\mu_A)$ (Theorems 7 and 8), which leads to the discussion of the additivity and σ-additivity of $\int_{\mathfrak{X}} \cdot\, d\mu$ (Theorems 9 and 10). In § 6, the relationship between $\int_{\mathfrak{X}} \cdot\, d\mu$, the Haupt-Aumann-Pauc integral and the Pettis integral is cleared up, and it is shown that $\int_{\mathfrak{X}} \cdot\, d\mu$ is a "pointwise" integral (Theorem 11). The relationship of $\int_{\mathfrak{X}} \cdot\, d\mu$ to the Bochner integral is discussed in § 7. In § 8, the considerations on iterated integrals as being made for the extended real line in [12] are translated into the present situation.

§ 5 Integration on subsets, σ-additivity of the $(\mu, \mathfrak{X})$-integral

In this Part II we write the direction $\leqq$ on $\Omega(\mu)$, defined in Definition 4, consistently as $\underline{\leqq}$ in order to distinguish it *in printing* clearly from the order relation $\leqq$ on the real line. Furthermore, for typographic reasons, we write ${}^{X}\!\int \cdot\, d\mu$ as $\int_X \cdot\, d\mu$ and occasionally the summation symbol $\sum_I$ (see § 0/g)) also as $\sum\limits_I$.

[1]) Teil I der Arbeit erschien in Band 9, Heft 6 (1990) dieser Zeitschrift.

We begin with the following

Definition 7: (a) For each non-empty $A \in \operatorname{Dmn}\mu$ and each $\mathfrak{x} \in \mathfrak{X}$, the set $\{A \cap X \mid X \in \mathfrak{x}\} \setminus \{\emptyset\}$ is called the *trace* $\mathfrak{x}_A$ *of* $\mathfrak{x}$ *in* A. (Remark: In particular, $\mathfrak{y}_X$ in Definition 4/f is the trace of $\mathfrak{y}$ in X.)

(b) For each non-empty $A \in \operatorname{Dmn}\mu$, the set $\{\mathfrak{x}_A \mid \mathfrak{x} \in \mathfrak{X}\}$ is called the *trace* $\mathfrak{X}_A$ *of* $\mathfrak{X}$ *in* A.

(c) For each $A \in \operatorname{Dmn}\mu$, let $\mu_A(B) := \mu(B)$ for all $B \in \operatorname{Dmn}\mu$ with $B \subseteq A$. For each non-empty $A \in \operatorname{Dmn}\mu$, $\leqq_A$ denotes the direction defined as in Definition 4 on the set $\Omega(\mu_A)$ of all μ_A-partitions on A, and $\vee_A$ denotes the binary operation in $\Omega(\mu_A)$ assigning to each ordered pair $(\mathfrak{x}_A, \mathfrak{y}_A)$ their $\leqq_A$-supremum; furthermore, for each $\mathfrak{z} \in \Omega(\mu_A)$, $B_A(f|A, \mathfrak{z})$ or $B_A(\mathfrak{z})$ denotes the Birkhoff sum of $f|A$ belonging to μ_A and $\mathfrak{z}$.

Proposition 23: *The trace* $\mathfrak{X}_A$ *of* $\mathfrak{X}$ *in* A *is a* μ_A*-partition system. The mapping* λ: $\mathfrak{X} \to \mathfrak{X}_A$ *with* $\lambda(\mathfrak{x}) = \mathfrak{x}_A$ *for all* $\mathfrak{x} \in \mathfrak{X}$ *is a* $(\leqq, \leqq_A)$*-homomorhism onto. If* $\mathfrak{X}$ *is* $\vee$*-closed, then* $\mathfrak{X}_A$ *is* $\vee_A$*-closed, and* λ *is a* $(\vee, \vee_A)$*-homomorphism.*

Proof: The first assertion is clear. Let $\mathfrak{X}$ be $\vee$-closed. Then, for each $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$, $\mathfrak{x}_A \vee_A \mathfrak{y}_A (= \{X \cap Y \mid X \in \mathfrak{x}_A \text{ and } Y \in \mathfrak{y}_A\} \setminus \{\emptyset\}) = (\mathfrak{x} \vee \mathfrak{y})_A \in \mathfrak{X}_A$ ∎

Theorem 7: *Let* $\mathfrak{X}$ *be* $\vee$*-closed,* $A \in \bigcup \mathfrak{X}$ *and let the function* f *be singleton-valued. Then* $\int^{\mathfrak{X}_A} (f|A)\, d(\mu_A) = \int^{\mathfrak{X}} (f\, 1_A)\, d\mu$, *where* 1_A: $F \to \mathcal{K}$ *denotes the characteristic function w.r.t. to* A. (For the terminology, see § 0/c.)

Proof: 1. Choose $a \in \mathfrak{X}$ such that $A \in \mathfrak{a}$. 2. Let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{a} \leqq \mathfrak{x}$. Then, for each $X \in \mathfrak{x}$, one has either $X \subseteq A$ (in this case, $(f\, 1_A)[X] = (f|A)[X]$) or $X \subseteq F \setminus A$ (in this case, $(f\, 1_A)[X] = \{0\}$, since f is singleton-valued), therefore $B(f\, 1_A, \mathfrak{x}) = B_A(f|A, \mathfrak{x}_A)$. 3. Using the $\vee$-closedness of $\mathfrak{X}$ and Proposition 23, one obtains the following: a) Given $\mathfrak{x}_0 \in \mathfrak{X}$, there is, for each $\mathfrak{y} \in \mathfrak{X}_A$ with $(\mathfrak{x}_0)_A \leqq_A \mathfrak{y}$, a $\mathfrak{z} \in \mathfrak{X}$ such that $\mathfrak{y} = \mathfrak{z}_A = (\mathfrak{z} \vee \mathfrak{x}_0 \vee \mathfrak{a})_A$. b) Given $\mathfrak{y}_0 \in \mathfrak{X}_A$, there is an $\mathfrak{x}_0 \in \mathfrak{X}$ such that $\mathfrak{y}_0 = (\mathfrak{x}_0)_A = (\mathfrak{x}_0 \vee \mathfrak{a})_A$; and, for each $\mathfrak{x} \in \mathfrak{X}$ such that $\mathfrak{x}_0 \vee \mathfrak{a} \leqq \mathfrak{x}$, one has $\mathfrak{y}_0 \leqq_A \mathfrak{x}_A$. 4. By means of the Corollary to Proposition 22, Part 3a (resp. Part 3b), and Part 2 of this proof, one obtains that $\int^{\mathfrak{X}} (f\, 1_A)\, d\mu \subseteq \int^{\mathfrak{X}_A} (f|A)\, d(\mu_A)$ (resp. that $\int^{\mathfrak{X}_A} (f|A)\, d(\mu_A) \subseteq \int^{\mathfrak{X}} (f\, 1_A)\, d\mu$) ∎

Remark 8: In Theorem 7, the supposition that f be singleton-valued is not allowed to be cancelled: Assume that $f(x) = \emptyset$ for some $x \in F \setminus A$. Then, by Proposition 18, $\int^{\mathfrak{X}} (f\, 1_A)\, d\mu = \emptyset$; but nevertheless it can happen that $\int^{\mathfrak{X}_A} (f|A)\, d(\mu_A) \neq \emptyset$ holds, as trivial examples show.

Proposition 24: *Let* $\mathfrak{x} \in \mathfrak{X}$ *and* $A \in (\operatorname{Dmn}\mu) \setminus \{\emptyset\}$. *If* $B(f, \mathfrak{x})$ *is non-empty and bounded, then* $B_A(f \mid A, \mathfrak{x}_A)$ *is non-empty and bounded.*

Proof (cf. [5, proof of 4.3.4]): Assume the premise within the assertion. Let $A \neq F$. Then, one has $\mathfrak{x} \leqq \mathfrak{x}_A \cup \mathfrak{x}_{F \setminus A} \in \Omega(\mu)$, thus (by Proposition 21) the set $B(f, \mathfrak{x}_A \cup \mathfrak{x}_{F \setminus A})$ is non-empty and bounded, therefore (by Propositions 8 and 14) so is the set $B_A(f \mid A, \mathfrak{x}_A)$ ∎

Theorem 8 (cf. [5, 4.4.5 and 4.2.3]): *Let* $A \in (\operatorname{Dmn}\mu) \setminus \{\emptyset\}$. *Then* $\int^{\mathfrak{X}} f\, d\mu \neq \emptyset$ *implies* $\int^{\mathfrak{X}_A} (f \mid A)\, d(\mu_A) \neq \emptyset$.

Proof (cf. [5, p. 58]): For abbreviation, put $f[X] - f[X] = g(X)$ for all $X \subseteq F$. Assume $\int^{\mathfrak{X}} f\,d\mu \neq \emptyset$ and let $\varepsilon > 0$. Then, there is (by Theorem 2) an $\mathfrak{x} \in \mathfrak{X}$ such that

(1) $\emptyset \neq B(\mathfrak{x}) - B(\mathfrak{x}) \subseteq B_{\varepsilon/2}(0)$,

thus (by Proposition 10)

(2) $\emptyset \neq B(\mathfrak{x}) - B(\mathfrak{x}) = \sum_{X \in \mathfrak{x}} g(X)\,\mu X$,

therefore (by Propositions 24 and 10)

(3) $\emptyset \neq B_A(f|A, \mathfrak{x}_A) - B_A(f|A, \mathfrak{x}_A) = \sum_{X \in \mathfrak{x}_A} g(X)\,\mu X$.

We assert that

(4) $\sum_{X \in \mathfrak{x}_A} g(X)\,\mu X \subseteq \sum_{X \in \mathfrak{x}} \operatorname{co}\big(g(X)\,\mu X\big)$.

Indeed, in view of (2), the set $\sum_{X \in \mathfrak{x}} \operatorname{co}(g(X)\,\mu X)$ is non-empty (by the Propositions 15 and 7/b). Let $x \in \sum_{X \in \mathfrak{x}_A} g(X)\,\mu X$. Then, there is a $\varphi \in \mathrm{P}_{X \in \mathfrak{x}_A}\, g(X)\,\mu X$ such that $x = \sum_{\mathfrak{x}_A} \varphi$. We define a mapping ψ by letting, for each $X \in \mathfrak{x}$, $\psi(X) = \varphi(X \cap A)$ if $X \cap A \neq \emptyset$, and $\psi(X) = 0$ (= zero vector of E) if $X \cap A = \emptyset$. Since $f[X] \neq \emptyset$ for each $X \in \mathfrak{x}$ (by (1) and Proposition 7), one has $\psi \in \mathrm{P}_{X \in \mathfrak{x}} \operatorname{co}\big(g(X)\,\mu X\big)$. Furthermore, $x = \sum_{\mathfrak{x}} \psi$.

By Proposition 15 and (1)–(4), one gets

$$\emptyset \neq B_A(f|A, \mathfrak{x}_A) - B_A(f|A, \mathfrak{x}_A) \subseteq \tau\operatorname{co}\big(B(\mathfrak{x}) - B(\mathfrak{x})\big) \subseteq B_\varepsilon(0).$$

By Theorem 2, $\int^{\mathfrak{X}_A} (f|A)\,d(\mu_A)$ is non-empty ∎

For the remainder of this section, we assume $\mathfrak{X}$ to be $\vee$-closed and $\mathfrak{a} \in \mathfrak{X}$.

Theorem 9 (cf. [5, 4.6.2 and 4.4.5]): *Let $\mathfrak{a}$ be finite. Then*

$$\int^{\mathfrak{X}} f\,d\mu = \sum_{A \in \mathfrak{a}} \int^{\mathfrak{X}_A} (f|A)\,d(\mu_A).$$

Proof: 1. For abbreviation, let C_1 denote the left-hand side of the asserted equation, C_6 its right-hand side. We define $C_2, \ldots, C_5$ as follows: $C_2 = {}^{\mathcal{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}}\, B(\mathfrak{x})$,

$$C_3 = {}^{\mathcal{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}}\, B(\mathfrak{x} \vee \mathfrak{a}), \qquad C_4 = {}^{\mathcal{F}\mathfrak{X}}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}} \sum_{A \in \mathfrak{a}} B_A(\mathfrak{x}_A), \qquad C_5 = \sum_{A \in \mathfrak{a}} {}^{\mathcal{F}(\mathfrak{X}_A)}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}_A}\, \dot{B}(\mathfrak{x}).$$

2. If there is an $x \in F$ such that $f(x) = \emptyset$, then $x \in A$ for some $A \in \mathfrak{a}$, therefore (by Proposition 18, applied twice) $C_1 = \emptyset = C_6$ holds. For the remainder of this proof, we assume f to be singleton-valued. Then, the equations $C_1 = C_2$ and $C_5 = C_6$ hold by the Corollary to Proposition 22. $C_2 = C_3$ holds, since $B(\mathfrak{x}) = B(\mathfrak{x} \vee \mathfrak{a})$ holds for $\mathcal{F}\mathfrak{X}$-almost all $\mathfrak{x} \in \mathfrak{X}$. The equation $C_3 = C_4$ holds by Proposition 11, since $\{\mathfrak{x}_A \mid A \in \mathfrak{a}\}$ is a finite partition of $\mathfrak{x} \vee \mathfrak{a}$ for each $\mathfrak{x} \in \mathfrak{X}$.

3. We assert that $C_5 \subseteq C_4$. Indeed, let $x \in C_5$, say $x = \sum_{A \in \mathfrak{a}} x_A$ (finite summation in E) with $x_A \in {}^{\mathcal{F}(\mathfrak{X}_A)}\mathrm{LIM}_{\mathfrak{x} \in \mathfrak{X}_A}\, B(\mathfrak{x})$ for each $A \in \mathfrak{a}$. Let $U \in \mathfrak{B}_\tau x$. Since $\sum_{\mathfrak{a}}$ is continuous, there is a family $(U_A)_{A \in \mathfrak{a}}$ with $U_A \in \mathfrak{B}_\tau x_A$ for all $A \in \mathfrak{a}$ such that $\sum_{A \in \mathfrak{a}} U_A \subseteq U$. For each

$A \in \mathfrak{a}$, there is a $\zeta(A) \in \mathfrak{X}_A$ such that

(1) $\emptyset \neq B_A(\mathfrak{x}) \subseteqq U_A$ for all $\mathfrak{x} \in \mathfrak{X}_A$ such that $\zeta(A) \leqq_A \mathfrak{x}$

and a $\xi(A) \in \mathfrak{X}$ such that $(\xi(A))_A = \zeta(A)$. Since $\mathfrak{a}$ is finite, there is an $\mathfrak{x}_0 \in \mathfrak{X}$ such that $\xi(A) \leqq \mathfrak{x}_0$ holds for all $A \in \mathfrak{a}$. Now let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_0 \leqq \mathfrak{x}$. Then, $\zeta(A) \leqq_A \mathfrak{x}_A \in \mathfrak{X}_A$ and therefore (by (1)) $\emptyset \neq B_A(\mathfrak{x}_A) \subseteqq U_A$ holds for all $A \in \mathfrak{a}$, thus $\emptyset \neq \sum_{A \in \mathfrak{a}} B_A(\mathfrak{x}_A) \subseteqq \sum_{A \in \mathfrak{a}} U_A \subseteqq U$. Thus $x \in C_4$.

4. We have proved that $C_6 \subseteqq C_1$. On the other hand, by Theorem 8, $C_1 \neq \emptyset$ implies $C_6 \neq \emptyset$, since $\mathfrak{a}$ is finite. Thus, by Proposition 1, $C_1 = C_6$ ∎

Without the supposition of the finiteness of $\mathfrak{a}$, the following remains true.

Theorem 10 (cf. [5, 4.6.3, 4.4.5 and 4.2.3]): *One has the inclusion*

$$\int^{\mathfrak{X}} f\, d\mu \subseteqq \sum_{A \in \mathfrak{a}} \int^{\mathfrak{X}_A} (f|A)\, d(\mu_A).$$

Proof (cf. [5, p. 60]): Put $J := \int^{\mathfrak{X}} f\, d\mu$ and, for each $A \in \mathfrak{a}$, $J_A := \int^{\mathfrak{X}_A} (f|A)\, d(\mu_A)$. Let $x \in J$; then (by Theorem 8) $J_A \neq \emptyset$ for all $A \in \mathfrak{a}$. As an auxiliary mapping we choose a one-to-one mapping χ on $\mathfrak{a}$ into $\mathbb{N}$. Let $\varepsilon > 0$. Since $J \neq \emptyset$, there is (by the Corollary to Proposition 22) a $\mathfrak{y} \in \mathfrak{X}$ such that

(1) $\emptyset \neq B(\mathfrak{y}) - J \subseteqq \tau B_\varepsilon(0)$.

For each $A \in \mathfrak{a}$, there is (by the same Corollary), because of $J_A \neq \emptyset$, a $\zeta(A) \in \mathfrak{X}_A$ such that

(2) $\emptyset \neq B_A(\zeta(A)) - J_A \subseteqq \tau B_{\delta(A)}(0)$, where $\delta(A) := \varepsilon 2^{-\chi(A)}$.

Let $\mathfrak{z} := \bigcup_{A \in \mathfrak{a}} \zeta(A)$; then $\mathfrak{z} \in \Omega(\mu)$. Let $\mathfrak{x} := \mathfrak{y} \vee \mathfrak{z}$; then

(3) $\mathfrak{x} = \bigcup_{A \in \mathfrak{a}} \mathfrak{x}_A$.

Since $\mathfrak{y} \leqq \mathfrak{x}$ and $\zeta(A) \leqq_A \mathfrak{x}_A$ for all $A \in \mathfrak{a}$, one has by Proposition 21, (1) and (2)

$$\emptyset \neq B(\mathfrak{x}) \subseteqq \tau \operatorname{co} B(\mathfrak{y}) \quad \text{and} \quad \emptyset \neq B_A(\mathfrak{x}_A) \subseteqq \tau \operatorname{co} B_A(\zeta(A))\ \forall A \in \mathfrak{a},$$

therefore (by (1))

(4) $\emptyset \neq B(\mathfrak{x}) - J \subseteqq \tau B_\varepsilon(0)$

and (by (2))

(5) $\emptyset \neq B_A(\mathfrak{x}_A) - J_A \subseteqq \tau B_{\delta(A)}(0) \qquad$ for all $\; A \in \mathfrak{a}$.

Since $\sum_{A \in \mathfrak{a}} \tau B_{\delta(A)}(0)$ is non-empty, one obtains in view of (5) and Proposition 7/b

(6) $\qquad \emptyset \neq \sum_{\mathfrak{a} \in A} \left(B_A(\mathfrak{x}_A) - J_A\right) \subseteqq \tau B_\varepsilon(0),$

while one gets, by (3), Proposition 11 and (4), $\emptyset \neq B(\mathfrak{x}) = \sum_{A \in \mathfrak{a}} B_A(\mathfrak{x}_A)$. Therefore, by Proposition 10,

$$\emptyset \neq B(\mathfrak{x}) - \sum_{A \in \mathfrak{a}} \left(B_A(\mathfrak{x}_A) - J_A\right) = \sum_{A \in \mathfrak{a}} \left(B_A(\mathfrak{x}_A) - \left(B_A(\mathfrak{x}_A) - J_A\right)\right).$$

Since $eJ_A \in B_A(\mathfrak{x}_A) - \left(B_A(\mathfrak{x}_A) - J_A\right)$ holds for all $A \in \mathfrak{a}$, one has consequently

$$\emptyset \neq \sum_{A \in \mathfrak{a}} J_A \subseteqq B(\mathfrak{x}) - \sum_{A \in \mathfrak{a}} \left(B_A(\mathfrak{x}_A) - J_A\right) \subseteqq J + \tau B_{2\varepsilon}(0)$$

(by (4) and (6)), thus, by the choice of x and ε, $x \in \sum_{A \in \mathfrak{a}} J_A$ ∎

§ 6 Haupt-Aumann-Pauc integral, Pettis integral and Birkhoff integral, some elementary properties of the $(\mu, \mathfrak{X})$-integral

Given an $\mathfrak{M}$-ary algebra (K, Θ), where $\mathfrak{M}$ is a non-empty class of filtered sets (for the terminology, see [13, p. 120]) and a non-empty set D, we define the $\mathfrak{M}$-ary operation $\Theta^{(D)}$ in K^D by letting, for each $g \in K^D$ and each $(h, I, \mathfrak{a}) \in \Phi(K^D)$ with $(I, \mathfrak{a}) \in \mathfrak{M}$,

$$g = \Theta^{(D)}(h, I, \mathfrak{a}) \quad \text{if} \quad g(d) = \Theta(\mathrm{pr}_d \circ h, I, \mathfrak{a}) \; \forall\, d \in D,$$

where pr_d denotes the d-th projection $\mathrm{pr}_d: K^D \to K$. In particular, let $(K, \Theta) = (\mathfrak{E}, \lim^\wedge)$; thus $\mathrm{Dmn}\,\Theta = \Phi\mathfrak{E}$. Then, the $\mathfrak{M}$-ary operation $(\lim^\wedge)^{(D)}$, where $\mathfrak{M}$ is now the class of all filtered sets, is a natural extension of the pointwise convergence in the Hausdorff space $(E, \tau)^D$ ($= D$-th power of the Hausdorff space (E, τ)), and one has $\mathrm{Dmn}\,(\lim^\wedge)^{(D)} = \Phi(\mathfrak{E}^D)$. (Recall, for the next, the terminology introduced in § 0/d).) Let $x \in E$. If one defines D by $D = Q\mathfrak{X}$, the mapping g_x by $g_x(\varphi) = \{x\}$ for all $\varphi \in Q\mathfrak{X}$, and the filtered family $(h, I, \mathfrak{a})$ by $I = \mathfrak{X}$, $\mathfrak{a} = \mathcal{F}\mathfrak{X}$ and $h(\varphi) = \big(R(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot))\big)_{\mathfrak{x}\in\mathfrak{X}}$ for all $\varphi \in Q\mathfrak{X}$, then

$$x \in \int^{\mathfrak{X}} f\, d\mu \quad \text{iff} \quad g_x = (\lim^\wedge)^{(D)}(h, I, \mathfrak{a}).$$

Expressed in other terms, this assertion says that the first equation in the following theorem holds:

Theorem 11 (cf. [5, 5.6.2]): *One has the equation*

$$\int^{\mathfrak{X}} f\, d\mu = \bigcap_{\varphi\in Q\mathfrak{X}} {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} R\big(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)\big).$$

Especially, $\int^{\mathfrak{X}} f\, d\mu \neq \emptyset$ *implies* $\int^{\mathfrak{X}} f\, d\mu = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} R\big(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)\big)$ *for all* $\varphi \in Q\mathfrak{X}$!

Proof: 1. The validity of $\subseteq$ instead of $=$ in the first assertion follows immediately from the definitions. (For details in another setting, cf. [12, proof of Satz 6].) 2. Let $x \in E \setminus \int^{\mathfrak{X}} f\, d\mu$. Then, there is a $U \in \mathfrak{B}_\tau x$ and a $G \in \mathcal{S}(\mathcal{F}\mathfrak{X}^{\#})$ (for the notation, see § 0/b)) such that

(1) $R(f, \mathfrak{x}, \psi) \cap U = \emptyset$ for all $(\mathfrak{x}, \psi) \in G$.

Choose, as auxiliary mappings, $\varkappa \in Q\mathfrak{X}$ and $\lambda: \mathrm{Dmn}\, G \to \mathrm{Rng}\, G$ such that $\lambda \subseteq G$ (use of the axiom of choice); consider that G is a relation. Define a special element φ of $Q\mathfrak{X}$ by letting, for each $(\mathfrak{x}, X) \in \mathcal{F}\mathfrak{X}$, $\varphi(\mathfrak{x}, X) = \big(\lambda(\mathfrak{x})\big)(X)$ if $\mathfrak{x} \in \mathrm{Dmn}\, G$ and $\varphi(\mathfrak{x}, X) = \varkappa(\mathfrak{x}, X)$ if $\mathfrak{x} \in (\mathfrak{X} \setminus \mathrm{Dmn}\, G)$. Then, in view of (1), $R\big(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)\big) \cap U = \emptyset$ holds for all $\mathfrak{x} \in \mathrm{Dmn}\, G$. Since $U \in \mathfrak{B}_\tau x$ and $\mathrm{Dmn}\, G \in \mathcal{S}(\mathcal{F}\mathfrak{X})$, we have showed that x belongs to $E \setminus {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} R\big(f, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)\big)$ ∎

Remark 9: The remarks preceding Theorem 11 justify to call the $(\mu, \mathfrak{X})$-integral a "pointwise integral", the choice functions φ being considered as "points" (see the final remark in [12, p. 94]). The special case in Theorem 11 may be interpreted also by saying that the choice functions φ can be "drawn out from" the limit process originally defining $\int^{\mathfrak{X}} f\, d\mu$ (cf. Definition 5/b).

The relationship to the classical theory of integration is clarified next.

For the remainder of this paper, let $\int^{\mathrm{H}} \cdot\, d\mu$ denote the *Haupt-Aumann-Pauc integral* (see [16, pp. 94–95]), being redefined in [12, p. 85 and p. 86, Beispiel 9; there called the "$(\mu, \mathfrak{X})$-Unterteilungsintegral", where $\mathfrak{X}$ denotes the set of all countable partitions $\mathfrak{x}$ of F such that $\mathfrak{x} \subseteq \operatorname{Dmn} \mu$], now being considered (in an obvious way) as an F-ary partial operation in the extended real line $\overline{\mathbb{R}}(= \mathbb{R} \cup \{+\infty, -\infty\})$. Let

$$M(\mu) := \left\{ g \mid g \in \mathbb{R}^F \text{ and } g \in \operatorname{Dmn}\left(\overset{\mathrm{H}}{\int} \cdot\, d\mu\right) \text{ and } \overset{\mathrm{H}}{\int} g\, d\mu < +\infty \right\}.$$

Expressed in other terms, $M(\mu)$ is the set of all functions $g: F \to \mathbb{R}$, which are μ-summable (in the terminology of HAUPT, AUMANN and PAUC [16, p. 95] "μ-summierbar über F"). (Realize that $\left(\overset{\mathrm{H}}{\int} \cdot\, d\mu\right) \Big| M(\mu)$ is an F-ary partial operation in $\mathbb{R}$; see § 0/a).)

Theorem 12: *Let* $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. *Then, one has* $\left(\left(\overset{\mathrm{H}}{\int} \cdot\, d\mu\right) \Big| M(\mu)\right)^{\wedge} = \overset{\Omega}{\int} \cdot\, d\mu$ *(for the terminology, see* § 0/a)).

Proof: Let $\mathfrak{Z} = \{\mathfrak{z} \mid \mathfrak{z} \subseteq \operatorname{Dmn} \mu$ and $\mathfrak{z}$ is a countable partition of $F\}$. (Observe that $\mathfrak{Z}$ plays here the role of the "Zerlegungssystem" $\mathfrak{X}$ occuring in [12, p. 85, Beispiel 9].) Let, for this proof, $\leqq_{\mathfrak{Z}}$ be the relation on $\mathfrak{Z}$ being defined by letting, for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Z}$, $\mathfrak{x} \leqq_{\mathfrak{Z}} \mathfrak{y}$ if, for all $Y \in \mathfrak{y}$, $Y \subseteq X$ holds for some $X \in \mathfrak{x}$. Clearly, $(\mathfrak{Z}, \leqq_{\mathfrak{Z}})$ is a directed set, and $(\leqq_{\mathfrak{Z}} | \Omega) = \leqq$. Recall that $\Omega \neq \emptyset$. Choose $\mathfrak{x}_0 \in \Omega$. Let $\mathfrak{Y} = \{\mathfrak{y} \in \Omega \mid \mathfrak{x}_0 \leqq \mathfrak{y}\}$. Then, one has $\mathfrak{Y} = \{\mathfrak{z} \in \mathfrak{Z} \mid \mathfrak{x}_0 \leqq_{\mathfrak{Z}} \mathfrak{z}\}$, thus by (0.2)

(1) $\mathfrak{Y} \in \mathcal{F}(\mathfrak{Z}, \leqq_{\mathfrak{Z}})$ and $\mathcal{F}\mathfrak{Y} = \mathcal{F}(\mathfrak{Z}, \leqq_{\mathfrak{Z}})_{\mathfrak{Y}}$.

Let $f: F \to \mathfrak{R}$ and $x \in \mathbb{R}$. Define the statements (2)—(5) in the following way:

(2) $x \in \left(\overset{\mathrm{H}}{\int} \cdot\, d\mu\right)^{\wedge}(f)$.

(3) $x \in {}^{\mathcal{F}\mathfrak{Y}}\lim\limits_{\mathfrak{y} \in \mathfrak{Y}} R(f, \mathfrak{y}, \varphi(\mathfrak{y}, \cdot))$ for all $\varphi \in \mathcal{Q}\mathfrak{Y}$.

(4) $x \in \overset{\mathfrak{Y}}{\int} f\, d\mu$.

(5) $x \in \overset{\Omega}{\int} f\, d\mu$.

Then, by Satz 22 in [12], (1) above, (0.1) and the remarks in § 0/d), the statement (2) is equivalent to the statement (3). Furthermore, by Theorem 11, (3) is equivalent to (4); by the definition of $\mathfrak{Y}$ and Proposition 19, (4) is equivalent to (5) ∎

Remark 10: Theorem 4 together with Theorem 11 might be even of interest in the theory of numerical integration: Given a classical situation as $(E, \|\cdot\|) := (\mathbb{R}, |\cdot|)$ and $\mathfrak{Y} := \Omega$. Let $g: F \to E$ and $(\mathfrak{x}_n)_{n \in \mathbb{N}}$ be a sequence of μ-partitions of F, say with $\mathfrak{x}_n \leqq \mathfrak{x}_{n+1}$ for all $n \in \mathbb{N}$, such that $\{\mathfrak{x}_n \mid n \in \mathbb{N}\}$ is a μ-partition system, say $\mathfrak{X}$. Then, if one can show that $\overset{\mathfrak{X}}{\int} g^{\sim}\, d\mu \neq \emptyset$ with $g^{\sim}(u) := \{g(u)\}$ for all $u \in F$, then Theorem 4 assures that $\overset{\mathfrak{X}}{\int} g^{\sim}\, d\mu = \overset{\Omega}{\int} g^{\sim}\, d\mu$. Conversely, if one assumes $\overset{\mathfrak{X}}{\int} g^{\sim} d\mu \neq \emptyset$, Theorem 11 provides us with a mean to approximate $e\left(\overset{\mathfrak{X}}{\int} g^{\sim}\, d\mu\right)$ numerically: Choose $\varphi \in \mathcal{Q}X$ and approximate ${}^{\mathcal{F}\mathfrak{X}}\lim\limits_{\mathfrak{x} \in \mathfrak{X}} R(g^{\sim}, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)) = \lim\limits_{n \in \mathbb{N}} R(g^{\sim}, \mathfrak{x}_n, \varphi(\mathfrak{x}_n, \cdot))$ (where $=$ holds, since $\mathfrak{x}_n \leqq \mathfrak{x}_{n+1}$ for all $n \in \mathbb{N}$). By Theorem 11, one has ${}^{\mathcal{F}\mathfrak{X}}\lim\limits_{\mathfrak{x} \in \mathfrak{X}} R(g^{\sim}, \mathfrak{x}, \varphi(\mathfrak{x}, \cdot)) = \overset{\mathfrak{X}}{\int} g^{\sim}\, d\mu$. — This procedure of approximation of an integral is well known from the classical Riemann integral. For illustration, take, in particular, F to be a non-empty compact subset of $\mathbb{R}^m (m \in \mathbb{N})$, μ to be the natural restriction of the Lebesgue measure λ_m in $\mathbb{R}^m$ to $(\mathfrak{P}F) \cap \cap (\operatorname{Dmn} \lambda_m)$, and g to be continuous. Then $\int^{\Omega} g^{\sim} \, d\mu$ is non-empty and $e\left(\int^{\Omega} g^{\sim} \, d\mu\right)$ is the Riemann integral of g over F (by Theorem 12 and in view of the relationship between the Riemann integral and $\int^{H} \cdot \, d\mu$ (see HAUPT, AUMANN and PAUC [16, pp. 189–190]). Assume that $\mathfrak{x}_n$ be finite for all $n \in \mathbb{N}$ and that $0 = \lim_{n \in \mathbb{N}} \max_{X \in \mathfrak{x}_n} \operatorname{diam} X$. Then, $\int^{\mathfrak{X}} g^{\sim} \, d\mu$ turns out to be non-empty, and, in view of the statements made above, given some $\varphi \in \Omega\mathfrak{X}$, one has $\lim_{n \in \mathbb{N}} R(g^{\sim}, \mathfrak{x}_n, \varphi(\mathfrak{x}_n, \cdot)) = \int^{\Omega} g^{\sim} \, d\mu$, as it is, essentially, well known (in the classical terminology) from any textbook on real analysis.

Proposition 25: *Let $g: F \to \mathfrak{E}$. Then, one has (a):*

(a) *If $f(u) \subseteqq g(u)$ holds for all $u \in F$, then $\int^{\mathfrak{X}} f \, d\mu \subseteqq \int^{\mathfrak{X}} g \, d\mu$.*

Proof: One uses (0.3), (0.9) and (0.4) ∎

Proposition 26: *If $g: F \to \mathfrak{E}$, then $\int^{\mathfrak{X}} f \, d\mu + \int^{\mathfrak{X}} g \, d\mu \subseteqq \int^{\mathfrak{X}} (f + g) \, d\mu$. (For the terminology, see § 0/c).)*

Proof: One uses that $(A + B) \alpha = A\alpha + B\alpha$ holds for all $A, B \in \mathfrak{E}$ and all $\alpha \in \mathcal{K}$, (0.3)–(0.5), and (0.10) ∎

Proposition 27: *One has the inclusion $\left(\int^{\mathfrak{X}} f \, d\mu\right) \alpha \subseteqq \int^{\mathfrak{X}} (f\alpha) \, d\mu$. (For the terminology, see § 0/c).)*

Proof: One uses (0.3), (0.4), (0.6), and (0.11) ∎

The $(\mu, \mathfrak{X})$--integral is preserved under continuous linear mappings on E into another Banach space E'; more precisely, one has

Proposition 28 (cf. [5, 4.5.1]): *Let E' be a Banach space and $\int'^{\mathfrak{X}} \cdot \, d\mu$ the $(\mu, \mathfrak{X})$-integral related to E' (instead of E). Let $\varphi: E \to E'$ be a continuous linear mapping. Then*

$$\varphi\left(\int^{\mathfrak{X}} f \, d\mu\right) \subseteqq \int'^{\mathfrak{X}} (\varphi \circ^* f) \, d\mu,$$

where $\varphi \circ^* f$ is defined by $(\varphi \circ^* f)(u) = \varphi(f(u))$ for all $u \in F$.

Proof: If $(g, K, \mathfrak{b}) \in \Phi\,\mathfrak{E}$ and $\psi: E \to E'$ is (τ, τ')-continuous, then

$$\psi\left({}^{\mathfrak{b}}\lim_{k \in K}{}_{\tau}{}^{\wedge}\, g(k)\right) \subseteqq {}^{\mathfrak{b}}\lim_{k \in K}{}_{\tau'}{}^{\wedge}\, \psi(g(k)),$$

where $\lim_{\tau'}$ denotes the limit operation w.r. to the norm topology τ' of E' and $\lim_{\tau'}{}^{\wedge}$ is defined analogously as $\lim_{\tau}{}^{\wedge}$. Use this fact for $\psi = \varphi$, furthermore (0.3), (0.4), and Definition 5, furthermore the linearity of φ ∎

Observe that in Proposition 28, $\varphi \circ^* f$ is not the usual composition of f and φ, since $\operatorname{Rng} f \subseteqq \mathfrak{E}$, but $\operatorname{Dmn} \varphi = E$.

Remark 11: We define a relation $\int^{P} \cdot \, d\mu$ between E^F and E by letting, for each $g: F \to E$ and each $x \in E$, $g\left(\int^{P} \cdot \, d\mu\right) x$ if one has (a):

(a) $\psi(x) \in \int^{\Omega} (\psi \circ g)^{\sim} \, d\mu$ for all $\psi \in E^*$ (= dual of E), where $(\psi \circ g)^{\sim}(y) = \{(\psi \circ g)(y)\}$ for all $y \in F$. If $\mathcal{K} = \mathbb{R}$, then, by Theorem 12, (a) is equivalent to (b):

(b) $\psi \circ g \in \operatorname{Dmn}\left(\overset{H}{\int} \cdot\, d\mu\right)$ and $\psi(x) = \overset{H}{\int} (\psi \circ g)\, d\mu$ for all $\psi \in E^*$.

The relation $\overset{P}{\int} \cdot\, d\mu$ is an F-ary partial operation in E being called the *Pettis integral* (see HILLE and PHILLIPS [17, p. 77]). Using Proposition 28 and Proposition 18, one obtains (c) and (d):

(c) $\overset{\Omega}{\int} f\, d\mu \subseteq \left(\overset{P}{\int} \cdot\, d\mu\right)^{\wedge}(f)$.

(d) If $(E, \|\cdot\|) = (\mathcal{K}, |\cdot|)$, then $\overset{\Omega}{\int} \cdot\, d\mu = \left(\overset{P}{\int} \cdot\, d\mu\right)^{\wedge}$.

Proposition 29: *One has* $\left\| \overset{\mathfrak{X}}{\int} f\, d\mu \right\| \leqq \overset{\mathfrak{X}}{\int} \|f\|\, d\mu$, *where the integration on the right-hand side is related to the Banach space* $\mathbb{R}$. (For the terminology, see § 0/c).)

Proof: By (0.7), (0.8) and (0.12), one has the chain

$$\left\| {}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(\mathfrak{x},\varphi)\in\mathfrak{X}^{\#}} R(f, \mathfrak{x}, \varphi) \right\| \subseteq {}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(\mathfrak{x},\varphi)\in\mathfrak{X}^{\#}} \|R(f, \mathfrak{x}, \varphi)\| \leqq {}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(\mathfrak{x},\varphi)\in\mathfrak{X}^{\#}} R(\|f\|, \mathfrak{x}, \varphi) \blacksquare$$

Proposition 30: *Beside* f, *let be given* $g\colon F \to \mathfrak{E}$. *Assume* f *and* g *to be singleton-valued and* $f(u) = g(u)$ *for* μ*-almost all* $u \in F$. *Then* $\overset{\Omega}{\int} f\, d\mu = \overset{\Omega}{\int} g\, d\mu$.

Proof: Omitting the trivial case $f = g$, we assume $f \neq g$. Then, there exists an $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$, $M \neq \emptyset$ and $f\,|(F \setminus M) = g|\,(F \setminus M)$. Let $x \in \overset{\Omega}{\int} f\, d\mu$ and $U \in \mathfrak{B}_\tau x$. Then, there is an $\mathfrak{x}_0 \in \Omega$ such that

(1) $\emptyset \neq R(f, \mathfrak{x}, \varphi) \subseteq U$ for all $\mathfrak{x} \in \Omega$ with $\mathfrak{x}_0 \leqq \mathfrak{x}$ and all $\varphi \in \mathcal{P}\mathfrak{x}$.

Let $\mathfrak{x}_1 = (\{X \cap M \mid X \in \mathfrak{x}_0\} \cup \{X \setminus M \mid X \in \mathfrak{x}_0\}) \setminus \{\emptyset\}$. Then, $\mathfrak{x}_1 \in \Omega(\mu)$ and $\mathfrak{x}_0 \leqq \mathfrak{x}_1$. Let $\mathfrak{x} \in \Omega$ with $\mathfrak{x}_1 \leqq \mathfrak{x}$ and $\varphi \in \mathcal{P}\mathfrak{x}$. Let, for abbreviation, $\mathfrak{y}(\mathfrak{x}) = \{X \in \mathfrak{x} \mid X \subseteq F \setminus M\}$. If $\mathfrak{y}(\mathfrak{x}) = \emptyset$, then $R(f, \mathfrak{x}, \varphi) = \{0\} = R(g, \mathfrak{x}, \varphi)$ (where 0 denotes the zero vector of E), since $\mu M = 0$ and f and g are singleton-valued. If $\mathfrak{y}(\mathfrak{x}) \neq \emptyset$, then

$$R(f, \mathfrak{x}, \varphi) = \sum_{X \in \mathfrak{y}(\mathfrak{x})} f(\varphi(X))\, \mu X = \sum_{X \in \mathfrak{y}(\mathfrak{x})} g(\varphi(X))\, \mu X = R(g, \mathfrak{x}, \varphi),$$

where we used that $\mu M = 0$, f and g are singleton-valued, $f|(F \setminus M) = g|\,(F \setminus M)$, and (2.1). By (1), one has therefore $\emptyset \neq R(g, \mathfrak{x}, \varphi) \subseteq U$, thus, by the choice of $\mathfrak{x}, \varphi$ and U, $x \in \overset{\Omega}{\int} g\, d\mu$ ∎

§ 7. Bochner integral and Birkhoff integral

In order to give a self-contained representation of the relationship between the Bochner integral and the Birkhoff integral (discussed by BIRKHOFF [2, p. 377] in a way based on a definition of the Bochner integral using "finite-valued" step functions) we discuss as a preparation, first, the question, in which way "convergence in mean w.r.t. to μ and $\mathfrak{X}$" is compatible with the integration $\overset{\mathfrak{X}}{\int} \cdot\, d\mu$. But we will refrain to introduce such a notion of convergence explicitly. — In this section, we agree that, for each $j\colon F \to \mathfrak{R}$ (see § 0/c)), $\overset{\mathfrak{X}}{\int} j\, d\mu$ denotes the $(\mu, \mathfrak{X})$-integral of j w.r.t. the Banach space $(\mathbb{R}, |\cdot|)$.

For this section, let $(I, \mathfrak{a})$ be a filtered set and $h = (g_i)_{i \in I}$ be a family of mappings $g_i: F \to \mathfrak{E}$. We denote, for abbreviation, by $H(f, h, \mathfrak{a})$ the statement:

$$H(f, h, \mathfrak{a}): \quad \text{If } 0 \in {}^{\mathfrak{a}}\lim_{i \in I} \int^{\mathfrak{X}} \|f - g_i\| \, d\mu, \text{ then } {}^{\mathfrak{a}}\lim_{i \in I} \int^{\mathfrak{X}} g_i \, d\mu \subseteqq \int^{\mathfrak{X}} f \, d\mu,$$

where the first sign lim stands for $(\lim_\sigma)^\wedge$ with the Euclidean topology σ of $\mathbb{R}$, $(\mathbb{R}, |\cdot|)$ taken as Banach space (for the terminology, see § 0/b) and c)).

Proposition 31: *Assume that $R(f, \mathfrak{x}, \varphi)$ is non-empty for $\mathcal{F}\mathfrak{X}^\#$-almost all $(\mathfrak{x}, \varphi) \in \mathfrak{X}^\#$. Then $H(f, h, \mathfrak{a})$ holds.*

Proof: By supposition, there is an $\mathfrak{x}_0 \in \mathfrak{X}$ such that

(1) $R(f, \mathfrak{x}, \varphi) \neq \emptyset$ for all $(\mathfrak{x}, \varphi) \in \mathfrak{X}^\#$ with $\mathfrak{x}_0 \leqq \mathfrak{x}$ and $\varphi \in \mathcal{P}\mathfrak{x}$.

Assume

(2) $0 \in {}^{\mathfrak{a}}\lim_{i \in I} \int^{\mathfrak{X}} \|f - g_i\| \, d\mu$.

Let $\varepsilon > 0$. Because of (2), there is a set $A_0 \in \mathfrak{a}$ such that

(3) $\emptyset \neq \int^{\mathfrak{X}} \|f - g_i\| \, d\mu < (\varepsilon/4)$ for all $i \in A_0$.

Therefore (use of the sign $\neq$ in (3)), for each $i \in A_0$ there is an $\mathfrak{y}_i \in \mathfrak{X}$ such that

(4) $\emptyset \neq \left| \int^{\mathfrak{X}} \|f - g_i\| \, d\mu - R(\|f - g_i\|, \mathfrak{x}, \varphi) \right| < (\varepsilon/4) \ \forall (\mathfrak{x}, \varphi): \mathfrak{y}_i \leqq \mathfrak{x} \in \mathfrak{X}, \varphi \in \mathcal{P}\mathfrak{x}$.

We choose such a family $(\mathfrak{y}_i)_{i \in A_0}$ now (axiom of choice). (3) and (4) imply that

(5) $\emptyset \neq R(\|f - g_i\|, \mathfrak{x}, \varphi) < (\varepsilon/2) \ \forall (i, \mathfrak{x}, \varphi): i \in A_0, \mathfrak{y}_i \leqq \mathfrak{x} \in \mathfrak{X}, \varphi \in \mathcal{P}\mathfrak{x}$.

Let $x \in {}^{\mathfrak{a}}\lim_{i \in I} \int^{\mathfrak{X}} g_i \, d\mu$. Then there exists an $A_1 \in \mathfrak{a}$ such that

(6) $\emptyset \neq \left\| \{x\} - \int^{\mathfrak{X}} g_i \, d\mu \right\| < (\varepsilon/4)$ for all $i \in A_1$.

For each $i \in A_1$, one has $\int^{\mathfrak{X}} g_i \, d\mu \neq \emptyset$ (by the sign $\neq$ in (6)), thus, there exists a $\mathfrak{z}_i \in \mathfrak{X}$ such that

(7) $\emptyset \neq \left\| \int^{\mathfrak{X}} g_i \, d\mu - R(g_i, \mathfrak{x}, \varphi) \right\| < (\varepsilon/4) \ \forall (\mathfrak{x}, \varphi): \mathfrak{z}_i \leqq \mathfrak{x} \in \mathfrak{X}, \varphi \in \mathcal{P}\mathfrak{x}$.

We choose such a family $(\mathfrak{z}_i)_{i \in A_1}$ now (axiom of choice). Choose $i \in A_0 \cap A_1$ (which is possible, since $\mathfrak{a}$ is a filter). Choose $\mathfrak{x}_1 \in \mathfrak{X}$ with $\mathfrak{x}_0 \leqq \mathfrak{x}_1$, $\mathfrak{y}_i \leqq \mathfrak{x}_1$, and $\mathfrak{z}_i \leqq \mathfrak{x}_1$. Let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_1 \leqq \mathfrak{x}$, and let $\varphi \in \mathcal{P}\mathfrak{x}$. By (1), one has $R(f, \mathfrak{x}, \varphi) \neq \emptyset$, by (7), one has $R(g_i, \mathfrak{x}, \varphi) \neq \emptyset$. Hence, one obtains (using (0.9)–(0.12)) from (5) the chain

(8) $\emptyset \neq \|R(f, \mathfrak{x}, \varphi) - R(g_i, \mathfrak{x}, \varphi)\| \subseteqq \|R(f - g_i, \mathfrak{x}, \varphi)\| \leqq_0 R(\|f - g_i\|, \mathfrak{x}, \varphi) < (\varepsilon/2)$

and therefore (observe the definition of the relation $\leqq_0$ in § 0)

(9) $\emptyset \neq \|R(f, \mathfrak{x}, \varphi) - R(g_i, \mathfrak{x}, \varphi)\| < (\varepsilon/2)$.

Combining (6), (7), and (9), one gets $\emptyset \neq \|\{x\} - R(f, \mathfrak{x}, \varphi)\| < \varepsilon$ ∎

Proposition 32: *If* (a) *or* (b), *then* $H(f, h, \mathfrak{a})$ *holds, where the statements* (a) *and* (b) *are defined as follows:*

(a) *Each* $\mathfrak{x} \in \mathfrak{X}$ *is finite.*

(b) *The linear space* E *is finite-dimensional.*

Proof: 1. Copy the proof of Proposition 31 beginning with "Assume (2)" and ending with the choice of $i \in A_0 \cap A_1$. Choose $\mathfrak{x}_1 \in \mathfrak{X}$ with $\mathfrak{y}_i \leqq \mathfrak{x}_1$ and $\mathfrak{z}_i \leqq \mathfrak{x}_1$. Let $\mathfrak{x} \in \mathfrak{X}$ with $\mathfrak{x}_1 \leqq \mathfrak{x}$ and $\varphi \in \mathscr{P}\mathfrak{x}$. Continue as follows. 2a. Assume (a). In view of (2), there is an $A_2 \in \mathfrak{a}$ such that, for all $i \in A_2$ and all $u \in F$, $\|f(u) - g_i(u)\| \neq \emptyset$, thus $f(u) \neq \emptyset$ (use of Proposition 18 in the Banach space $\mathbb{R}$), therefore $R(f, \mathfrak{x}, \varphi) \neq \emptyset$ since $\mathfrak{x}$ is finite. 2b. Assume (b). Since $R(g_i, \mathfrak{x}, \varphi) \neq \emptyset$ (by (7)) and E is finite-dimensional, one has $R(\|g_i\|, \mathfrak{x}, \varphi) \neq \emptyset$ (by the Dvoretzky-Rogers Theorem, see [22, p. 27]). On the other hand, (by (5)) one has $R(\|f - g_i\|, \mathfrak{x}, \varphi) \neq \emptyset$, thus $R(f, \mathfrak{x}, \varphi) \neq \emptyset$ (by Theorem 14 in [13]). 3. For the remainder of this proof, we assume "(a) or (b)". By (7), one has $R(g_i, \mathfrak{x}, \varphi) \neq \emptyset$. Hence, since $R(f, \mathfrak{x}, \varphi) \neq \emptyset$ (by Part 2a and 2b of this proof) one obtains from (5) the chain (8) in the proof of Proposition 31. Now, one copies the remainder of that proof word by word ∎

Example 2: If E is in particular the Banach space $\mathbb{R}$, the statement $H(f, h, \mathfrak{a})$ holds.

For each mapping $g: F \to E$, we put $g^{\sim} = e^{-1} \circ g$ (for the definition of the mapping e, see § 0/a)). One has $e \circ g^{\sim} = g$ and, if f is singleton-valued, $(e \circ f)^{\sim} = f$.

Definition 8 (see Hille and Phillips [17, p. 78]): We define a relation $\int^{B} \cdot\, d\mu$ by letting, for each $g: F \to E$ and each $x \in E$, $g\left(\int^{B} \cdot\, d\mu\right) x$ hold if there is a sequence $(g_n)_{n \in \mathbb{N}}$ of Ω-step functions $g_n: F \to E$ such that (a)–(d) hold:

(a) $\int^{\Omega} \|g_n^{\sim}\| \, d\mu \neq \emptyset \quad \forall\, n \in \mathbb{N}$; (b) $0 \in \lim\limits_{n \in \mathbb{N}} \int^{\Omega} \|g^{\sim} - g_n^{\sim}\| \, d\mu$;

(c) $g(u) = \lim\limits_{n \in \mathbb{N}} g_n(u)$ for μ-a. a. $u \in F$; (d) $x \in \lim\limits_{n \in \mathbb{N}} \int^{\Omega} g_n^{\sim} \, d\mu$.

The relation $\int^{B} \cdot\, d\mu$ (between E^F and E) is an F-ary partial operation in E, which is called *Bochner integral*, and we write $\left(\int^{B} \cdot\, d\mu\right)(g) = \int^{B} g\, d\mu$ if $g \in \operatorname{Dmn}\left(\int^{B} \cdot\, d\mu\right)$. Furthermore, $\int^{B\wedge} \cdot\, d\mu$ denotes the 1-point completion $\left(\int^{B} \cdot\, d\mu\right)^{\wedge}$ of $\int^{B} \cdot\, d\mu$.

Remark 12: In the proof of the fact that $\int^{B} \cdot\, d\mu$ is a mapping, one uses (b) twice. — From (b) follows that $e\left(\int^{\Omega} g_n^{\sim}\, d\mu\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in E; therefore, if (b) holds, then $\lim\limits_{n \in \mathbb{N}} \int^{\Omega} g_n^{\sim} d\mu$ is non-empty. (But in Definition 8, we did not need this conclusion.) — Just for convenience, we used the (μ, Ω)-integral of Ω-step functions in Definition 8. By Theorem 5, these integrals could be replaced by certain sums. So the definition of the Bochner integral does not depend in any way on the definition of $(\mu, \mathfrak{X})$-integral. — Using measure-theoretic arguments similar to those occuring in the proof of Proposition 14 in Dinculeanu's book [4, p. 130], Sätze 3 and 4 in Haupt, Aumann and Pauc's book [16, pp. 96 and 100], furthermore Theorem 12, one obtains that, for each $g \in E^F$ and each sequence $(g_n)_{n \in \mathbb{N}}$ in E^F, the statement (b) in Definition 8 implies the statement (e) formulated next, provided that the measure μ is complete.

(e) There is a subsequence (g_{n_k}) of (g_n) such that $g(u) = \lim\limits_{k \in \mathbb{N}} g_{n_k}(u)$ holds for μ-almost all $u \in F$.

Therefore, if μ is complete, in Definition 8 the condition (c) is allowed to be cancelled.

For the remainder of this section, let $g\colon F \to E$.

Proposition 33: *Assume the measure μ to be complete. If $\int^{\mathrm{B}\,\wedge} g^{\sim} d\mu$ is non-empty, then $\int^{\Omega} \|g^{\sim}\|\, d\mu$ is non-empty.* (For the terminology, see § 0/c).)

Proof: Assume the premise. Choose $x \in \int^{\mathrm{B}\,\wedge} g^{\sim}\, d\mu$. Then $\int^{\mathrm{B}} g\, d\mu$. Therefore, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ as in Definition 8 satisfying (a)—(d) there. By (c), $\|g(u)\| = \lim_{n\in\mathbb{N}} \|g_n(u)\|$ holds for μ-almost all $u \in F$. The functions $\|g_n\|$ are μ-measurable, thus (since μ is a complete measure) so is also the function $\|g\|$ (see Haupt, Aumann and Pauc [16, p. 80]), therefore there is an $r \in \overline{\mathbb{R}}$ such that $r = \int^{\mathrm{H}} \|g\|\, d\mu$ (for the terminology, see § 6, before Theorem 12). By (a) and (b) in Definition 8, there is an $n \in \mathbb{N}$ such that the sets $A = \int^{\Omega} \|g^{\sim} - g_n^{\sim}\|\, d\mu$ and $B = \int^{\Omega} \|g_n^{\sim}\|\, d\mu$ are non-empty. Thus, one has $eA = \int^{\mathrm{H}} \|g - g_n\|\, d\mu$, $eB = \int^{\mathrm{H}} \|g_n\|\, d\mu$ and $\int^{\mathrm{H}} \|g\|\, d\mu \leqq eA + eB < +\infty$, therefore, by Theorem 12, $\int^{\Omega} \|g^{\sim}\|\, d\mu$ is non-empty. ∎

Theorem 13: *Assume the measure μ to be complete. Then $\int^{\mathrm{B}\,\wedge} f\, d\mu \subseteqq \int^{\Omega} f\, d\mu$.*

Proof: Let x be a member of the left-hand side. Then, (by Definition 8) f is singleton-valued and $x = \int^{\mathrm{B}} (e \circ f)\, d\mu$. By Proposition 33, one has $\int^{\Omega} \|f\|\, d\mu \neq \emptyset$, therefore $R(\|f\|, \mathfrak{x}, \varphi) \neq \emptyset$, thus $R(f, \mathfrak{x}, \varphi) \neq \emptyset$ for $\mathcal{F}\mathfrak{X}^{\#}$-almost all $(\mathfrak{x}, \varphi) \in \mathfrak{X}^{\#}$ with $\mathfrak{X} = \Omega$. Choose a sequence $h^* = (g_n)_{n\in\mathbb{N}}$ as in Definition 8 (with $g = e \circ f$). Then, by Proposition 31, the statement $H(f, h, \mathcal{F}\mathbb{N})$ holds, where h^* denotes the sequence $(g_n^{\sim})_{n\in\mathbb{N}}$. Since (b) in Definition 8 and $x \in \lim_{n\in\mathbb{N}} \int^{\Omega} g_n^{\sim}\, d\mu$ hold, we have therefore $x \in \int^{\Omega} f\, d\mu$ ∎

Expressed in classical terms, Theorem 13 says that for complete measure μ the Bochner integral $\int^{\mathrm{B}} \cdot\, d\mu$ is a restriction of the F-ary partial operation $\left(\int^{\Omega} \cdot\, d\mu\right)^{\wedge}$.

§ 8 Iterated integrals

From now on, the symbol f is not anymore reserved for a mapping on F into $\mathfrak{E}$. Since the domain of the mapping $\int^{\mathfrak{X}} \cdot\, d\mu$ is the whole set $\mathfrak{E}^F$, our technique of working in $\mathfrak{E}$ instead of E turns out to be quite efficient in dealing with iterated integrals: If, namely, as we assume for the next, a non-empty measure space (G, ν) with a σ-finite measure ν and a ν-partition system $\mathfrak{Y}$ of G are given beside (F, μ) and $\mathfrak{X}$, the *iterated integral* $\int^{\mathfrak{X}} \int^{\mathfrak{Y}} \cdot\, d\nu\, d\mu$ is defined to be the $F \times G$-ary operation in $\mathfrak{E}$ assigning to each $f \in \mathfrak{E}^{F\times G}$ the set

$$\int^{\mathfrak{X}} g\, d\mu \quad \text{with} \quad g(u) = \int^{\mathfrak{Y}} f(u, \cdot)\, d\nu \quad \text{for all} \quad u \in F.$$

Expressed in other terms, we have

$$\int^{\mathfrak{X}}\int^{\mathfrak{Y}} f\,d\nu\,d\mu = \int^{\mathfrak{X}}_{U\in F}\left(\int^{\mathfrak{Y}}_{v\in G} f(u, v)\,d\nu\right)d\mu$$

for all $f \in \mathfrak{C}^{F\times G}$. For the following considerations, we fix some $f\colon F\times G \to \mathfrak{C}$ and we define for each $\varphi \in Q\mathfrak{X}$ and each $\psi \in Q\mathfrak{Y}$ a mapping $h_{\varphi\psi}$ by letting

$$h_{\varphi\psi}(\mathfrak{x}, \lambda) = \sum_{X\in\mathfrak{x}}\ \sum_{Y\in\lambda X} f\big(\varphi(\mathfrak{x}, X), \psi(\lambda X, Y)\big)\,\mu X\,\nu Y$$

for all $(\mathfrak{x}, \lambda) \in \mathop{\mathrm{S}}_{\mathfrak{z}\in\mathfrak{X}} \mathfrak{Y}^{\mathfrak{z}}$. For abbreviation, we set

$$A = \mathop{\mathrm{S}}_{\mathfrak{z}\in\mathfrak{X}} \mathfrak{Y}^{\mathfrak{z}} \quad\text{and}\quad \mathfrak{a} = \mathop{\mathrm{S}}^{\mathcal{F}\mathfrak{X}}_{\mathfrak{z}\in\mathfrak{X}} \left({}^{*}(\mathcal{F}\mathfrak{Y})^{\mathfrak{z}}\right)$$

(for the notation, see § 0/e)). We now consider the filtered family $(h_{\varphi\psi}, A, \mathfrak{a})$ for each $(\varphi, \psi) \in Q\mathfrak{X}\times Q\mathfrak{Y}$.

Essentially as a consequence of Theorem 11 in the present paper and Theorem 7 and Example 2 in [13], one obtains

Theorem 14: *The inclusion* $\int^{\mathfrak{X}}\int^{\mathfrak{Y}} f\,d\nu\,d\mu \subseteqq {}^{\mathfrak{a}}\lim_{(\mathfrak{x},\lambda)\in A} h_{\varphi\psi}(\mathfrak{x}, \lambda)$ *holds for all* $(\varphi, \psi) \in Q\mathfrak{X}\times Q\mathfrak{Y}$.

Proof: Let $(\varphi, \psi) \in Q\mathfrak{X}\times Q\mathfrak{Y}$. For abbreviation, we put $f\big(\varphi(\mathfrak{x}, X), \psi(\mathfrak{y}, Y)\big) = f(\mathfrak{x}, X, \mathfrak{y}, Y)$ with $\mathfrak{x}\in\mathfrak{X}$, $X \in \mathfrak{x}$, $\mathfrak{y}\in\mathfrak{Y}$; $Y \in \mathfrak{y}$. We define the sets $C_1 - C_4$ by

$$C_1 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} \sum_{X\in\mathfrak{x}} \left({}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}} \sum_{Y\in\mathfrak{y}} f(\mathfrak{x}, X, \mathfrak{y}, Y)\,\nu Y\right)\mu X;$$

$$C_2 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} \sum_{X\in\mathfrak{x}} {}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}} \sum_{Y\in\mathfrak{y}} f(\mathfrak{x}, X, \mathfrak{y}, Y)\,\mu X\,\nu Y;$$

$$C_3 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}}\ {}^{*(\mathcal{F}\mathfrak{Y})^{\mathfrak{x}}}\lim_{\lambda\in({}^{*}\mathfrak{Y}^{\mathfrak{x}})} h_{\varphi\psi}(\mathfrak{x}, \lambda);$$

$$C_4 = {}^{\mathfrak{a}}\lim_{(\mathfrak{x},\lambda)\in A} h_{\varphi\psi}(\mathfrak{x}, \lambda).$$

By Theorem 11 and Proposition 25, we have $\int^{\mathfrak{X}}\int^{\mathfrak{Y}} f\,d\nu\,d\mu \subseteqq C_1$; by (0.6), (0.11), (0.4), (0.9), we have $C_1 \subseteqq C_2$; by Theorem 7 in [13] together with Example 2 there and (0.4), we get $C_2 \subseteqq C_3$; by (2.4) in [10, p. 246], we obtain $C_3 \subseteqq C_4$ ∎

Remark 13: Theorem 14 says, roughly speaking, that each iterated integral can be approximated "pointwise" (the "points" being the ordered pairs (φ, ψ)) by "iterated Riemann sums" $h_{\varphi\psi}(\mathfrak{x}, \lambda)$ "belonging to" $\mathfrak{x}, \lambda, \varphi$, and ψ. Of course, this can be extended to n-fold iterated integrals ($n \in \mathbb{N}$, $3 \leqq n$).

We are going to discuss Theorem 14 in the case in which all $\mathfrak{x} \in \mathfrak{X}$ are finite. As an auxiliary theorem the following elementary considerations will be of use (for the notation, see § 0/e)).

Lemma 3: *Let S, T, M be sets, S consisting of finite non-empty sets. Let $\mathfrak{A}$, $\mathfrak{B}$ be filters on S, T, respectively. Let* $h\colon \mathop{\mathrm{S}}_{s\in S}(T^s) \to M$ *and* $j\colon S\times T \to M$ *be mappings such that* (a) *holds:*

(a) *If* $s \in S$, $t \in T$, $\lambda \in T^s$, *then*: $\lambda(\sigma) = t\ \forall\,\sigma \in s \Rightarrow h(s, \lambda) = j(s, t)$.

Then the filterbase $j(\mathfrak{A} \otimes \mathfrak{B})$ *is finer than the filterbase* $h\left({}^{\mathfrak{A}}\underset{s\in S}{\mathsf{S}}({}^{*}B^{s})\right)$.

Proof: Let $K \in {}^{\mathfrak{A}}\underset{s\in S}{\mathsf{S}}({}^{*}\mathfrak{B}^{s})$. Then, there exists a $C \in \mathfrak{A}$ and, for each $c \in C$, a mapping $\varkappa_c: c \to \mathfrak{B}$ such that

(1) $\underset{c\in C}{\mathsf{S}}\ \underset{\sigma\in c}{\mathsf{P}}\varkappa_c(\sigma) \subseteqq K$.

Since each $c \in C$ is finite and non-empty, $D(c) := \bigcap_{\sigma\in c} \varkappa_c(\sigma) \in \mathfrak{B}$. It follows that

(2) $\underset{c\in C}{\mathsf{S}}\ D(c) \in \mathfrak{A} \otimes \mathfrak{B}$.

We assert that $j\left(\underset{c\in C}{\mathsf{S}}(D(c))\right) \subseteqq h(K)$. Indeed, let $(s, t) \in \underset{c\in C}{\mathsf{S}}\ D(c)$. Define λ by $\lambda(\sigma) = t$ for all $\sigma \in s$. Then, by (a), $h(s, \lambda) = j(s, t)$. Since $t \in D(s)$, one has $\lambda \in \underset{\sigma\in s}{\mathsf{P}}\varkappa_s(\sigma)$. Therefore, by (1), $(s, \lambda) \in K$. Now, because of (2), the proof is complete ∎

For abbreviation, we introduce, for each $(\varphi, \psi) \in \mathcal{Q}\mathfrak{X} \times \mathcal{Q}\mathfrak{Y}$, a mapping $j_{\varphi\psi}$ by letting, for each $(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{X} \times \mathfrak{Y}$,

$$j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y}) = \sum_{X\in\mathfrak{x}} \sum_{Y\in\mathfrak{y}} f(\varphi(X), \psi(Y))\, \nu Y\, \mu X.$$

We are going to compare the filterbases $j_{\varphi\psi}(\mathcal{F}\mathfrak{X} \otimes \mathcal{F}\mathfrak{Y})$ and $h_{\varphi\psi}(\mathfrak{a})$ in the set $\mathfrak{E}$, where $\mathcal{F}\mathfrak{X} \otimes \mathcal{F}\mathfrak{Y}$ denotes the ordinal product of the filters $\mathcal{F}\mathfrak{X}$ and $\mathcal{F}\mathfrak{Y}$ (see § 0/e)). As an immediate consequence of Lemma 3, one has the

Corollary: *If each* $\mathfrak{x} \in \mathfrak{X}$ *is finite, then, for all* $(\varphi, \psi) \in \mathcal{Q}\mathfrak{X} \times \mathcal{Q}\mathfrak{Y}$, *the filterbase* $j_{\varphi\psi}(\mathcal{F}\mathfrak{X} \otimes \mathcal{F}\mathfrak{Y})$ *is finer than the filterbase* $h_{\varphi\psi}(\mathfrak{a})$.

If each $\mathfrak{x} \in \mathfrak{X}$ is finite, the situation in Theorem 14 simplifies to (a) in the following theorem, where, in (b), the symbol $\mathcal{F}\mathfrak{X}({}^{*}\times)\,\mathcal{F}\mathfrak{Y}$ denotes the cardinal product of the filters $\mathcal{F}\mathfrak{X}$ and $\mathcal{F}\mathfrak{Y}$ (see § 0/e)).

Theorem 15: *Let each set* $\mathfrak{x} \in \mathfrak{X}$ *be finite. Then, the next statements* (a) *and* (b) *hold for all* $(\varphi, \psi) \in \mathcal{Q}\mathfrak{X} \times \mathcal{Q}\mathfrak{Y}$.

(a) $\int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu \subseteqq {}^{\mathcal{F}\mathfrak{X}\otimes\mathcal{F}\mathfrak{Y}}\lim_{(\mathfrak{x},\mathfrak{y})\in\mathfrak{X}\times\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y})$.

(b) *If* ${}^{\mathcal{F}\mathfrak{X}({}^{*}\times)\mathcal{F}\mathfrak{Y}}\lim_{(\mathfrak{x},\mathfrak{y})\in\mathfrak{X}\times\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y})$ *is non-empty, then*

$$\int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu \subseteqq {}^{\mathcal{F}\mathfrak{X}({}^{*}\times)\mathcal{F}\mathfrak{Y}}\lim_{(\mathfrak{x},\mathfrak{y})\in\mathfrak{X}\times\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y}).$$

Proof: (a) implies (b), since the filter $\mathcal{F}\mathfrak{X} \otimes \mathcal{F}\mathfrak{Y}$ is finer than the filter $\mathcal{F}\mathfrak{X}({}^{*}\times)\mathcal{F}\mathfrak{Y}$. The validity of (a) follows from Theorem 14 and the Corollary to Lemma 3. One obtains a second proof of (a) by modifying the proof of Theorem 14 in the following way: Define C_3 and C_4, now, by

$$C_3 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} {}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y}) \quad \text{and} \quad C_4 = {}^{\mathcal{F}\mathfrak{X}\otimes\mathcal{F}\mathfrak{Y}}\lim_{(\mathfrak{x},\mathfrak{y})\in\mathfrak{X}\times\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x}, \mathfrak{y}).$$

By means of (0.5) and (0.4), one obtains $C_2 \subseteqq C_3$; by means of (2.4) in [10, p. 246] one gets $C_3 \subseteqq C_4$ ∎

By means of Part (b) of Theorem 15, one obtains the following theorem on the reversal of the order of integration.

Theorem 16: *Assume that all* $\mathfrak{x} \in \mathfrak{X}$ *and all* $\mathfrak{y} \in \mathfrak{Y}$ *are finite sets and that the sets* $\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu$, $\int_{\mathfrak{Y}}\int_{\mathfrak{X}} f\,d\mu\,d\nu$ *and* ${}^{\mathcal{F}\mathfrak{X}(\bullet\times)\mathcal{F}\mathfrak{Y}}\lim_{(\mathfrak{x},\mathfrak{y})\in\mathfrak{X}\times\mathfrak{Y}} j_{\varphi\psi}(\mathfrak{x},\mathfrak{y})$ *are non-empty, the last mentioned set for some* $(\varphi,\psi) \in \mathcal{Q}\mathfrak{X}\times\mathcal{Q}\mathfrak{Y}$. *Then* $\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu = \int_{\mathfrak{Y}}\int_{\mathfrak{X}} f\,d\mu\,d\nu$.

Proof: Copy the proof of Satz 13 in [12] with the following modifications: Replace "α" by "$\mathcal{Q}$". Refer to Theorem 15 instead of Satz 12 and justify the equation

$$k_{\varphi\psi}(\mathfrak{y},\mathfrak{x}) = \sum_{Y\in\mathfrak{y}}\sum_{X\in\mathfrak{x}} f\big(\varphi(\mathfrak{x},X),\psi(\mathfrak{y},Y)\big)\,\nu Y\,\mu X$$

for all $(\varphi,\psi)\in\mathcal{Q}\mathfrak{X}\times\mathcal{Q}\mathfrak{Y}$ by the fact that $(\mathfrak{E}, +^\wedge)$ is a commutative semigroup and the multiplication in $\mathbb{R}$ is commutative ∎

Remark 14: The authors do not know at the present time if one of the suppositions made in this theorem is redundant. Of course, one gets an equivalent statement if, for instance, $\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu \neq \emptyset$ is not required but if the assertion of the theorem is replaced by $\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu \subseteq \int_{\mathfrak{Y}}\int_{\mathfrak{X}} f\,d\mu\,d\nu$. Theorem 16 can be considered as a substitute for a Fubini theorem. In view of an example given by Birkhoff in [2, p. 376] at least without any restrictions made for (F,μ), (G,ν) or f, the Fubini theorem does not hold for the $(\mu\times\nu, \Omega(\mu\times\nu))$-integral of f.

If one reconsiders Theorem 16, the question arises whether this theorem can be modified in such a way that also infinite $\mathfrak{x}$ and infinite $\mathfrak{y}$ are admitted there. In preparation of an answer we define the set $s\mathfrak{X}$ and the filter $s^0\mathfrak{X}$ on $s\mathfrak{X}$ by $s\mathfrak{X} = \mathrm{S}_{\mathfrak{x}\in\mathfrak{X}}\, e\mathfrak{x}$ and $s^0\mathfrak{X} = {}^{\mathcal{F}\mathfrak{X}}\mathrm{S}_{\mathfrak{x}\in\mathfrak{X}}\, e^0\mathfrak{x}$ (for the notation, see § 0/e)) and analogously $s\mathfrak{Y}$ and $s^0\mathfrak{Y}$. By means of Theorem 11, (0.3), the statement (2.4) in [10, p. 246] and (0.4), one obtains, for each mapping $g\colon F\to\mathfrak{E}$, the validity of

Proposition 34: *The inclusion* $\int_{\mathfrak{X}} g\,d\mu \subseteq {}^{s^0\mathfrak{X}}\lim_{(\mathfrak{x},K)\in s\mathfrak{X}} \sum_{X\in K} g\big(\varphi(\mathfrak{x},X)\big)\,\mu X$ *holds for all* $\varphi\in\mathcal{Q}\mathfrak{X}$.

The analogue to Theorem 16 we were looking for is prepared by the following

Theorem 17: *For all* $(\varphi,\psi)\in\mathcal{Q}\mathfrak{X}\times\mathcal{Q}\mathfrak{Y}$ *the statements* (a) *and* (b) *formulated next hold with the abbreviation*

$$T(\mathfrak{x},K,\varphi,\mathfrak{y},L,\psi) = \sum_{X\in K}\sum_{Y\in L} f\big(\varphi(\mathfrak{x},X),\psi(\mathfrak{y},Y)\big)\,\mu X\,\nu Y.$$

(a) $\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu \subseteq {}^{s^0\mathfrak{X}\otimes s^0\mathfrak{Y}}\lim_{((\mathfrak{x},K),(\mathfrak{y},L))\in s\mathfrak{X}\times s\mathfrak{Y}} T(\mathfrak{x},K,\varphi,\mathfrak{y},L,\psi)$.

(b) *If* ${}^{s^0\mathfrak{X}(\bullet\times)s^0\mathfrak{Y}}\lim_{((\mathfrak{x},K),(\mathfrak{y},L))\in s\mathfrak{X}\times s\mathfrak{Y}} T(\mathfrak{x},K,\varphi,\mathfrak{y},L,\psi)$ *is non-empty, then*

$$\int_{\mathfrak{X}}\int_{\mathfrak{Y}} f\,d\nu\,d\mu \subseteq {}^{s^0\mathfrak{X}(\bullet\times)s^0\mathfrak{Y}}\lim_{((\mathfrak{x},K),(\mathfrak{y},L))\in s\mathfrak{X}\times s\mathfrak{Y}} T(\mathfrak{x},K,\varphi,\mathfrak{y},L,\psi).$$

Proof: We follow the proof of Theorem 14 down to the definition of C_2 and define $C_3 - C_6$ now in the following way:

$$C_3 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}}\ {}^{e^0\mathfrak{x}}\lim_{K\in e\mathfrak{x}} \sum_{X\in K} {}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}}\ {}^{e^0\mathfrak{y}}\lim_{L\in e\mathfrak{y}} \sum_{Y\in L} f(\mathfrak{x},X,\mathfrak{y},Y)\,\mu X\,\nu Y;$$

$$C_4 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}}\ {}^{e^0\mathfrak{x}}\lim_{K\in e\mathfrak{x}}\ {}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}} \sum_{X\in K} {}^{e^0\mathfrak{y}}\lim_{L\in e\mathfrak{y}} \sum_{Y\in L} f(\mathfrak{x},X,\mathfrak{y},Y)\,\mu X\,\nu Y;$$

$$C_5 = {}^{\mathcal{F}\mathfrak{X}}\lim_{\mathfrak{x}\in\mathfrak{X}} {}^{e^*\mathfrak{x}}\lim_{K\in e\mathfrak{x}} {}^{\mathcal{F}\mathfrak{Y}}\lim_{\mathfrak{y}\in\mathfrak{Y}} {}^{e^*\mathfrak{y}}\lim_{L\in e\mathfrak{y}} T(\mathfrak{x}, K, \varphi, \mathfrak{y}, L, \psi);$$

$$C_6 = {}^{s^0\mathfrak{X}\otimes s^0\mathfrak{Y}}\lim_{((\mathfrak{x},K),(\mathfrak{y},L))\in s\mathfrak{X}\times s\mathfrak{Y}} T(\mathfrak{x}, K, \varphi, \mathfrak{y}, L, \psi).$$

As in the proof of Theorem 14, one gets $\int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu \subseteqq C_1 \subseteqq C_2$. By (0.3), one obtains $C_2 = C_3$. By (0.4) and (0.5), $C_3 \subseteqq C_4 \subseteqq C_5$ holds. By (2.4) in [10, p. 246] and (0.4), we get $C_5 \subseteqq C_6$. Therefore, (a) holds. In order to prove (b) by means of (a), one uses that $s^0\mathfrak{X}(^*\times)\, s^0\mathfrak{Y} \subseteqq s^0\mathfrak{X} \otimes s^0\mathfrak{Y}$ ∎

Now we obtain (with the abbreviation introduced in Theorem 17) the desired analogue to Theorem 16 as

Theorem 18: *Assume that the sets* $\int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu$, $\int^{\mathfrak{Y}} \int^{\mathfrak{X}} f\, d\mu\, d\nu$ *and*

$$ {}^{s^0\mathfrak{X}(^*\times)s^0\mathfrak{Y}}\lim_{((\mathfrak{x},K),(\mathfrak{y},L))\in s\mathfrak{X}\times s\mathfrak{Y}} T(\mathfrak{x}, K, \varphi, \mathfrak{y}, L, \psi)$$

are non-empty, the last mentioned set for some $(\varphi, \psi) \in Q\mathfrak{X} \times Q\mathfrak{Y}$. *Then* $\int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu = \int^{\mathfrak{Y}} \int^{\mathfrak{X}} f\, d\mu\, d\nu$.

Proof: Follow the proof of Satz 15 in [12] word by word except for the following modifications: Replace "α" by "Q". Now, the equation $A_{\varphi,\psi}(\mathfrak{x}, K, \mathfrak{y}, L) = B_{\varphi,\psi}(\mathfrak{y}, L, \mathfrak{x}, K)$ (see [12, p. 85]) (properly interpreted here) holds, because $(\mathfrak{E}, +^\wedge)$ and $(\mathbb{R}, \cdot)$ are commutative semigroups. Refer to Theorem 17, now, instead of Satz 14 in [12] ∎

Remark 15: Despite the "nicety" of Theorem 18, one should realize that the iterated sums $T(\mathfrak{x}, K, \varphi, \mathfrak{y}, L, \psi)$ are not iterated Riemann sums; they are iterated "partial sums" (in a terminology used in the elementary analysis) of iterated Riemann sums. — Also it should be remarked critically that the finite sums occuring in Proposition 34 are not Riemann sums but "partial sums" of Riemann sums.

As an immediate consequence of Theorem 4 and Proposition 25, one obtains

Theorem 19: *If* $\mathfrak{X}'$ *and* $\mathfrak{Y}'$ *are* μ*- and* ν*-partition systems (respectively), then*

$$\mathfrak{X}' \subseteqq \mathfrak{X} \text{ and } \mathfrak{Y}' \subseteqq \mathfrak{Y} \textit{ implies } \int^{\mathfrak{X}'} \int^{\mathfrak{Y}'} f\, d\nu\, d\mu \subseteqq \int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\mu\, d\nu.$$

Example 3: Let $\mu F < +\infty$ and $\nu G < +\infty$. If $\mathfrak{X}'$ is the set of all finite μ-partitions of F and $\mathfrak{Y}'$ the set of all finite ν-partitions of G and $\mathfrak{X} = \Omega(\mu)$, $\mathfrak{Y} = \Omega(\nu)$, then (by Theorem 19) (a) implies (b):

(a) $\emptyset \neq \int^{\mathfrak{X}'} \int^{\mathfrak{Y}'} f\, d\nu\, d\mu = \int^{\mathfrak{Y}'} \int^{\mathfrak{X}'} f\, d\mu\, d\nu$; (b) $\emptyset \neq \int^{\mathfrak{X}} \int^{\mathfrak{Y}} f\, d\nu\, d\mu = \int^{\mathfrak{Y}} \int^{\mathfrak{X}} f\, d\mu\, d\nu$.

Thus, using Theorem 16 for $\mathfrak{X}'$ and $\mathfrak{Y}'$, one obtains (by means of Theorem 19) sufficient conditions for the validity of (b).

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[2]) On p. 94, line 11, replace "$C \in \mathfrak{R}$" by "$C \in \mathfrak{R} \setminus \{\emptyset\}$".
[3]) In Example 4 on p. 119 replace "Then $\sup_I$ is ..." by "Then $\sup_I$ is neither an α-summation-like nor a β-summation-like I-ary partial operation on $\mathbb{R}$."