Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 1, p. 63–72

Duality for Nonlinear Abstract Evolution Differential Equations

A. Nowakowski and A. Rogowski

Es werden Existenz periodischer Lösungen und Dualitätssätze für eine abstrakte Differentialgleichung in einem Hilbert-Raum hergeleitet. Das basiert auf der dualen Methode der Variationsrechnung und einer Modifikation der Tolandschen Dualitätstheorie.

Доказана существование периодических решений и предложения двойственности для одного абстрактного дифференциального уравнения в гильбертовом пространстве. Для этого использованы двойственный метод вариационного исчисления и одна модификация теории двойственности Толанда.

The existence of periodic solutions and duality results for an abstract differential equation in Hilbert space are established. The dual variational method and a modification of the duality theory of Toland are used.

1. Introduction and statement of the main results

We shall be dealing with the following T-periodic abstract problem in a real separable Hilbert space X:

$$d\,\partial\psi\big(t, x'(t)\big)/dt + \partial\varphi\big(t, x(t)\big) \ni 0, \tag{1.1a}$$

$$x(t+T) = x(t) \tag{1.1b}$$

for almost all t in $\mathbb{R}$, where T is a given positive number, $\partial\psi$ and $\partial\varphi$ are the subdifferentials of convex lower semicontinuous functions $\psi(t,\cdot), \varphi(t,\cdot)$; $\psi, \varphi\colon \mathbb{R}\times X \to \mathbb{R}$ are T-periodic in t and $\mathbb{L}\times\mathbb{B}$-measurable, i.e. measurable with respect to the σ-algebra generated in $\mathbb{R}\times X$ by products of Lebesgue sets in $\mathbb{R}$ and Borel sets in X.

To obtain some results for (1.1), we shall consider two functionals: primal

$$J\big(x(\cdot)\big) = \int_0^T \Big(-\varphi\big(t, x(t)\big) + \psi\big(t, x'(t)\big)\Big)\,dt \tag{1.2}$$

and dual to it

$$J_D\big(p(\cdot)\big) = \int_0^T \Big(\varphi^*\big(t, -p'(t)\big) - \psi^*\big(t, p(t)\big)\Big)\,dt + l\big(p(0), p(T)\big), \tag{1.3}$$

$$l(a,b) = l_1(a) + l_2(b), \tag{1.4}$$

$$l_1(a) = \begin{cases} 0 & \text{if } a = 0,\\ +\infty & \text{if } a \neq 0,\end{cases} \qquad l_2(b) = \begin{cases} 0 & \text{if } b = 0,\\ +\infty & \text{if } b \neq 0\end{cases}$$

(φ^* and ψ^* are the Fenchel conjugates of $\varphi(t,\cdot), \psi(t,\cdot)$) both defined on the space $A(X)$ of absolutely continuous functions from $[0, T]$ to X whose squares of norms of derivatives are integrable.

Note that (1.1a) is the generalized Euler-Lagrange equation for (1.2) and, further, that the integrand in (1.2) is a concave-convex function. The last fact makes (1.2) indefinite, i.e. it is, in general, unbounded from below and from above. This is the reason why we use the dual functional (1.3) to study (1.1) which is, under some growth conditions on φ and ψ, bounded from below in $A(X)$.

The applying of a modification of the duality theory of TOLAND [11, 12] (see also AUCHMUTY [1]) allows us to reveal the relationships between (1.1), (1.2) and (1.3).

Since the pioneer works of CLARKE and EKELAND [4, 5] dual variational methods have been extensively used in the study of differential equations (see survey lectures [2, 8]). Our duality principle (see Theorem 1.1 below) differs from that of [1, 2, 4, 8, 12]. However, it has been pointed out to us by the referee that our Theorem 3.1 may be obtained as a direct application of Tolands duality formula from [11]; but as to the necessary and sufficient optimality conditions compare [7]. But to prove that J_D attains its minimum in $A(X)$ (note that J_D is not weakly lower semicontinuous in $A(X)$), we shall consider a functional I (see Theorem 3.3) which corresponds in some sense to the dual least action principle for (1.2) (or (1.3)) devised by Clarke. Indeed, the Hamiltonian associated with the integrand of J is $H(t, x, p) = \varphi(t, x) + \psi^*(t, x)$; the dual Hamiltonian is $H^*(t, \dot{x}, \dot{p}) = \varphi^*(t, \dot{p}) + \psi(t, \dot{x})$ and it is the main part of the functional I. That explains the remark of BREZIS [2, p. 415 (bottom)] on the relations between the idea of Toland and that of Clarke and Ekeland.

Let $|\cdot|$ denote the norm of X and $\langle\cdot,\cdot\rangle$ the scalar product in X. A measurable function $u\colon \mathbb{R} \to X$ will be called T-periodic if $u(t + T) = u(t)$ for almost all t in $\mathbb{R}$. The set of all such functions will be denoted by $P(X)$, and $A^T(X)$ stands for the subspace of $P(X)$ consisting of all functions which are absolutely continuous in $(iT, (i + 1)\,T)$, $i \in \mathbb{Z}$, and whose squares of norms of derivatives are integrable.

Definition: We say that $x \in A^T(X)$ is a *solution* to (1.1) when there exists some $p \in A^T(X)$ such that

$$-dp(t)/dt \in \partial\varphi(t, x(t)) \text{ a.e.,} \tag{1.5a}$$

$$p(t) \in \partial\psi(t, x'(t)) \quad \text{a.e.} \tag{1.5b}$$

Now, we state the main result of the paper.

Theorem 1.1: *Assume that the following hypotheses are satisfied:*

(a) φ, ψ *satisfy the growth conditions*

$$\varphi(t, x) \leqq (4l)^{-1}\,|x|^2 - d(t), \quad \psi(t, z) \geqq (4k)^{-1}\,|z|^2 - e(t)$$

for $(t, x), (t, z) \in \mathbb{R} \times X$; $d, e\colon \mathbb{R} \to \mathbb{R}$ *are T-periodic and summable in* $[0, T]$; $k > 0$ *and* $\pi^2 l > kT^2$; $v \to \int_0^T \varphi^*(t, v(t))\,dt$ *is finite on* $L^\infty(0, T; B)$, *where* $B = \{v \in X : |v| \leqq j\}$ *for some* $j > 0$; *there exists* $z \in L^2(0, T; X)$ *such that* $\int_0^T \psi(t, z(t))\,dt < \infty$.

(b) φ, ψ *satisfy the representation*

$$\psi(t, z) = \psi_1(t, z) + 2^{-1}\,|z|^2 \text{ and } \varphi^*(t, v) = \varphi_1(t, v) + a\,|v|^2,$$

$a \geqq T^2/2\pi^2$, *where* $z \to \int_0^T \psi_1(t, z(t))\,dt$ *and* $v \to \int_0^T \varphi_1(t, v(t))\,dt$ *are sequentially weakly lower semicontinuous in* $L^2(0, T; X)$.

Then there exists a pair $(\overline{x}_T, \overline{p}_T)$, *being a solution to* (1.1), *whose restriction* $(\overline{x}, \overline{p})$ *to* $[0, T]$ *satisfies*

$$J_D(\overline{p}) = \inf_{p \in A(X)} J_D(p) = \inf_{x' \in L^2}\ \sup_{x(0) \in X} J(x) = J(\overline{x}). \tag{1.6}$$

Conversely, for any $\overline{p}_T \in A^T(X)$ for which there exists $\tilde{x}_T \in A^T(X)$ such that their restriction to $[0, T]$ satisfies (1.6), one can choose $\overline{x}_T \in A^T(X)$ satisfying, together with $\overline{p}_T$, (1.5) and, consequently, (1.1).

2. Auxiliary results

Let $L^2(0, T; X)$ and $L^\infty(0, T; X)$ denote the usual Banach spaces. It is known [3] that, for each $x \in A(X)$, the derivative $x'(t)$ exists almost everywhere in $[0, T]$. Therefore $A(X)$ can be identified with $X \oplus L^2(0, T; X)$ normed by $\|x\|_{A(X)} = |x(0)| + \|x'\|_{L^2}$. Here $A_0(X)$ denotes the subset of $A(X)$ of those x for which $x(0) = x(T) = 0$. Let $B^2(X)$ be the linear space $X \oplus L^2(0, T; X)$ with the norm $\|c, v\|_2 = \max \{|c|, \|v\|_{L^2}\}$. The dual $A^*(X)$ of $A(X)$ will be identified with $B^2(X)$ under the pairing

$$\langle x, (c, v)\rangle = \langle x(0), c\rangle + \int_0^T \langle x'(t), v(t)\rangle\, dt. \tag{2.1}$$

The conjugate of a function $g\colon X \to [-\infty, +\infty]$ is the function $g^*\colon X \to [-\infty, +\infty]$ defined by $g^*(x^*) = \sup \{\langle x^*, x\rangle - g(x)\colon x \in X\}$; it is lower semicontinuous and convex.

For a concave-convex function $G\colon X \times X \to \mathbb{R}$, $\partial G(z; p)$ is the set of all $(u, v) \in X \times X$ such that

$$G(z, \overline{p}) \geqq G(z, p) + \langle v, \overline{p} - p\rangle \quad \text{for all } \overline{p} \in X,$$
$$G(\overline{z}, p) \leqq G(z, p) + \langle u, \overline{z} - z\rangle \quad \text{for all } \overline{z} \in X.$$

Theorem 2.1 [9]: *Let $g\colon [0, T] \times X \to \mathbb{R}$ be $\mathbb{L} \times \mathbb{B}$-measurable and let $g(t, \cdot)$, $t \in [0, T]$, be lower semicontinuous and convex. Assume that $x \to \int_0^T g(t, x(t))\, dt$ is finite on $L^\infty(0, T; B)$, $B = \{x \in X\colon |x| \leqq j\}$ for a certain $j > 0$. Then $t \to \sup \{g(t, x)\colon |x| \leqq j\}$ is summable in $[0, T]$.*

Theorem 2.2 [10]: *Let g be the same as in Theorem 2.1 and assume that there is a $p \in L^2(0, T; X)$ such that $\int_0^T g^*(t, p(t))\, dt < +\infty$. Then the functionals $v \to \int_0^T g(t, v(t))\, dt$ and $p \to \int_0^T g^*(t, p(t))\, dt$ are convex and lower semicontinuous in $L^2(0, T; X)$ and they are in duality with respect to the pairing $\langle v, p\rangle = \int_0^T \langle v(t), p(t)\rangle\, dt$.*

Lemma 2.1: *If $x \in A_0(X)$, then*

$$\int_0^T |x(t)|^2\, dt \leqq (T^2/\pi^2) \int_0^T |x'(t)|^2\, dt. \tag{2.2}$$

Proof: Let $x_k = (1/T) \int_0^T x'(t) \exp(-2k\pi i t/T)\, dt$, $k \in \mathbb{Z}$.

Then expanding $x' \in L^2(0, T; X)$ in a Fourier series, we get

$$x'(t) = \sum_{k \in \mathbb{Z}} x_k \exp(2k\pi i t/T), \quad t \in [0, T]. \tag{2.3}$$

Since $x \in A_0(X)$, therefore $\int_0^T x'(t)\, dt = 0$, so $x_0 = 0$. By integrating (2.3) termwise,

$x(t) = \sum_{0 \neq k \in \mathbb{Z}} (T/2k\pi i) x_k(\exp(2k\pi i t/T) - 1)$. Hence

$$\|x\|_{L^2}^2 = \int_0^T |x(t)|^2 \, dt = \sum_{0 \neq k \in \mathbb{Z}} (T/2k\pi)^2 |x_k|^2 + \Big| \sum_{0 \neq k \in \mathbb{Z}} (T/2k\pi) x_k \Big|^2$$
$$\leqq (T^2/\pi^2) \sum_{k \in \mathbb{Z}} |x_k|^2 = (T^2/\pi^2) \|x'\|_{L^2}^2,$$

and so we have (2.2) ∎

Lemma 2.2: *Let $c > 0$, $cT^2 \leqq \pi^2$. Then the function*

$$x \to Q(x) = \int_0^T \big(|x'(t)|^2 - c\,|x(t)|^2\big)\, dt$$

is convex and lower semicontinuous in $A_0(X)$.

Proof: Let $x, v \in A_0(X)$ and let $\alpha \in [0, T]$. Then, using Lemma 2.1, a direct calculation shows

$$(1 - \alpha) Q(x) + \alpha Q(v) - Q\big((1 - \alpha) x + \alpha v\big) = (1 - \alpha) \alpha Q(x - v)$$
$$\geqq (1 - \alpha) \alpha \big((\pi^2/T^2) - c\big) \|x - v\|_{L^2}^2 \geqq 0,$$

and thus, the convexity of Q. Since each of the summands of Q is continuous in $A_0(X)$ thus Q is also lower semicontinuous in $A_0(X)$ ∎

Suppose further that all assumptions of Theorem 1.1 are satisfied.

Lemma 2.3: *Let $S_b = \{p \in A(X): J_D(p) \leqq b\}$, $b > 0$. For sufficiently large b, S_b are non-empty and bounded in the supremum norm $\|\cdot\|_C$. Moreover, J_D is bounded from below.*

Proof: Fix $b > 0$ and take any $p \in A(X)$ such that $J_D(p) \leqq b$ (such b and p exist by the assumptions on φ and ψ). Then, of course, $p(0) = p(T) = 0$ and, by Theorem 1.1/(b),

$$b \geqq J_D(p) \geqq l \int_0^T |p'(t)|^2 \, dt + \int_0^T e(t)\, dt - k \int_0^T |p(t)|^2 \, dt - \int_0^T d(t)\, dt.$$

By Lemma 2.1 and the inequality $|p(t)|^2 \leqq T \int_0^T |p'(t)|^2 \, dt$ we further obtain

$$b + \int_0^T \big(d(t) - e(t)\big)\, dt \geqq (l - kT^2/\pi^2) \int_0^T |p'(t)|^2 \, dt \geqq (l/T - kT/\pi^2)\, |p(t)|^2$$

for all t in $[0, T]$. Hence we infer the assertions of the lemma ∎

Lemma 2.4: *Let*

$$f\big(t, z(t)\big) = l\,|p'(t)|^2 + e(t) + (4k)^{-1} |x'(t)|^2 - d(t) - E\,|x'(t)|,$$

where $z(t) = \big(x'(t), p'(t)\big)$ and $E > 0$. Then the sets

$$Z_C = \{z \in L^2(0, T; X \times X): \int_0^T f\big(t, z(t)\big)\, dt \leqq c\}, \quad c \in \mathbb{R},$$

are either empty or relatively sequentially weakly compact in $L^2(0, T; X \times X)$.

Proof: Since $L^2(0, T; X \times X)$ is reflexive, it suffices to note that $\int_0^T f\big(t, z(t)\big)\, dt / \|z\|_{L^2} \to +\infty$ as $\|z\|_{L^2} \to +\infty$ ($\|z\|_{L^2} = \|x'\|_{L^2} + \|p'\|_{L^2}$) ∎

3. Duality results

In what follows we assume that the hypotheses of Theorem 1.1 are satisfied. We define, for each $p \in A(X)$, the perturbation of J_D as

$$J_{Dp}(a, g) = -l_1(p(0) + a) - l_2(p(T)) + \int_0^T \left(-\varphi^*(t, -p'(t)) + \psi^*(t, p(t) + g(t))\right) dt$$

for $(a, g) \in B^2(X)$. Of course, $J_{Dp}(0) = -J_D(p)$. Next, define, for $p, x \in A(X)$,

$$J^{\#}_{Dp}(x) = \sup_{g \in L^2} \left\{ \int_0^T \langle g(t), x'(t)\rangle\, dt - \int_0^T \left(\psi^*(t, p(t) + g(t)) - \varphi^*(t, -p'(t))\right) dt + l_2(p(T)) \right\} + \inf_{a \in X} \{\langle a, x(0)\rangle + l_1(p(0) + a)\}.$$

A direct calculation gives

$$J^{\#}_{Dp}(x) = -\langle x(T), p(T)\rangle + l_2(p(T)) + \int_0^T \langle x(t), p'(t)\rangle\, dt + \int_0^T \varphi^*(t, -p'(t))\, dt + \int_0^T \psi(t, x'(t))\, dt. \tag{3.1}$$

Thus

$$\begin{aligned}\sup_{p \in A(X)} \{-J^{\#}_{Dp}(x)\} &= \sup_{b \in X} \{\langle b, x(T)\rangle - l_2(b)\} + \sup_{v \in L^2} \left\{ \int_0^T \langle v(t), x(t)\rangle\, dt - \int_0^T \varphi^*(t, v(t))\, dt \right\} - \int_0^T \psi(t, x'(t))\, dt \\ &= -\int_0^T \left(-\varphi(t, x(t)) + \psi(t, x'(t))\right) dt = -J(x)\end{aligned} \tag{3.2}$$

for $x \in A(X)$. For $p \in A(X)$ and $(a, g) \in B^2(X)$, define

$$J^{\#\#}_{Dp}(a, g) = \sup_{w \in L^2} \left\{ \int_0^T \langle g(t), w(t)\rangle\, dt - \int_0^T \langle -p(t), w(t)\rangle\, dt - \int_0^T \psi(t, w(t))\, dt - l_2(p(T)) - \int_0^T \varphi^*(t, -p'(t))\, dt \right\} + \inf_{x(0) \in X} \{\langle x(0), a\rangle + \langle p(0), x(0)\rangle\}.$$

We see that $J^{\#\#}_{Dp}(0) = J_{Dp}(0)$ for all $p \in A(X)$. Using the minimax theorem [3], (3.1) and (3.2), we calculate that

$$\begin{aligned}\sup_{p \in A(X)} J^{\#\#}_{Dp}(0) &= \sup_{p \in A(X)} \sup_{x' \in L^2} \inf_{x(0) \in X} \{-J^{\#}_{Dp}(x)\} \\ &= \sup_{x' \in L^2} \inf_{x(0) \in X} \{-J(x)\} = -\inf_{x' \in L^2} \sup_{x(0) \in X} J(x).\end{aligned} \tag{3.3}$$

A direct consequence of the above considerations is the following

Theorem 3.1: *Let J, J_D be as above. Then* $\inf_{p \in A(X)} J_D(p) = \inf_{x' \in L^2} \sup_{x(0) \in X} J(x)$.

Theorem 3.2: *Let $\overline{p} \in A(X)$ be a minimizer for J_D and let $\partial J_{D\overline{p}}(0)$ be non-empty. Then there exists $(\overline{x}(0), \overline{x}') \in \partial J_{D\overline{p}}(0)$, where $\overline{x}' \in L^2(0, T; X)$ and $\overline{x}(t) = \overline{x}(0) + \int_0^t \overline{x}'(s)\, ds$, such that $\overline{x} \in A(X)$ and $J(\overline{x}) = \inf_{x' \in L^2} \sup_{x(0) \in X} J(x)$. Furthermore,*

$$J_{D\overline{p}}(0) + J^{\#}_{D\overline{p}}(\overline{x}) = 0, \qquad J(\overline{x}) - J^{\#}_{D\overline{p}}(\overline{x}) = 0. \tag{3.4}$$

Proof: The assumptions on φ^* and ψ^* imply that, for the minimizer $\overline{p}$, $\overline{p}(0) = \overline{p}(T) = 0$. We shall prove that $J_D(\overline{p}) \geqq \sup \{J(\tilde{x}) : x(0) \in X\} = J(\overline{x})$, where $\tilde{x}(t) = x(0) + \int_0^t \overline{x}'(s)\, ds$ and $(\overline{x}(0), \overline{x}') \in \partial J_{D\overline{p}}(0)$. Put $J_D(\overline{p}) = i_D > -\infty$. By the definition of ∂J_{Dp}, for the concave-convex function $J_{Dp}(\cdot, \cdot)$ we have, for each $(x(0), \overline{x}') \in \partial J_{D\overline{p}}(0)$,

$$-l_1(a) \leqq \langle x(0), a \rangle \quad \forall\, a \in X, \quad x(0) \in X,$$

$$\int_0^T \psi^*(t, \overline{p}(t) + g(t))\, dt \geqq \int_0^T \psi^*(t, \overline{p}(t))\, dt + \int_0^T \langle g(t), \overline{x}'(t) \rangle\, dt$$

for all $g \in L^2(0, T; X)$. The last inequality implies

$$J_{D\overline{p}}(0, g) \geqq -i_D + \int_0^T \langle g(t), \overline{x}'(t) \rangle\, dt \qquad \forall\; g \in L^2(0, T; X).$$

Further, for $\tilde{x}(t) = x(0) + \int_0^t \overline{x}'(s)\, ds$, $x(0) \in X$, we obtain

$$J^{\#}_{D\overline{p}}(\tilde{x}) = \sup_{g \in L^2} \left\{ \int_0^T \langle g(t), \overline{x}'(t) \rangle\, dt - J_{D\overline{p}}(0, g) \right\} \leqq i_D .$$

Therefore $-J(\tilde{x}) = \sup_{p \in A(X)} \{-J^{\#}_{Dp}(\tilde{x})\} \geqq -i_D$, thus $\sup_{x(0) \in X} J(\tilde{x}) \leqq i_D$.

In view of the assumptions on φ and the definition of φ^*, we notice that $\varphi(t, \tilde{x}(t)) + r(t) \geqq j\, |\tilde{x}(t)|$, $r(t) := \sup \{\varphi^*(t, v) : |v| \leqq j\}$, for t in $[0, T]$. The function r is summable in $[0, T]$ (see Theorem 2.1). Observing that $\int_0^T |\tilde{x}(t)| \geqq T \left(|x(0)| - \int_0^T |\overline{x}'(t)|\, dt \right)$, we infer that $\int_0^T \varphi(t, \tilde{x}(t))\, dt \to +\infty$ as $|x(0)| \to +\infty$. Thus, in virtue of the convexity and lower semicontinuity of $x(0) \to \int_0^T \varphi(t, \tilde{x}(t))\, dt$, we conclude that there exists $\overline{x}(0)$ in X such that

$$\int_0^T \varphi(t, \overline{x}(t))\, dt = \min_{x(0) \in X} \int_0^T \varphi(t, \tilde{x}(t))\, dt,$$

where

$$\overline{x}(t) = \overline{x}(0) + \int_0^t \overline{x}'(s)\, ds,$$

i.e. $\sup \{J(\tilde{x}) : x(0) \in X\} = J(\overline{x})$. This proves the first assertion of the theorem.

Since $J_{D\overline{p}}(0) = -i_D$ and $J(\overline{x}) = i_D$, therefore $J_{D\overline{p}}(0) + J(\overline{x}) = 0$. The inclusion $(\overline{x}(0), \overline{x}) \in \partial J_{D\overline{p}}(0)$ together with the former equality gives (3.4) ∎

Corollary 3.1: *Let $\bar{p} \in A_0(X)$ minimize J_D over $A(X)$. Then there exists $\bar{x} \in A(X)$, $\bar{x}(t) = \bar{x}(0) + \int_0^t \bar{x}'(s)\,ds$, such that $\bar{p}(t) \in \partial\psi(t, \bar{x}'(t))$ and $-\bar{p}'(t) \in \partial\varphi(t, \bar{x}(t))$ for almost all t in $[0, T]$, and*

$$J(\bar{x}) = \inf_{x' \in L^2} \sup_{x(0) \in X} J(x) = J_D(\bar{p}). \tag{3.5}$$

Proof: Since $g \to \int_0^T \psi^*(t, \bar{p}(t) + g(t))\,dt$ is finite in $L^2(0, T; X)$, thus, by [6, Prop. I.5.2], we see that $\partial J_{D\bar{p}}(0)$ is non-empty. Hence and by Theorem 3.2, there exists $\bar{x} \in A(X)$ satisfying, along with $\bar{p} \in A(X)$, (3.4) and (3.5). Rewriting (3.4) explicitly, we obtain the inclusions for $\bar{p}$ and $-\bar{p}'$, as asserted ∎

Theorem 3.3: *Let*

$$I(x', p) = l_1(p(0)) + l_2(p(T)) - \int_0^T \langle p(t), x'(t)\rangle\,dt + \int_0^T \varphi^*(t, -p'(t))\,dt + \int_0^T \psi(t, x'(t))\,dt.$$

Then

$$\inf_{(x',p) \in L^2 \times A(X)} I(x', p) = \inf_{p \in A(X)} J_D(p) = \inf_{x' \in L^2} \sup_{x(0) \in X} J(x)$$

$\left(x(t) = x(0) + \int_0^t x'(s)\,ds\right)$. *Moreover, if $(\bar{x}', \bar{p})$ is a minimizer for I, then $\bar{p}$ is a minimizer for J_D.*

Proof: By the form of I and Theorem 2.2, $\inf\{I(x', p): x' \in L^2\} = J_D(p)$ for all $p \in A(X)$, and so we have the first equality. The second one follows from Theorem 3.1.

Suppose that $(\bar{x}', \bar{p})$ minimizes I on $L^2(0, T; X) \times A(X)$. Then $I(\bar{x}', \bar{p}) = \inf\{J_D(p): p \in A(X)\}$. By the first part of the proof, $J_D(\bar{p}) = \inf\{I(x', \bar{p}): x' \in L^2\} \leq I(\bar{x}', \bar{p})$. Therefore $J_D(\bar{p}) = \inf\{J_D(p): p \in A(X)\}$ ∎

4. The proof of Theorem 1.1

We begin with showing that the I defined in Theorem 3.3 is sequentially weakly lower semicontinuous in $L^2(0, T; X) \times A_0(X)$.

By our assumptions, $\varphi^*(t, v) = \varphi_1(t, v) + a\,|v|^2$ and $\psi(t, u) = \psi_1(t, u) + 2^{-1}\,|u|^2$, where $p' \to \int_0^T \varphi_1(t, p'(t))\,dt$ and $x' \to \int_0^T \psi_1(t, x'(t))\,dt$ are sequentially weakly lower semicontinuous in $L^2(0, T; X)$. The function $(p, v) \to 2^{-1}\,|p - v|^2$ is convex and lower semicontinuous in $X \times X$, so $(x', p) \to 2^{-1} \int_0^T |p(t) - x'(t)|^2\,dt$ is sequentially weakly lower semicontinuous in $L^2(0, T; X) \times A_0(X)$. In virtue of Lemma 2.2,

$p \to a \int_0^T \left(|p'(t)|^2 - (2a)^{-1} |p(t)|^2\right) dt$ is also such a function in $A_0(X)$. Hence we see that

$$I(x', p) = l_1(p(0)) + l_2(p(T)) + 2^{-1} \int_0^T |p(t) - x'(t)|^2 \, dt + \int_0^T (a|p'(t)|^2 - 2^{-1} |p(t)|^2) \, dt + \int_0^T \varphi_1(t, p'(t)) \, dt + \int_0^T \psi_1(t, x'(t)) \, dt$$

is, really, sequentially weakly lower semicontinuous in $L^2(0, T; X) \times A_0(X)$.

Let now $E > 0$ be sufficiently large. Put $D = \{p \in X : |p| \leqq E\}$ and $h(p, u) = \langle p, u\rangle$ if $(p, u) \in D \times X$, $h(p, u) = +\infty$ otherwise. Denote by I_h the functional I with the term $\int_0^T h(p(t), x'(t)) \, dt$ instead of $\int_0^T \langle x'(t), p(t)\rangle \, dt$. In view of Lemma 2.3 and Theorem 3.3, $\inf I(x', p) = \inf I_h(x', p)$ on $L^2 \times A(X)$. Let us take any $(x_b', p_b) \in L^2(0, T; X) \times A_0(X)$ and put $b = I(x_b', p_b)$. Of course, $I(x_b', p_b) \geqq J_D(p)$ for all $p \in A_0(X)$. Next, choose $E > 0$ so large that each $p \in S_b$ (S_b defined as in Lemma 2.3) is contained in the interior of D; this is possible by Lemma 2.3. We easily check that $I_h(x', p) \geqq \int_0^T f(t, z(t)) \, dt$ for all $(x', p) \in L^2(0, T; X) \times A(X)$, where f is defined in Lemma 2.4. By this lemma, the set

$$M = \{(x', p) \in L^2(0, T; X) \times A(X) : I_h(x', p) \leqq b\}$$

is relatively sequentially weakly compact in $L^2(0, T; X) \times A(X)$ (in fact, for all p from this set, $p(0) = p(T) = 0$, and then, the weak compactness of p' in $L^2(0, T; X)$ implies that of p in $A(X)$). Let $\{(x_n', p_n)\} \subset M$ be a minimizing sequence for I_h. Then, by the above construction, it is also minimizing for I, and $I(x_n', p_n) = I_h(x_n', p_n)$, $n \in \mathbb{N}$. Since I is sequentially weakly lower semicontinuous and $\{(x_n', p_n)\}$ is relatively sequentially weakly compact, there exists a subsequence of $\{(x_n', p_n)\}$ converging to some $(\bar{x}', \bar{p})$ in M being a minimizer for I. Hence and by Theorem 3.3, $\bar{p}$ is a minimizer for J_D.

Further, from Corollary 3.1 we obtain that there exists $\bar{x} \in A(X)$ such that $\bar{p}(t) \in \partial\psi((t, \bar{x}'(t))$ and $-\bar{p}'(t) \in \partial\varphi(t, \bar{x}(t))$ almost everywhere in $[0, T]$, and (3.5) holds.

In view of the T-periodicity of the functions $\varphi^*(\cdot, x)$, $\psi^*(\cdot, x)$, $x \in X$, the functions $\varphi^*(\cdot, x)$, $\psi(\cdot, x)$ are T-periodic, too. Thus each minimizer $\bar{p}$ of J_D can be identified with some $\bar{p}_T \in A^T(X)$ restricted to $[0, T]$. This means that if we consider the functional

$$\int_{iT}^{(i+1)T} \left(\varphi^*(t, -p'(t)) - \psi^*(t, p(t))\right) dt + l\left(p(iT), p((i + 1) T)\right),$$

$i \in \mathbb{Z}$, instead of J_D, then $\bar{p}_T$ restricted to $[iT, (i + 1) T]$ is a minimizer for it in the space of absolutely continuous functions $p: [iT, (i + 1) T] \to X$. Therefore the $\bar{x}' \in L^2(0, T; X)$ from the assertion of Theorem 3.2 can be identified with some function $\bar{x}_T' \in P(X)$ restricted to $[0, T]$. Define

$$\bar{x}_T(t) = \bar{x}(0) + \int_{iT}^t \bar{x}_T'(s) \, ds \qquad \left(t \in [iT, (i + 1) T); \; i \in \mathbb{Z}\right).$$

Of course, $\bar{x}_T \in A^T(X)$. Corollary 3.1 implies that $\bar{x}_T$, $\bar{p}_T$ satisfy (1.5). The proof of Theorem 1.1 is completed ∎

5. An example of equation (1.1)

Let $j\colon \mathbb{R} \times \mathbb{R} \to (-\infty, \infty)$ be bounded, $\mathbb{L}$-measurable and T-periodic with respect to the first variable, continuously differentiable and convex with respect to the second variable and such that $j(t, r) \geqq e(t)$, $(t, r)' \in [0, T] \times \mathbb{R}$, $e\colon \mathbb{R} \to \mathbb{R}$ is T-periodic and summable in T. We put $\psi_1(t, z) = \int_{\mathbb{R}/U\mathbb{Z}} j(t, z(u))\, du$ for $(t, z) \in \mathbb{R} \times L_U^2(\mathbb{R}, \mathbb{R})$ ($L_U^2(\mathbb{R}, \mathbb{R})$ is the space of U-periodic functions, $U > 0$, whose restrictions to $[0, U]$ belong to $L^2(0, U; \mathbb{R})$). Let $X = H_U^1(\mathbb{R}, \mathbb{R})$ ($H_U^1(\mathbb{R}, \mathbb{R})$ is the Hilbert space of all U-periodic functions whose restrictions to $(-M, M)$ belong to the Sobolev space $H^1(-M, M; \mathbb{R})$ for all $M > 0$) with the norm $|z|^2 = \int_0^U (z_u(u))^2\, du + \int_0^U (z(u))^2\, du$. $\psi_1(t, \cdot)$ is convex and lower semicontinuous in X, $t \in [0, T]$; ψ_1 is also $\mathbb{L} \times \mathbb{B}$-measurable in $\mathbb{R} \times X$. In consequence, $z \to \int_0^T \psi_1(t, z(t))\, dt$ is sequentially weakly lower semicontinuous in $L^2(0, T; X)$. We set $\psi(t, z) = (4k)^{-1}|z|^2 + \psi_1(t, z)$, $k > 0$, for $(t, z) \in \mathbb{R} \times X$. Thus all assumptions of Theorem 1.1 concerning ψ are satisfied. Now, take $\varphi(t, x) = 2^{-1}|x|^2$ and assume that k and T are such that $\pi^2 > 2kT^2$ and $\pi \geqq T$. Then we have fulfilled all assumptions of Theorem 1.1. From (1.5a) we calculate for our φ that $-p(t) = \int_0^t x(s)\, ds$. Therefore by Theorem 1.1 there exists $\bar{x}_T \in A^T(X)$ whose restriction $\bar{x}$ to $[0, T]$ together with $\bar{p}(t) = -\int_0^t \bar{x}(s)\, ds$, $t \in [0, T]$, satisfy (1.6) for our φ, ψ and it is a solution to the problem

$$(d/dt)\Big((\partial/\partial r)\, j(t, x_t(t, u))\Big) + (2k)^{-1}\,(x_t(t, u) - \Delta x_t(t, u))$$
$$- \Delta x(t, u) + x(t, u) = 0,$$
$$x(t + T, u) = x(t, u), \quad x(t, u + U) = x(t, u), \quad (t, u) \in \mathbb{R} \times \mathbb{R},$$
$$\int_0^T x(t, u)\, dt = 0, \quad u \in \mathbb{R},$$

where Δ is Laplace's operator in u.

What is essentially new here is that $\bar{x}$ satisfies (1.6). This is also interesting in its own right as it is not easily to prove directly that the functional

$$J(x) = \int_0^T \int_0^U \Big(j(t, x_t(t, u)) + (4k)^{-1}\big((x_{tu}(t, u))^2 + (x_t(t, u))^2\big)$$
$$- 2^{-1}\big((x_u(t, u))^2 + (x(t, u))^2\big)\Big)\, du\, dt$$

attains its minimum in any reasonable space of functions without boundary conditions.

REFERENCES

[1] AUCHMUTY, G.: Duality for non-convex variational principles. J. Diff. Equ. 50 (1983), 80 to 145.

[2] BREZIS, H.: Periodic solutions of nonlinear vibrating strings and duality principles. Bull. Amer. Math. Soc. 8 (1983), 409–426.

[3] BREZIS, H.: Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert (Math. Studies: Vol. 5). Amsterdam: North-Holland 1973.

[4] CLARKE, F. H.: Periodic solution to Hamiltonian inclusion. J. Diff. Equ. 40 (1981), 1–6.

[5] CLARKE, F. H., and I. EKELAND: Hamiltonian trajectories having prescribed minimal period. Comm. Pure Appl. Math. 33 (1980), 103–116.

[6] Ekeland, I., and R. Temam: Convex Analysis and Variational Problems. Amsterdam — Oxford: North-Holland Publ. Comp. 1976.
[7] Hiriart-Urruty, J.-B.: From convex optimization to nonconvex optimization. Part I: Necessary and sufficient conditions for global optimality. In: Nonsmooth Optimization and Related Topics (Eds.: F. H. Clarke, V. F. Demyanov and F. Giannessi). New York: Plenum Press 1989.
[8] Mawhin, J.: Problèmes de Dirichlet Variationnels Non Linéaires (Sém. Math. Sup. Univ. Montréal 104). Montréal: Les presses de l'Université 1987.
[9] Nowakowski, A.: Note on convex integral functionals. Demonstratio Math. 21 (1988), 827—834.
[10] Rockafellar, R. T.: Convex integral functionals and duality. In: Contributions to Nonlinear Functional Analysis (Ed.: E. Zarantonello). New York: Academic Press 1971.
[11] Toland, J. F.: A duality principle for non-convex optimization and the calculus of variations. Arch. Rat. Mech. Anal. 71 (1979), 41—61.
[12] Toland, J. F.: Duality in nonconvex optimization. J. Math. Anal. Appl. 66 (1978), 399 to 415.

Manuskripteingang: 30. 11. 1988; in revidierter Fassung 01. 11. 1989

VERFASSER:

Dr. Andrzej Nowakowski und Dr. Andrzej Rogowski
University of Łódź
Institute of Mathematics
ul. S. Banacha 22
PL — 90-238 Łódź
Poland