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Double Walsh Series with Coefficients of Bounded Variation[1])

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Es werden die Konvergenzeigenschaften Walshscher Doppelreihen untersucht, deren Koeffizienten eine Nullfolge mit beschränkter Variation bilden. Diese Reihen konvergieren regulär in allen Punkten aus $(0, 1) \times (0, 1)$ und in der Pseudometrik des L^r für alle $r \in (0, 1)$. Außerdem werden mit Differenzen zweiter Ordnung der Koeffizienten hinreichende Konvergenzbedingungen angegeben.

Исследуются свойства сходимости двойных рядов Уолша, коэффициенты которых образуют сходящуюся к нулю последовательность с ограниченной вариацией. Такие ряды сходятся регулярно во всех точках из $(0, 1) \times (0, 1)$ и в псевдометрике пространства L^r для всех $r \in (0, 1)$. Кроме того, с помощью разностей второго порядка для коэффициентов выводятся достаточные условия сходимости.

Convergence properties of double Walsh series are studied whose coefficients form a null sequence of bounded variation. These series converge regularly at all points of $(0, 1) \times (0, 1)$ and converge in the pseudometric of L^r for all $r \in (0, 1)$. Sufficient conditions for convergence are also proved which involve the second-order differences of the coefficients.

1. Introduction. We will study the convergence behaviour of double Walsh series of the form

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} w_j(x)\, w_k(y), \tag{1.1}$$

where $\{a_{jk}\}$ is a null sequence of complex (or real) numbers and $\{w_j\}$ is the well-known Walsh orthonormal system defined on the interval $I = [0, 1)$ and considered in the Paley enumeration (see, e.g., [1, p. 60]). Thus, series (1.1) is considered on the unit square $I^2 = [0, 1) \times [0, 1)$. The pointwise *convergence* of (1.1) is usually defined *in Pringsheim's sense* (see, e. g., [6, Vol. 2, Ch. 17]). This means that we form the rectangular partial sums $s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} w_j(x)\, w_k(y)$, then let both m and n tend to ∞, independently of one another, and assign the limit $f(x, y)$ (if it exists) to series (1.1) as its sum. Following Hardy [3], series (1.1) is said to be *regularly convergent* if it converges in Pringsheim's sense, and, in addition, each "row series" of (1.1) (i.e., when we delete $\sum_{k=0}^{\infty}$ in (1.1) and the summation is done only with respect to j for each fixed k) as well as each "column series" converges in the ordinary sense of convergence of single series. The notion of regular convergence was rediscovered in [4], where it was

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defined by the following equivalent condition: the sums

$$s(Q; x, y) = \sum_{j=m}^{M} \sum_{k=n}^{N} a_{jk} w_j(x)\, w_k(y) \tag{1.2}$$

tend to zero as $\max(m, n) \to \infty$, independently of the choices of M ($\geqq m$) and N ($\geqq n$), where $Q = \{(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0 : m \leqq j \leqq M \text{ and } n \leqq k \leqq N\}$.

2. Main results. We remind the reader that the differences Δ_{pq} of a double sequence $\{a_{jk}\}$ are defined for any non-negative integers p and q as follows:

$$\Delta_{00} a_{jk} = a_{jk}, \; \Delta_{pq} a_{jk} = \begin{cases} \Delta_{p-1,q} a_{jk} - \Delta_{p-1,q} a_{j+1,k} & \text{if } p \geqq 1, \\ \Delta_{p,q-1} a_{jk} - \Delta_{p,q-1} a_{j,k+1} & \text{if } q \geqq 1. \end{cases}$$

As is well known, the two right-hand sides coincide if $\min(p, q) \geqq 1$. We mention that a double induction argument gives

$$\Delta_{pq} a_{mn} = \sum_{j=0}^{p} \sum_{k=0}^{q} (-1)^{j+k} \binom{p}{j} \binom{q}{k} a_{m+j,n+k}.$$

We will prove convergence results for the cases $p = q = 1$ and $p = q = 2$.

Theorem 1: *If a double sequence $\{a_{jk}\}$ is such that*

$$a_{jk} \to 0 \quad \text{as} \quad \max(j, k) \to \infty \tag{2.1}$$

and

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| < \infty, \tag{2.2}$$

then

(i) *series* (1.1) *converges regularly to some function $f = f(x, y)$ for all $0 < x, y < 1$;*

(ii) *for all $0 < r < 1$,*

$$\|s_{mn} - f\|_r \to 0 \quad \text{as} \quad \min(m, n) \to \infty, \tag{2.3}$$

where $\|\cdot\|_r$ means the pseudonorm in $L^r(I^2)$ defined by $\|g\|_r = \int_0^1 \int_0^1 |g(x, y)|^r \, dx\, dy$.

If condition (2.2) is satisfied, $\{a_{jk}\}$ is said to be of *bounded variation.* We note that an analogous theorem was proved in [5] for double trigonometric series.

Theorem 2: *If a double sequence $\{a_{jk}\}$ is such that condition* (2.1) *is satisfied and*

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{22} a_{jk}| < \infty, \tag{2.4}$$

$$\sum_{j=0}^{\infty} |\Delta_{20} a_{jk}| \text{ is finite for each } k \text{ and tends to } 0 \text{ as } k \to \infty, \tag{2.5}$$

$$\sum_{k=0}^{\infty} |\Delta_{02} a_{jk}| \text{ is finite for each } j \text{ and tends to } 0 \text{ as } j \to \infty, \tag{2.6}$$

then conclusion (i) *in Theorem 1, except possibly when x or y is a dyadic rational, and conclusion* (ii) *for all $0 < r < 1/2$ hold true.*

3. Auxiliary results. We need the following three lemmas.

Lemma 1: *If $\{a_{jk}\}$ satisfies condition* (2.1) *and for some $p, q \geqq 1$,*

$$C_{pq} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{pq} a_{jk}| < \infty, \tag{3.1}$$

then

$$\sum_{j=0}^{\infty} |\Delta_{p,q-1} a_{jk}| \leqq C_{pq} \qquad (k = 0, 1, \ldots), \tag{3.2}$$

$$\sum_{j=0}^{\infty} |\Delta_{p,q-1} a_{jk}| \to 0 \quad as \quad k \to \infty, \tag{3.3}$$

$$\sup_{k} \sum_{j=m}^{\infty} |\Delta_{p,q-1} a_{jk}| \to 0 \quad as \quad m \to \infty. \tag{3.4}$$

Analogous statements hold true for $\Delta_{p-1,q} a_{jk}$ under the same conditions (2.1) *and* (3.1) *if the roles of j and k are interchanged.*

Proof: By (2.1), $\Delta_{p,q-1} a_{jk_0} = \sum_{k=k_0}^{\infty} \Delta_{pq} a_{jk}$, whence $\sum_{j=0}^{\infty} |\Delta_{p,q-1} a_{jk_0}| \leqq \sum_{j=0}^{\infty} \sum_{k=k_0}^{\infty} |\Delta_{pq} a_{jk}|$. Clearly, (3.1) implies both (3.2) and (3.3). Finally, (3.4) is a consequence of (3.3) (applied for large values of k) and (3.2) (applied for small values of k) ∎

Now we consider another double sequence $\{b_{jk}\}$ of numbers with rectangular partial sums $B_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{jk}$ $(m, n = 0, 1, \ldots)$. The next two lemmas can easily be verified by performing double summations by parts.

Lemma 2: *For all $0 \leqq m \leqq M$ and $0 \leqq n \leqq N$,*

$$\begin{aligned} \sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk} a_{jk} = {} & \sum_{j=m}^{M} \sum_{k=n}^{N} B_{jk} \Delta_{11} a_{jk} + \sum_{j=m}^{M} B_{jN} \Delta_{10} a_{j,N+1} \\ & - \sum_{j=m}^{M} B_{j,n-1} \Delta_{10} a_{jn} + \sum_{k=n}^{N} B_{Mk} \Delta_{01} a_{M+1,k} \\ & - \sum_{k=n}^{N} B_{m-1,k} \Delta_{01} a_{mk} + B_{MN} a_{M+1,N+1} \\ & - B_{M,n-1} a_{M+1,n} - B_{m-1,N} a_{m,N+1} + B_{m-1,n-1} a_{mn}. \end{aligned}$$

We introduce the notation

$$R_{mn} = \{(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0 \colon \text{either } j \geqq m+1 \text{ or } k \geqq n+1\} \tag{3.5}$$

and let $\sum_{R_{mn}} \ldots$ stand for $\sum\sum_{(j,k) \in R_{mn}} \ldots$.

Lemma 3: *If $\{a_{jk}\}$ satisfies condition* (2.1), *then, for all $m, n \geqq 0$,*

$$\begin{aligned} \sum_{R_{mn}} b_{jk} a_{jk} = {} & \sum_{R_{mn}} B_{jk} \Delta_{11} a_{jk} - \sum_{j=0}^{m}{}' B_{jn} \Delta_{10} a_{j,n+1} \\ & - \sum_{k=0}^{n} B_{mk} \Delta_{01} a_{m+1,k} - B_{mn} a_{m+1,n+1}. \end{aligned}$$

4. Proofs of Theorems 1 and 2. We recall that $D_m(x) = \sum_{j=0}^{m} w_j(x)$ is the *Dirichlet kernel*, while $F_m(x) = (m+1)^{-1} \sum_{j=0}^{m} D_j(x)$ is the *Fejér kernel* for the Walsh system. The following estimates are well known (see [2]):

$$|D_m(x)| < 2/x \qquad (m = 0, 1, \ldots; 0 < x < 1), \tag{4.1}$$

and for all integers $m \geqq 0$, $p \geqq 1$ and for all $0 < x < 1$, except possibly when x is a dyadic rational,

$$(m+1)\,|F_m(x)| < \frac{4}{x(x-2^{-p})} + \frac{4}{x^2} = C(x) \quad \text{if} \quad 2^{-p} < x < 2^{-p+1}. \tag{4.2}$$

Proof of Theorem 1: *Pointwise convergence.* Let $0 \leqq m \leqq M$ and $0 \leqq n \leqq N$. Keeping notation (1.2) in mind, by Lemma 2 we can write that

$$\begin{aligned} s(Q; x, y) &= \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x)\, D_k(y)\, \Delta_{11} a_{jk} \\ &\quad + \sum_{j=m}^{M} D_j(x)\, D_N(y)\, \Delta_{10} a_{j,N+1} - \sum_{j=m}^{M} D_j(x)\, D_{n-1}(y)\, \Delta_{10} a_{jn} \\ &\quad + \sum_{k=n}^{N} D_M(x)\, D_k(y)\, \Delta_{01} a_{M+1,k} - \sum_{k=n}^{N} D_{m-1}(x)\, D_k(y)\, \Delta_{01} a_{mk} \\ &\quad + a_{M+1,N+1} D_M(x)\, D_N(y) - a_{M+1,n} D_M(x)\, D_{n-1}(y) \\ &\quad - a_{m,N+1} D_{m-1}(x)\, D_N(y) + a_{mn} D_{m-1}(x)\, D_{n-1}(y). \end{aligned} \tag{4.3}$$

By (4.1), for $0 < x, y < 1$ we get that

$$\begin{aligned} 4^{-1}xy\, |s(Q; x, y)| &\leqq \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{11} a_{jk}| \\ &\quad + \sum_{j=m}^{M} [|\Delta_{10} a_{j,N+1}| + |\Delta_{10} a_{jn}|] + \sum_{k=n}^{N} [|\Delta_{01} a_{M+1,k}| + |\Delta_{01} a_{mk}|] \\ &\quad + |a_{M+1,N+1}| + |a_{M+1,n}| + |a_{m,N+1}| + |a_{mn}|. \end{aligned}$$

Making use of Lemma 1 (with $p = q = 1$) and (2.1), we can see that each term on the right-hand side tends to zero as $\max(m, n) \to \infty$. Thus, the sum $f(x, y)$ of series (1.1) exists for all $0 < x, y < 1$.

$L^r(I^2)$-convergence. It is plain that

$$f(x, y) - s_{mn}(x, y) = \sum_{R_{mn}} a_{jk} w_j(x)\, w_k(y),$$

where R_{mn} is defined by (3.5). By Lemma 3,

$$\begin{aligned} &f(x, y) - s_{mn}(x, y) \\ &= \sum_{R_{mn}} D_j(x)\, D_k(y)\, \Delta_{11} a_{jk} - \sum_{j=0}^{m} D_j(x)\, D_n(y)\, \Delta_{10} a_{j,n+1} \\ &\quad - \sum_{k=0}^{n} D_m(x)\, D_k(y)\, \Delta_{01} a_{m+1,k} - D_m(x)\, D_n(y)\, a_{m+1,n+1}. \end{aligned} \tag{4.4}$$

Using (4.1) gives, for all $0 < x, y < 1$,

$$4^{-1}xy\,|f(x, y) - s_{mn}(x, y)|$$

$$\leqq \sum_{R_{mn}} |\Delta_{11}a_{jk}| + \sum_{j=0}^{m} |\Delta_{10}a_{j,n+1}| + \sum_{k=0}^{n} |\Delta_{01}a_{m+1,k}| + |a_{m+1,n+1}|$$

$$\leqq 2 \sum_{R_{mn}} |\Delta_{11}a_{jk}|. \tag{4.5}$$

Hence

$$\|f - s_{mn}\|_r \leqq 8^r \Big(\sum_{R_{mn}} |\Delta_{11}a_{jk}|\Big)^r \int_0^1 \int_0^1 \frac{dx\,dy}{x^r y^r}.$$

Due to (2.2) and $0 < r < 1$, (2.3) follows immediately ∎

Proof of Theorem 2: *Pointwise convergence.* We start with (4.3). We apply Lemma 2 again to the double sum on the right-hand side of (4.3) to obtain

$$\sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x)\, D_k(y)\, \Delta_{11}a_{jk}$$

$$= \sum_{j=m}^{M} \sum_{k=n}^{N} F^*_{jk}(x, y)\, \Delta_{22}a_{jk} + \sum_{j=m}^{M} F^*_{jN}(x, y)\, \Delta_{21}a_{j,N+1}$$

$$- \sum_{j=m}^{M} F^*_{j,n-1}(x, y)\, \Delta_{21}a_{jn} + \sum_{k=n}^{N} F^*_{Mk}(x, y)\, \Delta_{12}a_{M+1,k}$$

$$- \sum_{k=n}^{N} F^*_{m-1,k}(x, y)\, \Delta_{12}a_{mk} + F^*_{MN}(x, y)\, \Delta_{11}a_{M+1,N+1}$$

$$- F^*_{M,n-1}(x, y)\, \Delta_{11}a_{M+1,n} - F^*_{m-1,N}(x, y)\, \Delta_{11}a_{m,N+1}$$

$$+ F^*_{m-1,n-1}(x, y)\, \Delta_{11}a_{mn},$$

where

$$F^*_{mn}(x, y) = (m + 1)\,(n + 1)\, F_m(x)\, F_n(y). \tag{4.6}$$

By (4.2), we can conclude for all $0 < x, y < 1$, except possibly when x or y is a dyadic rational,

$$\big(C(x)\, C(y)\big)^{-1} \left| \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x)\, D_k(y)\, \Delta_{11}a_{jk} \right|$$

$$\leqq \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{22}a_{jk}| + \sum_{j=m}^{M} [|\Delta_{21}a_{j,N+1}| + |\Delta_{21}a_{jn}|]$$

$$+ \sum_{k=n}^{N} [|\Delta_{12}a_{M+1,k}| + |\Delta_{12}a_{mk}|] + |\Delta_{11}a_{M+1,N+1}|$$

$$+ |\Delta_{11}a_{M+1,n}| + |\Delta_{11}a_{m,N+1}| + |\Delta_{11}a_{mn}|.$$

By virtue of Lemma 1 (with $p = q = 2$) and (2.1), each term on the right-hand side tends to zero as $\max(m, n) \to \infty$.

We have four single sums on the right-hand side of (4.3). We claim that each of them tends to zero as $\max(m, n) \to \infty$, for all $0 < x, y < 1$. We show this in the case of the first single sum. A single summation by parts yields

$$\sum_{j=m}^{M} D_j(x)\, D_N(y)\, \Delta_{10} a_{j,n+1}$$

$$= \sum_{j=m}^{M} (j+1)\, F_j(x)\, D_N(y)\, \Delta_{20} a_{j,N+1}$$

$$+ (M+1)\, F_M(x)\, D_N(y)\, \Delta_{10} a_{M,N+1} - m F_{m-1}(x)\, D_N(y)\, \Delta_{10} a_{m,N+1}. \qquad (4.7)$$

Hence, by (4.1) and (4.2), for all $0 < x, y < 1$, except possibly when x is a dyadic rational,

$$\left| \sum_{j=m}^{M} D_j(x)\, D_N(y)\, \Delta_{10} a_{j,N+1} \right|$$

$$\leq 2y^{-1} C(x) \left\{ \sum_{j=m}^{M} |\Delta_{20} a_{j,N+1}| + |\Delta_{10} a_{M,N+1}| + |\Delta_{10} a_{m,N+1}| \right\}.$$

Thanks to conditions (2.1) and (2.5), each term on the right-hand side tends to zero as $\max(m, n) \to \infty$. The other three single sums on the right-hand side of (4.3) can be estimated analogously. Finally, by (2.1) and (4.1), the four single terms on the right-hand side of (4.3) tend to zero as $\max(m, n) \to \infty$, for all $0 < x, y < 1$.

$L^r(I^2)$-convergence. Now we start with (4.4). We apply Lemma 3 once more to the double sum on the right-hand side of (4.4). As a result we get that

$$\sum_{R_{mn}} D_j(x)\, D_k(y)\, \Delta_{11} a_{jk} = \sum_{R_{mn}} F^*_{jk}(x, y)\, \Delta_{22} a_{jk} - \sum_{j=0}^{m} F^*_{jn}(x, y)\, \Delta_{21} a_{j,n+1}$$

$$- \sum_{k=0}^{n} F^*_{mk}(x, y)\, \Delta_{12} a_{m+1,k} - F^*_{mn}(x, y)\, \Delta_{11} a_{m+1,n+1}$$

where we used notation (4.6). By (4.2),

$$\big(C(x)\, C(y)\big)^{-1} \left| \sum_{R_{mn}} D_j(x)\, D_k(y)\, \Delta_{11} a_{jk} \right|$$

$$\leq \sum_{R_{mn}} |\Delta_{22} a_{jk}| + \sum_{j=0}^{m} |\Delta_{21} a_{j,n+1}| + \sum_{k=0}^{n} |\Delta_{12} a_{m+1,k}| + |\Delta_{11} a_{m+1,n+1}|$$

$$\leq 4 \sum_{R_{mn}} |\Delta_{22} a_{jk}|. \qquad (4.8)$$

Similarly to (4.7), a single summation by parts gives

$$\sum_{j=0}^{m} D_j(x)\, D_n(y)\, \Delta_{10} a_{j,n+1}$$

$$= \sum_{j=0}^{m} (j+1)\, F_j(x)\, D_n(y)\, \Delta_{20} a_{j,n+1} + (m+1)\, F_m(x)\, D_n(y)\, \Delta_{10} a_{m,n+1}.$$

Hence, by (4.1) and (4.2),

$$\left| \sum_{j=0}^{m} D_j(x)\, D_n(y)\, \Delta_{10} a_{j,n+1} \right| \leq 2y^{-1} C(x) \left\{ \sum_{j=0}^{m} |\Delta_{20} a_{j,n+1}| + |\Delta_{10} a_{m,n+1}| \right\}$$

$$\leq 2y^{-1} C(x) \sum_{j=0}^{\infty} |\Delta_{20} a_{j,n+1}|. \qquad (4.9)$$

Analogously,

$$\left|\sum_{k=0}^{n} D_m(x)\, D_k(y)\, \Delta_{01} a_{m+1,k}\right| \leqq 2x^{-1}C(y) \sum_{k=0}^{\infty} |\Delta_{02} a_{m+1,k}|. \tag{4.10}$$

Combining (4.4), (4.8)–(4.10) and (2.4)–(2.6) yields (2.3) for all $0 < r < 1/2$ if we take into account that, by (4.2),

$$\int_0^1 C^r(x)\, dx \leqq \sum_{p=1}^{\infty} \int_{2^{-p}}^{2^{-p+1}} \left(4^r/x^r(x - 2^{-p})^r\right) dx + \int_0^1 (4^r/x^{2r})\, dx$$

$$\leqq \sum_{p=1}^{\infty} \left(4^r/(1-r)\right) 2^{p(2r-1)} + 4^r/(1-2r) < \infty.$$

5. Concluding remarks. In the case of Theorem 1/(ii) we can prove somewhat more than (2.3) for $0 < r < 1$. To present this, let "meas" denote the planar Lebesgue measure and let $\ln^+ u = \max(1, \ln u)$.

Theorem 3: *If a double sequence $\mathcal{A} = \{a_{jk}\}$ satisfies conditions* (2.1) *and* (2.2), *then, for every $\varepsilon > 0$,*

$$\mu = \operatorname{meas}\left\{(x, y) \in I^2 : \sup_{m,n \geqq 0} |s_{mn}(x, y)| \geqq \varepsilon\right\} \leqq \frac{4\|\mathcal{A}\|}{\varepsilon}\left(1 + \ln^+ \frac{\varepsilon}{4\|\mathcal{A}\|}\right), \tag{5.1}$$

where

$$\|\mathcal{A}\| = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}|. \tag{5.2}$$

Since the space $c_2 \cap BV_2$ of double null sequences of bounded variation endowed with norm (5.2) is a Banach space, condition (5.1) is only slightly weaker than the condition that the mapping $\mathcal{A} \to f$ is of weak type (1, 1), where $f = f(x, y)$ is the sum of series (1.1) (see Theorem 1).

Proof of Theorem 3: Similarly to (4.3),

$$s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} D_j(x)\, D_k(y)\, \Delta_{11} a_{jk} + \sum_{j=0}^{m} D_j(x)\, D_n(y)\, \Delta_{10} a_{j,n+1}$$

$$+ \sum_{k=0}^{n} D_m(x)\, D_k(y)\, \Delta_{01} a_{m+1,k} + a_{m+1,n+1} D_m(x)\, D_n(y).$$

By (4.1), for all $0 < x, y < 1$, we get that

$$4^{-1}xy\, |s_{mn}(x, y)| \leqq \sum_{j=0}^{m} \sum_{k=0}^{n} |\Delta_{11} a_{jk}| + \sum_{j=0}^{m} |\Delta_{10} a_{j,n+1}|$$

$$+ \sum_{k=0}^{n} |\Delta_{01} a_{m+1,k}| + |a_{m+1,n+1}|,$$

whence, by (2.1) (cf. the proof of Lemma 1),

$$4^{-1}xy\, |s_{mn}(x, y)| \leqq \sum_{j=0}^{m} \sum_{k=0}^{n} |\Delta_{11} a_{jk}| + \sum_{j=0}^{m} \sum_{k=n+1}^{\infty} |\Delta_{11} a_{jk}|$$

$$+ \sum_{j=m+1}^{\infty} \sum_{k=0}^{n} |\Delta_{11} a_{jk}| + \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} |\Delta_{11} a_{jk}| = \|\mathcal{A}\|.$$

Now for every $\varepsilon \geqq 4\|\mathcal{A}\|$, $\mu \leqq \operatorname{meas}\{(x, y) \in I^2 : xy \leqq \gamma\} = \gamma + \gamma \ln(1/\gamma)$, where $\gamma = 4\|\mathcal{A}\|/\varepsilon$ ∎

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