Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 1, p. 73–82

Optimal Control of a Nonlinear Singular Integral Equation Arising in Electrochemical Machining

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In der Arbeit wird ein optimales Steuerproblem für eine nichtlineare singuläre Integralgleichung vom Cauchy-Typ untersucht. Die Existenz mindestens einer optimalen Lösung wird nachgewiesen und eine notwendige Optimalitätsbedingung hergeleitet.

В данной работе исследуется задача оптимального управления для нелинейного сингулярного интегрального уравнения типа Коши. Доказывается существование по меньшей мере одного оптимального решения и выводится необходимое условие оптимальности.

The paper is concerned with an optimal control problem for a nonlinear singular integral equation of Cauchy type. The existence of at least one optimal solution is proved and a necessary optimality condition is derived.

1. Problem statement and introductory remarks

We start with the description of the optimal control problem (**P**) considered in this paper. Let there be given two bounded closed intervals $[a, b]$ and $[c, d]$ with $b < c$, positive constants l, d, m and M with $ml \leqq d \leqq Ml$ (cf. (3)), positive constants c_1, c_2 and an arbitrary real number c_0 with $b < c_0 < c$ (cf. (4c)). In the whole paper, as usual, $C^\nu[a, b]$, $\nu \in (0, 1]$ to be specified below, denotes the space of all ν-Hölder continuous functions y equipped with the norm $\|y\|_\nu = \max |y(t)| + \sup \{|y(t) - y(s)| / |t - s|^\nu\}$, and $C^{1,\nu}[a, b]$ denotes the space of all continuously differentiable functions y whose derivative y' belongs to $C^\nu[a, b]$ and which is equipped with the norm $\|y\|_{1,\nu} = \max |y(t)| + \|y'\|_\nu$. We introduce the nonlinear Nemytskij operator (superposition operator)

$$\mathrm{G}y = g(y), \quad g(y)\,(t) = g\big(y(t)\big), \quad t \in [a, b], \tag{1}$$

generated by a given function $g = g(x)$, $x \in [0, l]$, and the linear singular integral operator of Cauchy type

$$\mathrm{S}y(t) = \frac{1}{\pi} \int_a^b \frac{y(s)}{s - t}\, ds, \quad t \in [a, b] \cup [c, d]. \tag{2}$$

With these data and abbreviations we define the *set of admissible controls*

$$G_{\mathrm{ad}} = \{g \in C^2[0, l]: g(0) = 0, g(l) = d, m \leqq g' \leqq M\}, \tag{3}$$

the *state equation*

$$\mathrm{G}y(t) - \mathrm{S}y(t) - D = p(t), \quad t \in [a, b], \tag{4a}$$

$$y(a) = y(b) = 0, \tag{4b}$$

with the given right-hand side

$$p(t) = \frac{l}{\pi(b-a)}\,[(b-a) + (t-b)\ln|t-b| - (t-a)\ln|t-a|]$$

$$- c_1 \ln|c_0 - t| + c_2 \int_{c_0}^{d} \ln|s-t| / \sqrt{(s^2 - c_0^2)\,(d^2 - s^2)}\, ds, \tag{4c}$$

$t \in [a, b] \cup [c, d]$, and the *cost functional*

$$\mathbf{J}(g) = \int_c^d h^2(t)\, dt, \quad h = Sy + D + p - q, \tag{5}$$

where $q \in C^r[c, d]$ is given. For fixed $g \in G_{ad}$ the state equation (4) is a *nonlinear* singular integral equation of Cauchy type containing the free parameter $D \in \mathbb{R}$, which must be determined together with the function

$$y \in C_0^r[a, b] = \{y \in C^r[a, b]: y(a) = y(b) = 0\},$$

such that the pair $w := \{y, D\} \in C_0^r[a, b] \times \mathbb{R} =: W_0^r$ satisfies (4a). Summing up we can write our control problem in the following form:

(P) Find $\inf\{\mathbf{J}(g): g \in G_{ad}, w \in W_0^r \text{ satisfies (4a)}\}$.

For this problem we will, in Section 2, discuss the existence of optimal solutions and, in Section 3, derive necessary optimality conditions.

To our best knowledge there are only very few papers dealing with control problems governed by a singular integral equation. This is rather surprising because both linear and nonlinear singular integral equations have a lot of applications in different branches of sciences and technology (cf. [3, 13, 14, 19, 24]). M. GOEBEL and L. v. WOLFERSDORF [8] have considered control problems with linear singular integral equations of both Hilbert and Cauchy type. Existence theorems and necessary and sufficient optimality conditions have been proved on the basis of [20], where v. WOLFERSDORF has dealt with control processes in Banach spaces with Noetherian operator equations acting as state equation. The generalization of this theory to control problems with nonlinear operator equations, where the linearized equation is supposed to be Noetherian, as presented by M. GOEBEL and L. v. WOLFERSDORF [9] (see also [5, 21] for short summaries), has enabled L. v. WOLFERSDORF [21] to outline some results concerning control problems with a linear singular integral equation of Cauchy type in weighted Lebesgue spaces and with nonlinear singular integral equations of Hilbert and Cauchy type in Hölder spaces, respectively. An isoperimetric variational problem involving a linear singular integral equation of Cauchy type has been discussed by T. YAO-TSU-WU and A. K. WHITNEY [24].

The control problem (P) is a slight generalization of a problem arising in electrochemical machining (abbreviated by ECM). Roughly speaking the control function g represents the shape of the cathode (with or without its isolating parts) and the given function q (cf. (5)) the wanted shape of the anode. If the control g is fixed and if $\{y, D\}$ denotes a solution to the state equation (4) related to g, then the term $Sy + D + p$ occuring in the cost functional (5) represents the shape of the anode caused by just this control g. That means, the problem (P) consists in finding such an admissible shape of the cathode that the corresponding shape of the anode approximates its wanted shape as good as possible (in the sense of L_2-norm). In other words, the control problem (P) is the output least squares formulation of the inverse ECM problem. For more detailed information concerning the technical background we refer the reader to D. OESTREICH [15, 16], M. GOEBEL and the papers cited there. Additionally we mention the nice monograph by J.-F. RODRIGUES [18], in which the ECM problem is derived and solved using the theory of variational inequalities. In [16] another approach to the inverse ECM pro-

blem has been outlined via the theory of Fredholm integral equations. We want to point out that the results presented in this paper could be also applied to certain optimization problems for the two-dimensional fluid flow through a nonlinearly shaped dam with a toe drain (cf. D. OESTREICH [14]).

2. On the existence of optimal controls

Clearly, the starting point of our investigations has to be the state equation (4), which, because of what follows below, will be considered for any $g \in G$,

$$G = \{g \in C^{1,1}[0,l]: g(0) = 0, \quad g(l) = d, \quad m \leqq g' \leqq M\}. \tag{6}$$

We introduce the Banach space

$$W^\nu = C^\nu[a,b] \times \mathbb{R} = \{w = \{y, D\}\}, \quad \|w\|_\nu = \|y\|_\nu + |D|,$$

$\nu \in (0,1]$; $W_0^\nu = C_0^\nu[a,b] \times \mathbb{R}$ is a subspace in W^ν.

Theorem 1: *Suppose $\nu \in (0,\lambda)$, where $\lambda = 1/2 - \arctan M/\pi$. Then for any $g \in G$ there exists a unique solution $w = w(g) \in W_0^\nu$ to the state equation (4) and, furthermore,*

$$\|w(g)\|_\nu \leqq \text{const} \ \forall\, g \in G. \tag{7}$$

Proof: Since the proof of the first statement is more or less completely the same as those given in D. OESTREICH [14] on the basis of L. v. WOLFERSDORF [23], we can restrict ourselves to some remarks concerning mainly the proof of (7). To this end, in (4a) $g \in G$ is replaced by a smooth extension $\tilde{g} \in C^1(\mathbb{R})$ with $\tilde{g}'(x) = g'(0)$ for $x \in (-\infty, 0)$ and $\tilde{g}'(x) = g'(l)$ for $x \in (l, \infty)$. Then by differentiating of (4a) we obtain a formally linear singular integral equation for which the solution can be given explicitly. Integrating this solution we come to a fixed point equation $y = \mathbf{P}y$ with known operator $\mathbf{P}$, see [23: § 2.2]. The estimations in [23: § 4.1] show that $\mathbf{P}: C_0^\nu[a,b] \to C_0^\nu[a,b]$ and that there exists a constant $c_0 > 0$ with $\|\mathbf{P}y\|_\lambda \leqq c_0$ for all $y \in C_0^\nu[a,b]$ and all $g \in G$, where ν and λ are given as above. This means, $\mathbf{P}$ maps the whole space $C_0^\nu[a,b]$ into its convex compact subset $Q = \{y \in C_0^\nu[a,b]: \|y\|_\lambda \leqq c_0\}$. Since $\mathbf{P}$ maps Q into itself continuously with respect to $\|\cdot\|_\nu$ (see [22]), the Schauder fixed point theorem yields the existence of at least one $y \in Q$ with $y = \mathbf{P}y$. Like in [14] it can be shown that $y(t) \in [0,l]$ for $t \in [a,b]$. Therefore, because the fixed point equation $y = \mathbf{P}y$ is equivalent to (4), problem (4) also has a solution $y \in Q$, for which holds

$$\|y\|_\lambda \leqq c_0 \quad \forall g \in G. \tag{8}$$

At this the parameter D is given by

$$D = g\big(y(a)\big) - \mathbf{S}y(a) - p(a) = -\mathbf{S}y(a) - p(a) \quad \big(= -\mathbf{S}y(b) - p(b)\big).$$

Due to (8), $\mathbf{S} \in \mathfrak{L}(C_0^\nu[a,b], C^\nu[a,b])$ (see, for example, S. PRÖSSDORF [17: § 3.4.1]) and $|\mathbf{S}y(a)| \leqq \|\mathbf{S}y\|_\nu$ we obtain (7) ∎

From now on, let $\nu \in (0,\lambda)$ be fixed.

It seems to be impossible to prove an existence theorem for our control problem (P). Therefore, we modify it by replacing the set of admissible controls G_{ad} by

$$G_{ad}^* = \{g \in C^{1,1}[0,l]: \ g(0) = 0, g(l) = d, m \leqq g'(x) \leqq M \ \forall x,$$
$$|g'(x) - g'(y)| \leqq k\,|x - y| \ \forall x, y\}, \tag{9}$$

where $k > 0$ is a given constant. The new optimal control problem will be called (**P***) and its set of admissible triples T^*_{ad}, i.e.,

$$T^*_{ad} = \{(g, y, D) : g \in G^*_{ad},\ w = (y, D) \in W_0^{\nu} \text{ solves (4)}\}.$$

In virtue of Theorem 1 we have $T^*_{ad} \neq \emptyset$. In the following lemma we prove the compactness of T^*_{ad} in a suitable space.

Lemma 1: *The set* $T^*_{ad} \subset C^{1,\mu}[0, l] \times C_0^{\varkappa}[a, b] \times \mathbb{R}$ *is compact for any fixed* $\mu \in (0, 1)$ *and* $\varkappa \in (0, \nu)$.

Proof: Because of the definition of G^*_{ad} and Theorem 1 the set T^*_{ad} considered in $C^{1,1}[0, l] \times C_0^{\varkappa}[a, b] \times \mathbb{R}$ is bounded and, hence, relatively compact in $C^{1,\mu}[0, l] \times C_0^{\varkappa}[a, b] \times \mathbb{R}$, where $\mu \in (0, 1)$ and $\varkappa \in (0, \nu)$. We show that T^*_{ad} is also closed in this space. Let $\{(g_n, y_n, D_n)\} \subset T^*_{ad}$ denote an arbitrary sequence converging to (g_0, y_0, D_0) in $C^{1,\mu}[0, l] \times C_0^{\varkappa}[a, b] \times \mathbb{R}$. The properties $g_0(0) = 0$, $g_0(l) = d$, $m \leqq g_0'(x) \leqq M$ for all x are evident. Since from

$$\begin{aligned} &|g_0'(x) - g_0'(y)| \\ &\leqq |g_0'(x) - g_n'(x)| + |g_n'(x) - g_n'(y) + |g_n'(y) - g_0'(y)| \\ &\leqq k\,|x - y| + |g_0'(x) - g_n'(x)| + |g_n'(y) - g_0'(y)| \end{aligned}$$

for all $x, y \in [0, l]$ and $n \in \mathbb{N}$ it follows that g_0' is Lipschitz continuous with the Lipschitz constant k, we find $g_0 \in G^*_{ad}$. Finally, since, because of $S \in \mathfrak{L}(C_0^{\varkappa}[a, b], C^{\varkappa}[a, b])$ and

$$\begin{aligned} &\big|g_n(y_n(t)) - g_0(y_0(t))\big| \\ &\leqq \big|g_n(y_n(t)) - g_n(y_0(t))\big| + \big|g_n(y_0(t)) - g_0(y_0(t))\big| \\ &\leqq M\,|y_n(t) - y_0(t)| + \big|g_n(y_0(t)) - g_0(y_0(t))\big| \end{aligned}$$

for all $t \in [a, b]$ and $n \in \mathbb{N}$, the relations $g_n(y_n(t)) - Sy_n(t) - D_n = p(t)$ imply $g_0(y_0(t)) - Sy_0(t) - D_0 = p(t)$, $t \in [a, b]$, we get $(g_0, y_0, D_0) \in T^*_{ad}$ which completes the proof ∎.

Theorem 2: *The optimal control problem* (**P***) *characterized by* (4), (5), (9) *has at least one optimal solution.*

Due to the above Lemma 1 and the obvious fact that the cost functional J maps $C_0^{\varkappa}[a, b] \times \mathbb{R}$ continuously into $\mathbb{R}$, the proof of this existence theorem is now standard and thus omitted.

3. Necessary optimality condition

This section is the main part of the paper. It is devoted the optimality conditions to be satisfied by each optimal solution

$$(g_0, w_0) \in G_{ad} \times W_0^{\nu}, \quad w_0 = (y_0, D_0)$$

to our original control problem (**P**). Unfortunately, it is not possible to apply to (**P**) some general method to be found, e.g., in [10] and in [4, 9] and the references cited there, since these theories usually need the partial Fréchet derivatives of the operator defining the state equation and the adjoint state space to describe the adjoint state.

Note that in our case the corresponding operator

$$\mathbf{F}(g, w) = \mathbf{G}y - \mathbf{S}y - D - p, \quad \mathbf{F}: C^2[0, l] \times W_0^\nu \to C^\nu[a, b],$$

is even not defined in a neighbourhood of $\{g_0, w_0\}$ and that $C_0^\nu[a, b]^*$ (and consequently $W_0^{\nu *}$) is at least very unconvenient, which is one of the reasons that the theory of linear singular integral equations of Cauchy type works only with the Hölder space and not with its dual, cf. [12, 17]. All this has led us to use the concept of directional derivative for finding necessary optimality conditions. As essential tools we use the theory of linear singular integral equations developed in MUSCHELISCHWILI [12: Kap. V] and continuity and differentiability properties of certain Nemytskij operators acting in Hölder spaces, which we have proved recently in [6]; concerning some other interesting properties of such operators we refer to [1, 2].

For arbitrarily fixed $g \in G_{ad}$ we define the convex linear combination $g_\varepsilon = (1 - \varepsilon) g_0 + \varepsilon g$, $\varepsilon \in [0, 1]$, and denote by $\mathbf{G}_\varepsilon$ the Nemytskij operator generated by g_ε (cf. (1)), and by $w_\varepsilon = \{y_\varepsilon, D_\varepsilon\} \in W_0^\nu$ the unique solution to (4) related to g_ε (cf. Theorem 1). Provided the directional derivative

$$\delta_+ \mathbf{J}(g_0; g - g_0) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \big(\Phi(\varepsilon) - \Phi(0)\big), \quad \Phi(\varepsilon) = \mathbf{J}(g_\varepsilon),$$

exists we have the obvious necessary optimality condition

$$\delta_+ \mathbf{J}(g_0; g - g_0) \geqq 0 \quad \forall g \in G_{ad}. \tag{10}$$

Therefore, in the following our main task is to calculate this directional derivative, which requires to study the behaviour of $\varepsilon^{-1}(w_\varepsilon - w_0)$ as $\varepsilon \downarrow 0$. First, however, we recall (cf. [12: Kap. V])

Lemma 2: *For each $f \in C^\nu[a, b]$, $\nu \in (0, \lambda)$, there exists a unique solution $\{y, D\} \in W_0^\nu$ to the linear singular integral equation*

$$g_0'\big(y_0(t)\big)\, y(t) - \mathbf{S}y(t) - D = f(t), \quad t \in [a, b].$$

Theorem 3: *Let $w = \{y, D\} \in W_0^\nu$ be the unique solution to*

$$g_0'\big(y_0(t)\big)\, y(t) - \mathbf{S}y(t) - D = \mathbf{G}_0 y_0(t) - \mathbf{G}y_0(t), \quad t \in [a, b]. \tag{11}$$

Then there exists an abstract function $\omega_\varepsilon = \{\varrho_\varepsilon, \chi_\varepsilon\} \in W_0^\nu$, $\varepsilon \in (0, \varepsilon_0)$ and sufficiently small, such that

$$w_\varepsilon = w_0 + \varepsilon w + \omega_\varepsilon \; \forall \varepsilon \in (0, \varepsilon_0), \quad \|\omega_\varepsilon\|_\nu = o(\varepsilon) \quad as \quad \varepsilon \downarrow 0. \tag{12}$$

Proof: 1. Let $\tilde{g}_0, \tilde{g} \in C^2(\mathbb{R})$ be arbitrary extensions of $g_0, g \in G_{ad}$. We define $\tilde{g}_\varepsilon = (1 - \varepsilon)\, \tilde{g}_0 + \varepsilon \tilde{g}$, $\varepsilon \in \mathbb{R}$, and introduce the Nemytskij operator $\tilde{\mathbf{G}}_\varepsilon$ by setting $\tilde{\mathbf{G}}_\varepsilon y = \tilde{g}_\varepsilon(y)$. From [6: Theorem 2] we know the following:

At each $z \in C^\nu[a, b]$, the operator $\tilde{\mathbf{G}}_\varepsilon: C^\nu[a, b] \to C^\nu[a, b]$ has a continuous Fréchet derivative $\tilde{\mathbf{G}}_\varepsilon'(z)\, y(t) = \tilde{g}_\varepsilon'\big(z(t)\big)\, y(t)$ $\forall y \in C^\nu[a, b]$.

Obviously, for any $\varepsilon \in [0, 1]$, $\tilde{\mathbf{G}}_\varepsilon$ is an extension of $\mathbf{G}_\varepsilon$ defined above. That means, setting $\mathfrak{D} = \{y \in C^\nu[a, b]: 0 \leqq y \leqq l\}$ we have

$$\tilde{\mathbf{G}}_\varepsilon' y = \mathbf{G}_\varepsilon y \; \forall y \in \mathfrak{D}; \quad \forall \varepsilon \in [0, 1]. \tag{14}$$

Consider the nonlinear singular integral equation

$$\tilde{\mathbf{G}}_\varepsilon y(t) - \mathbf{S}y(t) - D = p(t), \quad t \in [a, b],$$

which using the operator

$$\mathbf{F}: \mathbb{R} \times W_0^r \to C^r[a, b], \quad \mathbf{F}(\varepsilon, w) = \tilde{\mathbf{G}}_\varepsilon y - \mathbf{S}y - D - p$$

can be written as operator equation

$$\mathbf{F}(\varepsilon, w) = 0. \tag{15}$$

For this equation and its defining operator $\mathbf{F}$ we can establish the properties listed below:

(a) Because of Theorem 1 and (14), for each $\varepsilon \in [0, 1]$ the pair $w_\varepsilon = \{y_\varepsilon, D_\varepsilon\} \in W_0^r$ is the unique solution of (15). Particularly, $\mathbf{F}(0, w_0) = 0$.

(b) Essentially, due to (13), in each point $\{\delta, v\} \in \mathbb{R} \times W_0^r$, $v = \{z, E\}$, the operator $\mathbf{F}$ has a continuous partial Fréchet derivative $\mathbf{F}_w(\delta, v)$ given by

$$\mathbf{F}_w(\delta, v)\, w(t) = \tilde{g}_\delta'\big(z(t)\big)\, y(t) - \mathbf{S}y(t) - D \;\forall w = \{y, D\} \in W_0^r.$$

In particular we have

$$\mathbf{F}_w(0, w_0)\, w(t) = g_0'\big(y_0(t)\big)\, y(t) - \mathbf{S}y(t) - D \;\forall w = \{y, D\} \in W_0^r.$$

By Lemma 2 and a known theorem due to Banach (see., e.g., [25: Chap. III, §5] or [11: Kap. XII, §1]) the operator $\mathbf{F}_w(0, w_0)$ has a continuous inverse $\mathbf{F}_w(0, w_0)^{-1} \in \mathfrak{L}(C^r[a, b], W_0^r)$.

(c) Clearly, $\mathbf{F}$ has also a continuous partial Fréchet derivative with respect to ε at each point of $\mathbb{R} \times W_0^r$.

As a consequence of these properties the implicit function theorem to be found for example in [11: Kap. XVII, §4] can be applied to equation (15). Hence, there is an abstract function $w = w(\varepsilon)$ defined on $(-\varepsilon_1, \varepsilon_1)$, $\varepsilon_1 > 0$ sufficiently small, with the following two properties:

(d) $\mathbf{F}\big(\varepsilon, w(\varepsilon)\big) = 0 \;\forall \varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $w(0) = w_0$.

(e) At $\varepsilon = 0$, $w = w(\varepsilon)$ has a Frechet derivative.

In other words, after setting $w'(0) = w = \{y, D\}$ we can write

$$w(\varepsilon) = w(0) + \varepsilon w + \omega_\varepsilon \;\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

where $\varepsilon_0 \in (0, \varepsilon_1)$ is sufficiently small and $\omega_\varepsilon = \{\varrho_\varepsilon, \chi_\varepsilon\} \in W_0^r$ with $\|\omega_\varepsilon\|_r = o(\varepsilon)$ as $\varepsilon \downarrow 0$. Since, because of (a), $w(\varepsilon) = w_\varepsilon$ for $\varepsilon \in [0, \varepsilon_1) \wedge [0, 1]$, we have proved (12).

2. In virtue of Lemma 2 the linear integral equation (11) is uniquely solvable in W_0^r. Therefore, the theorem is completely proved after showing that its solution is just given by $w = \{y, D\}$ introduced in the first part of the proof.

By definition of w_0 and w_ε we have the identities

$$\mathbf{G}_\varepsilon y_\varepsilon - \mathbf{S}y_\varepsilon - D_\varepsilon = p \quad \text{and} \quad \mathbf{G}_0 y_0 - \mathbf{S}y_0 - D_0 = p,$$

$\varepsilon \in (0, 1]$, on $[a, b]$ from which with $\Delta_\varepsilon = \varepsilon^{-1}(\mathbf{G}_\varepsilon y_\varepsilon - \mathbf{G}_0 y_0)$ we get $\Delta_\varepsilon - \varepsilon^{-1}\mathbf{S}(y_\varepsilon - y_0) - \varepsilon^{-1}(D_\varepsilon - D_0) = 0$ on $[a, b]$. Because of $\mathbf{S} \in \mathfrak{L}(C_0^r[a, b], C^r[a, b])$, cf. [17: § 3.4.1], and relation (12), which is already proved, we find

$$\varepsilon^{-1}\mathbf{S}(y_\varepsilon - y_0) - \varepsilon^{-1}(D_\varepsilon - D_0) \to \mathbf{S}y - D \quad \text{in } C^r[a, b]$$

as $\varepsilon \downarrow 0$. Hence, it remains to show

$$\Delta_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} g_0'(y_0)\, y - \mathbf{G}_0 y_0 + \mathbf{G}y_0 \quad \text{in } C^r[a, b]. \tag{16}$$

Let us introduce a Nemytskij operator $\mathbf{G}^*$ by setting $\mathbf{G}^* y = g_0'(y)$, $y \in \mathfrak{D}$. We notice that

$$\mathbf{G}_0, \mathbf{G}, \mathbf{G}^*: \mathfrak{D} \to C^r[a, b] \quad \text{continuously}, \tag{17}$$

which is an immediate consequence of [6: Theorem 1]. Now, because of

$$\begin{aligned} \mathbf{G}_\varepsilon y_\varepsilon - \mathbf{G}_0 y_0 &= (1 - \varepsilon)\, \mathbf{G}_0 y + \varepsilon \mathbf{G} y_\varepsilon - \mathbf{G}_0 y_0 \\ &= (\mathbf{G}_0 y_\varepsilon - \mathbf{G}_0 y_0) + \varepsilon(\mathbf{G} y_0 - \mathbf{G}_0 y_0) \\ &\quad - \varepsilon(\mathbf{G}_0 y_\varepsilon - \mathbf{G}_0 y_0) + \varepsilon(\mathbf{G} y_\varepsilon - \mathbf{G} y_0), \quad \varepsilon \in (0, 1], \end{aligned}$$

we can write for $\varepsilon \in (0, 1]$

$$\begin{aligned} &\Delta_\varepsilon(t) - g_0'\big(y_0(t)\big)\, y(t) + \mathbf{G}_0 y_0(t) - \mathbf{G} y_0(t) \\ &= \big[\varepsilon^{-1}\big(\mathbf{G}_0 y_\varepsilon(t) - \mathbf{G}_0 y_0(t)\big) - y(t)\, \mathbf{G}^* y_0(t)\big] \\ &\quad + \big(\mathbf{G}_0 y_\varepsilon(t) - \mathbf{G}_0 y_0(t)\big) + \big(\mathbf{G} y_\varepsilon(t) - \mathbf{G} y_0(t)\big). \end{aligned} \tag{18}$$

If here ε tends to $+0$, then, by (12) and (17), the last two (...)-terms on the right-han side converge to zero in $C^r[a, b]$. Concerning the first expression in [...] we have

$$\begin{aligned} &\varepsilon^{-1}\big(\mathbf{G}_0 y_\varepsilon(t) - \mathbf{G}_0 y_0(t)\big) - y(t)\, \mathbf{G}^* y_0(t) \\ &= \varepsilon^{-1}\big(y_\varepsilon(t) - y_0(t)\big) \int_0^1 g_0'\big(y_0(t) + \tau(y_\varepsilon(t) - y_0(t))\big)\, d\tau - y(t)\, \mathbf{G}^* y_0(t) \\ &= \varepsilon^{-1}\big(y_\varepsilon(t) - y_0(t)\big) \int_0^1 \big[\mathbf{G}^*\big(y_0 + \tau(y_\varepsilon - y_0)\big)(t) - \mathbf{G}^* y_0(t)\big]\, d\tau + \varepsilon^{-1} \varrho_\varepsilon(t)\, \mathbf{G}^* y_0(t), \end{aligned}$$

$\varepsilon \in (0, \varepsilon_0)$, and therefore

$$\begin{aligned} &\|\varepsilon^{-1}(\mathbf{G} y_\varepsilon - \mathbf{G}_0 y_0) - y \mathbf{G}^* y_0\|_r \\ &\leqq \varepsilon^{-1} \|y_\varepsilon - y_0\|_r \int_0^1 \big\|\mathbf{G}^*\big(y_0 + \tau(y_\varepsilon - y_0)\big) - \mathbf{G}^* y_0\big\|_r\, d\tau + \varepsilon^{-1}\, \|\varrho_\varepsilon\|_r\, \|\mathbf{G}^* y_0\|_r, \end{aligned}$$

where we have used a lemma proved in [6] and the fact that $C^r[a, b]$ is a Banach algebra. From this, again in virtue of (12) and (17), conclude also we that the [...] — term of (18) tends to zero in $C^r[a, b]$ provided $\varepsilon \downarrow 0$. Thus, (16) is shown and the proof is complete ∎

Next we want to define the adjoint state. For this we introduce the set $H^*(a, b)$ of all functions z on $[a, b]$ which are Hölder continuous on each subinterval of (a, b) and for which there are two functions ζ_1 and ζ_2 that are Hölder continuous in a neighbourhood of $c_1 = a$ and $c_2 = b$, respectively, such that in the corresponding neighbourhood $z(t) = \zeta_i(t)/|t - c_i|^{\nu_i}$, $0 \leqq \nu_i < 1$ $(i = 1, 2)$. Consider the linear singular integral equation

$$g_0'\big(y_0(t)\big)\, z(t) + \mathbf{S} z(t) = \frac{2}{\pi} \int_c^d \frac{h_0(s)}{s - t}\, ds, \quad t \in (a, b); \tag{19a}$$

together with the additional condition

$$\int_a^b z(t)\, dt = 2 \int_c^d h_0(t)\, dt, \tag{19b}$$

where h_0 denotes the known function

$$h_0 = Sy_0 + D_0 + p - q. \tag{20}$$

From Muschelischwili [12: Kap. V] it can be seen that in $H^*(a, b)$ the solution to (19a) is uniquely determined apart from a constant. Because this constant can be chosen in such a way that (19b) is also satisfied (cf. [23: § 2.2]) there is a unique solution $z_0 \in H^*(a, b)$ to (19). In the sequel z_0 will be called *adjoint state* (to $\{g_0, y_0, D_0\}$).

Theorem 4: *The directional derivative of* $\mathbf{J}: C^2[0, l] \to \mathbb{R}$ *at* $g_0 \in G_{ad}$ *in the direction of* $g - g_0$, $g \in G_{ad}$, *is given by*

$$\delta_+\mathbf{J}(g_0; g - g_0) = \int_c^d \left[g\big(y_0(t)\big) - g_0\big(y_0(t)\big)\right] z_0(t)\, dt,$$

where $z_0 \in H^*(a, b)$ *denotes the adjoint state.*

Proof: With the notations introduced at the beginning of this section we have to show

$$\lim_{\varepsilon \downarrow 0} \frac{\Phi(\varepsilon) - \Phi(0)}{\varepsilon} = \int_c^d \left[g\big(y_0(t)\big) - g_0\big(y_0(t)\big)\right] z_0(t)\, dt. \tag{21}$$

Straightforward calculation leads to

$$\Phi(\varepsilon) - \Phi(0) = \mathbf{J}(g_\varepsilon) - \mathbf{J}(g_0)$$

$$= 2 \int_c^d h_0(t)\, [\mathrm{S}(y_\varepsilon - y_0)\,(t) + (D_\varepsilon - D_0)]\, dt + \delta(\varepsilon),$$

$\varepsilon \in (0, 1]$, with $\delta(\varepsilon) = \int_c^d [\mathrm{S}(y_\varepsilon - y_0)\,(t) + (D_\varepsilon - D_0)]^2\, dt$ and the function h_0 given in (20). Because of

$$\int_c^d h_0(t)\, \mathrm{S}y(t)\, dt = -\frac{1}{\pi} \int_a^b y(t) \int_c^d \frac{h_0(s)}{s - t}\, ds\, dt \quad \forall y \in C_0^1[a, b],$$

we get

$$\Phi(\varepsilon) - \Phi(0) = -\frac{2}{\pi} \int_a^b \big(y_\varepsilon(t) - y_0(t)\big) \int_c^d \frac{h_0(s)}{s - t}\, ds\, dt$$

$$+ 2(D_\varepsilon - D_0) \int_c^d h_0(t)\, dt + \delta(\varepsilon),$$

and by means of the definition of $z_0 \in H^*(a, b)$ as the unique solution to (19) we come to

$$\Phi(\varepsilon) - \Phi(0) = -\int_a^b \big(y_\varepsilon(t) - y_0(t)\big) \big(g_0'\big(y_0(t)\big)\, z_0(t) + \mathrm{S}z_0(t)\big)\, dt$$

$$+ (D_\varepsilon - D_0) \int_a^b z_0(t)\, dt + \delta(\varepsilon)$$

$$= -\int_a^b \left[g_0'\big(y_0(t)\big) \big(y_\varepsilon(t) - y_0(t)\big) - \mathrm{S}(y_\varepsilon - y_0)\,(t) - (D_\varepsilon - D_0)\right]$$

$$\times\, z_0(t)\, dt + \delta(\varepsilon).$$

Here, in the last step we have used the formula (cf., e.g., [12: § 96])

$$\int_a^b y(t)\,\mathrm{S}z_0(t)\,dt = -\int_a^b z_0(t)\,\mathrm{S}y(t)\,dt \quad \forall y \in C_0^1[a,b].$$

If now $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is the same as in Theorem 3, we can apply (11) and (12). Doing this we obtain

$$\Phi(\varepsilon) - \Phi(0) = \varepsilon \int_a^b \left[g\big(y_0(t)\big) - g_0\big(y_0(t)\big)\right] z_0(t)\,dt + \gamma(\varepsilon) + \delta(\varepsilon), \tag{22}$$

$\varepsilon \in (0, \varepsilon_0)$, with

$$\gamma(\varepsilon) = -\int_a^b \left[g_0'\big(y_0(t)\big)\,\varrho_\varepsilon(t) - \mathrm{S}\varrho_\varepsilon(t) - \chi_\varepsilon\right] z_0(t)\,dt.$$

Because of $|\gamma(\varepsilon)| \leqq \text{const}\,(\|\varrho_\varepsilon\|_1 + \|\mathrm{S}\varrho_\varepsilon\|_1 + |\chi_\varepsilon|)$, $\varepsilon \in (0, \varepsilon_0)$, we see that $\gamma(\varepsilon) = o(\varepsilon)$ as $\varepsilon \downarrow 0$. Since the same is true for $\delta(\varepsilon)$ the wanted relation (21) follows from (22) ∎

Because of (10), Theorem 4 yields at once a necessary optimality condition formulated as

Theorem 5: *If* $\{g_0, y_0, D_0\} \in G_{\mathrm{ad}} \times C_0^1[a,b] \times \mathbb{R}$ *is optimal to the control problem* (P), *then*

$$\int_c^d \left[g\big(y_0(t)\big) - g_0\big(y_0(t)\big)\right] z_0(t)\,dt \geqq 0 \quad \forall g \in G_{\mathrm{ad}}, \tag{23}$$

where $z_0 \in H^*(a,b)$ *is the adjoint state defined by* (19).

Condition (23) can be named Pontryagin minimum principle. Clearly, Theorem 5 remains valid for any other convex set of admissible controls $G_{\mathrm{ad}} \subset C^2[0,l]$. The proof of Theorem 5 for the modified control problem (P*), which we have considered in Section 2, is still open.

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Manuskripteingang: 22. 11. 1988

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