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On an Iterative Algorithm for Solving Nonlinear Operator Equations[1])

I. K. Argyros

Es wird die Sekantenmethode der Lösung nichtlinearer Operatorgleichungen in Banach-Räumen betrachtet. Unter der Voraussetzung nur der Hölder-Stetigkeit der Fréchet-Ableitung des nichtlinearen Operators wird gezeigt, daß die Sekanteniteration zu der lokal eindeutigen Lösung konvergiert. Es werden Beispiele betrachtet, in denen die vorgeschlagene Methode, nicht aber entsprechende andere Methoden aus der Literatur anwendbar sind.

Рассматривается метод секущих решения нелинейных операторных уравнений в банаховых пространствах. При предположении, что производная Фреше нелинейного оператора только непрерывна по Гельдеру, доказывается, что итерация секущих сходится к локально единственному решению. Даны примеры, в которых предложенный метод применим, а соответствующие другие методы из литературы отказываются.

The Secant method for solving nonlinear operator equations in Banach spaces is considered. By assuming that the Fréchet derivative of a nonlinear operator is only Hölder continuous we show that the secant iteration converges to a locally unique solution. Examples are also given where our results apply and some related ones already in the literature fail.

1. Introduction

Let F be a nonlinear operator defined on a convex subset D of a Banach space E with values in a Banach space $\hat{E}$. The Secant method for solving the equation

$$F(x) = 0 \tag{1}$$

can be written under the form

$$x_{n+1} = x_n - \delta F(x_n, x_{n-1})^{-1} F(x_n), \qquad n \in \mathbb{N}_0, \tag{2}$$

where, for each $x_{n-1}, x_n \in D$, $\delta F(x_n, x_{n-1})$ is a bounded linear operator from E to $\hat{E}$ (i.e. $\delta F(x_n, x_{n-1}) \in L(E, E)$, $n \in \mathbb{N}_0$) which is a consistent approximation of the Fréchet derivative of F.

The method of Euler-Chebysheff and the method of Halley which were generalized in Banach spaces by M. T. Necepurenko [6] and M. A. Mertvecova [5], respectively, are the best known cubically convergent iterative procedures for solving nonlinear equations. These methods have little practical value because they require an evaluation of the second Fréchet derivative at each step. That is, it requires a number of function evaluations being proportional with the cube to the dimension of the space. S. Ul'm used generalized divided differences of second order instead of the second Fréchet derivative and obtained order of convergence 1.839... [12]. But the use of generalized divided differences of second order which are bilinear operators are not

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easy to handle in practice. F. POTRA obtained the same order of convergence using only generalized divided differences of first order which are linear operators [8]! However, the above results cannot be applied when the Fréchet derivative of F is only (c, p)-Hölder continuous (to be precised later). In the present paper we study (2) under the above weaker assumption deriving semilocal and local convergence theorems. Some examples are provided where the hypotheses of the previous methods are not satisfied but ours are.

2. Convergence results

Definition 1: Let F be a nonlinear operator and L_0 a boundedly invertible operator defined on a subset D of a Banach space E with values in a Banach space $\hat{E}$. We say that the Fréchet derivative F' of F is *(c, p)-Hölder continuous on* $D \subset E$ if, for some $c > 0$ and $p \in [0, 1]$,

$$\|L_0^{-1}(F'(x) - F'(y))\| \leqq c \, \|x - y\|^p \qquad \text{for all } x, y \in D. \tag{3}$$

We then say that $F' \in H_D(c, p)$.

It is well known (see, e.g., [4]) that if D is convex, then

$$\|L_0^{-1}(F(x) - F(y) - F'(x)(x - y))\| \leqq \frac{c}{1 + p} \|x - y\|^{1+p} \quad \text{for all} \quad x, y \in D. \tag{4}$$

Definition 2: Let F be a nonlinear operator defined on a subset D of a linear space E with values in a linear space $\hat{E}$ and let v, w be two points of D. A linear operator from E into $\hat{E}$ which is denoted by $\delta F(v, w)$ and satisfies the condition

$$\delta F(v, w)(v - w) = F(v) - F(w) \tag{5}$$

is called a *divided difference of F at the points v and w*.

We will assume that $\delta F(v, w) \in L(E, \hat{E})$. Note that (5) does not uniquely determine the divided difference with the exception of the case when E is one-dimensional.

From now on we assume that E, $\hat{E}$ are Banach spaces, $\delta F(v, w) \in L(E, \hat{E})$ and $F' \in H_D(c, p)$ for some open convex set $D \subset E$. We shall assume that the divided differences of F satisfy Lipschitz conditions of the form ($d_1, d_2 \geqq 0$ and $p \in (0, 1]$)

$$\|\delta F(v, w) - \delta F(v, z)\| \leqq d_1 \|w - z\|^p \qquad (v, w, z \in D), \tag{6}$$

$$\|\delta F(v, w) - \delta F(z, w)\| \leqq d_2 \|v - z\|^p \qquad (v, w, z \in D). \tag{7}$$

We can now prove the following lemma.

Lemma 1: *Let us assume that the divided difference operator δF satisfies the conditions* (6) *and* (7). *Then*

(a) $\delta F(x, x) = F'(x)$; $x \in \operatorname{Int} D$;
(b) $F' \in H_D[d_1 + d_2, p]$ *for any fixed* $p \in (0, 1]$.

Proof: (a) Let us choose $x \in \operatorname{int} D$ and $\delta > 0$ such that $U(x, \delta) = \{y \in E \mid \|x - y\| < \delta\} \subset D$. For $\|\Delta x\| < \delta$, we get by (5), (7) $\|F(x + \Delta x) - F(x) - \delta F(x, x)(\Delta x)\| = \|[\delta F(x + \Delta x, x) - \delta F(x, x)](\Delta x)\|$, which is $\leqq \|\delta F(x + \Delta x, x) - \delta F(x, x)\| \, \|\Delta x\| \leqq d_2 \|\Delta x\|^p \|\Delta x\|$. The above inequality proves (a) if $\|\Delta x\| > 0$, $d_2 \neq 0$. If $d_2 = 0$, we choose another $d_2' > 0$ in the inequality (7).

(b) Let $v, w \in D$. Then, by (6) and (7),

$$\|F'(v) - F'(w)\| \leqq \|\delta F(v, v) - \delta F(v, w)\| + \|\delta F(v, w) - \delta F(w, w)\| \leqq (d_1 + d_2) \|v - w\|^p . \blacksquare$$

Let us consider the space $\mathbb{R}^m$ equipped with the max-norm. A linear operator $L \in L(\mathbb{R}^m, \mathbb{R}^m)$ will be represented by a matrix with entries l_{ij}; its norm is given by $\|L\| = \max_i \left\{ \sum_j |l_{ij}| \right\}$. Let B be an open ball of $\mathbb{R}^m$ and let F be an operator defined on B with values in $\mathbb{R}^m$. Let us denote by $F_1, \ldots, F_m$ the components of F. For each $\in B$ we can write $F(v) = (F_1(v), \ldots, F_m(v))^{\mathrm{tr}}$. We set

$$P_j F_i(v) = \partial F_i(v)/\partial v_j, \tag{8}$$

provided that $\partial F_i(v)/\partial v_j$ exist for all $i, j = 1, \ldots, m$. Let $v, w \in B$ and define $\delta F(v, w)$ by the matrix with entries $(v_j \neq w_j)$

$$\delta F(v, w)_{ij} = \frac{1}{v_j - w_j} \big(F_i(v_1, \ldots, v_j, w_{j+1}, \ldots, w_m) - F_i(v_1, \ldots, v_{j-1}, w_j, \ldots, w_m)\big). \tag{9}$$

It can easily be seen that the operator defined by (9) satisfies (5) and $\delta F(v, w) \in L(\mathbb{R}^m, \mathbb{R}^m)$. We can now show the following lemma.

Lemma 2: *If the partial derivatives $P_j F_i$ given by (8) exist and satisfy some Hölder conditions of the form*

$$|P_j F_i(v_1, \ldots, v_k + t, \ldots, v_m) - P_j F_i(v_1, \ldots, v_k, \ldots, v_m)| \leqq b^i_{jk} |t|^p, \tag{10}$$

then conditions (6) *and* (7) *are satisfied with*

$$d_1 = \max_{1 \leqq i \leqq m} \left\{ \frac{1}{p+1} \sum_{j=1}^m \left(b^i_{jj} + \sum_{k=j+1}^m b^i_{jk} \right) \right\}, \quad d_2 = \max_{1 \leqq i \leqq m} \left\{ \frac{1}{p+1} \sum_{j=1}^m \left(b^i_{jj} + \sum_{k=1}^{j-1} b^i_{jk} \right) \right\}. \tag{11}$$

Proof: Let $v, w, z \in B$. We can get

$$\delta F(v, w)_{ij} - \delta F(v, z)_{ij} = \sum_{k=1}^m \left\{ \delta F\big(v, (w_1, \ldots, w_k, z_{k+1}, \ldots, z_m)\big)_{ij} - \delta F\big(v, (w_1, \ldots, w_{k-1}, z_k, \ldots, z_m)\big)_{ij} \right\}.$$

If $k < j$, we get for the summands S_k of this series

$$S_k = \frac{1}{v_j - z_j} \{F_i(v_1, \ldots, v_j, z_{j+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, z_j, \ldots, z_m)\} - \frac{1}{v_j - z_j} \{F_i(v_1, \ldots, v_j, z_{j+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, z_j, \ldots, z_m)\} = 0.$$

For $k = j$ we get for the summand S_j

$$|S_j| = \left| \frac{1}{v_j - w_j} \{F_i(v_1, \ldots, v_j, z_{j+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, w_j, z_{j+1}, \ldots, z_m)\} - \frac{1}{v_j - z_j} \{F_i(v_1, \ldots, v_j, z_{j+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, z_j, \ldots, z_m)\} \right|$$

$$= \left| \int_0^1 \{P_j F_i(v_1, \ldots, v_j, w_j + t(v_j - w_j), z_{j+1}, \ldots, z_m) - P_j F_i(v_1, \ldots, v_j, z_j + t(v_j - z_j), z_{j+1}, \ldots, z_m)\} \, dt \right|$$

$$\leqq |w_j - z_j|^p \, b_{jj}^i \int_0^1 t^p \, dt = \frac{1}{p+1} |w_j - z_j|^p \, b_{jj}^i.$$

For $k > j$ we finally can get

$$|S_k| = \left| \frac{1}{v_j - w_j} \{F_i(v_1, \ldots, v_j, w_{j+1}, \ldots, w_k, z_{k+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, w_j, \ldots, w_k, z_{k+1}, \ldots, z_m) - F_i(v_1, \ldots, v_j, w_{j+1}, \ldots, w_{k-1}, z_k, \ldots, z_m) + F_i(v_1, \ldots, v_{j-1}, w_j, \ldots, w_{k-1}, z_k, \ldots, z_m)\} \right|$$

$$= \left| \int_0^1 \{F_i(v_1, \ldots, v_{j-1}, w_j + t(v_j - w_j), w_{j+1}, \ldots, w_k, z_{k+1}, \ldots, z_m) - F_i(v_1, \ldots, v_{j-1}, w_j + t(v_j - w_j), w_{j+1}, \ldots, w_{k-1}, z_k, \ldots, z_m)\} \, dt \right|$$

$$\leqq \|w_k - z_k\|^p \, b_{jk}^i.$$

Therefore, we have

$$|\delta F(v, w)_{ij} - \delta F(v, z)_{ij}| \leqq \frac{1}{p+1} |w_j - z_j|^p \, b_{jj}^i + \sum_{k=j+1}^{m} |w_k - z_k|^p \, b_{jk}^i$$

$$\leqq \|w - z\|^p \left\{ \frac{1}{p+1} b_{jj}^i + \sum_{k=j+1}^{m} b_{jk}^i \right\}.$$

Hence (6) is satisfied with d_1 given by (11). Similarly we show (7) with d_2 given by (11) ∎

We can now show the following result on the local convergence of the iterative algorithm (2) to a solution x^* of equation (1).

Theorem 1: *Let F be a nonlinear operator defined on an open convex subset D of a Banach space E with values in a Banach space $\hat{E}$. Assume that*

(a) *$F(x) = 0$ has a solution $x^* \in D$ at which the Frechet derivative $F'(x^*)$ exists and is boundedly invertible,*

(b) *F has divided differences satisfying the Hölder conditions ($p \in (0, 1]$)*

$$\|F'(x^*)^{-1} (\delta F(v, w) - \delta F(u, z))\| \leqq d_3(\|v - u\|^p + \|w - z\|^p), \tag{12}$$

(c) *$B = U(x^*, r_0) \subset D$ with $r_0 \in (0, (1/3d)^{1/p})$.*

Then the iterative algorithm $x_{n+1} = x_n - \delta F(x_n, x_{n-1})^{-1} F(x_n)$, ($n \in \mathbb{N}_0$, x_{-1}, $x_0 \in B$) is well-defined and generates a sequence $\{x_n\}_{n \geq 0}$ which remains in B, converges to x^* and satisfies the inequality

$$\|x_{n+1} - x^*\| \leqq d_3 \frac{\|x_{n-1} - x^*\|^p}{1 - d_3(\|x_n - x^*\|^p + \|x_{n-1} - x^*\|^p)} \|x_n - x^*\|. \tag{13}$$

Proof: Let us denote by $L = L(v, w)$ the linear operator

$$L = \delta F(v, w) \quad \text{with} \quad v, w \in B. \tag{14}$$

Then, by (12), we get

$$\|I - F'(x^*)^{-1} L\| = \|F'(x^*)^{-1} (\delta F(x^*, x^*) - \delta F(v, x^*) + \delta F(v, x^*) - \delta F(v, w))\|$$
$$\leqq d_3(\|v - x^*\|^p + \|w - x^*\|^p) \leqq 2d_3 r_0^p < 1,$$

by the choice of r_0. By the Banach lemma on invertible operators it follows that L is invertible and

$$\|L^{-1}F'(x^*)\| \leqq \left(1 - d_3(\|v - x^*\|^p + \|w - x^*\|^p)\right)^{-1}. \tag{15}$$

Let us now suppose that $x_{n-1}, x_n \in B$. Set $L_n = L(x_n, x_{n-1})$. Then L_n is invertible and we can write

$$\|x_{n+1} - x^*\| = \|x_n - x^* - L_n^{-1}(F(x_n) - F(x^*))\|$$
$$= \|-L_n^{-1}(\delta F(x_n, x^*) - L_n)(x_n - x^*)\|$$
$$\leqq \|L_n^{-1}F'(x^*)\| \, \|F'(x^*)^{-1} (\delta F(x_n, x^*) - L_n)\| \, \|x_n - x^*\|. \tag{16}$$

By (14) and (12) we get

$$\|F'(x^*)^{-1} (\delta F(x_n, x^*) - L_n)\| = \|F'(x^*)^{-1} (\delta F(x_n, x^*) - \delta F(x_n, x_{n-1}))\|$$
$$\leqq d_3 \|x_{n-1} - x^*\|^p. \tag{17}$$

The inequality (13) follows immediately from (15)—(17). From (14) and the choice of r_0 it follows that

$$\|x_{n+1} - x^*\| < \|x_n - x^*\| < r_0, \qquad n \in \mathbb{N}_0. \tag{18}$$

Thus, the iterative algorithm (2) is well-defined and the sequence generated by it remains in B. From (13) and (18) it follows that $\|x_n - x^*\| \to 0$ ∎

We can now show the following result on the semilocal convergence of the iterative algorithm (2) to a solution x^* of equation (1).

Theorem 2: *Let F be a nonlinear operator defined on an open convex subset D of a Banach space E with values in a Banach space $\hat{E}$. Assume that*

(a) *the linear operator L_0 is equal to $\delta F(x_0, x_{-1})$ where x_{-1}, x_0 are two given points from D is invertible,*

(b) *d_4, d_5 and d_6 are three nonnegative numbers such that*

$$\|x_{-1} - x_0\| \leqq d_4, \qquad \|L_0^{-1}F(x_0)\| \leqq d_5 \tag{19}$$

and $(p \in (0, 1])$

$$\|L_0^{-1}(\delta F(v, w) - \delta F(u, z))\| \leqq d_6(\|v - u\|^p + \|w - z\|^p), \tag{20}$$

(c) *r_1 is a nonnegative number such that*

$$r_1 > d_5/(1 - \gamma),\ 2(r_1 + d_4)^p + r_1^p < d_6^{-1} \tag{21}$$

with $\gamma = (r_1 + d_4)^p d_6/(1 - [r_1^p + (r_1 + d_4)^p] d_6)$ *(note that* $0 \leqq \gamma < 1$*),*

(d) *the closed ball $\overline{U}(x_0, r_1)$ is included in D.*

Then:

(i) *The real sequence $\{t_n\}_{n \geqq -1}$ defined by*

$$t_{-1} = r_1 + d_4, \qquad t_0 = r_1, \qquad t_1 = r_1 - d_5 \tag{22}$$

and, for $k \geqq 0$,

$$t_{k+1} - t_{k+2} = \frac{d_6(t_{k-1} - t_{k+1})^p}{1 - d_6[(t_0 - t_{k+1})^p + (t_{-1} - t_k)^p]}(t_k - t_{k+1}) = A_{k+2}(t_k - t_{k+1}), \tag{23}$$

is nonnegative and decreasingly converging to some $t^* \in \mathbb{R}$ *such that* $r_1 - d_5/(1-\gamma) \leqq t^* < t_{-1}$.

(ii) *The iterative algorithm* (2) *is well-defined, remains in* $\overline{U}(x_0, r_1)$, *and converges to a solution* $x^* \in \overline{U}(x_0, r_1)$ *of the equation* $F(x) = 0$. *Moreover, the following estimates are true:*

$$\|x_n - x^*\| \leqq t_n - t^* \tag{24}$$

and, for $n \geqq 1$,

$$\|x_n - x^*\| \leqq \frac{d_6(t_{n-2} - t_n)^p}{1 - d_6[(t_{-1} - t_0)^p + (t_0 - t_n)^p + (t_0 - t^*)^p]}(t_{n-1} - t_n). \tag{25}$$

Proof: (i) One can easily see that it suffices to show by induction the inequalities ($n \in \mathbb{N}_0$)

$$t_{n+1} \geqq t_{n+2} \geqq r_1 - \frac{1-\gamma^{n+2}}{1-\gamma} d_5 \geqq 0 \quad \text{and} \quad A_{n+2} \leqq \gamma. \tag{26}$$

Using (23) for $k = 0$ we obtain $t_2 \leqq t_1$, $t_2 \geqq r_1 - \big((1-\gamma^2)/(1-\gamma)\big) d_5 \geqq 0$ and $A_2 \leqq \gamma$ by (21), which shows (26) for $n = 0$. Let us assume that the inequalities (26) are true for $k = 0, 1, \ldots, n-1$. We will show that they are true for $k = n$. Using (23) we obtain $t_{k+1} \geqq t_{k+2}$ and since $t_{k+1} \geqq 0$ by the induction hypothesis we get

$$A_{k+2} = \frac{(t_{k-1} - t_{k+1})^p d_6}{1 - d_6[(t_0 - t_{k+1})^p + (t_{-1} - t_k)^p]} \leqq \frac{t_{k-1}^p d_6}{1 - d_6[t_0^p + t_{-1}^p]} \leqq \gamma.$$

Finally, by (23) and the induction hypothesis,

$$t_{k+2} \geqq r_1 - \frac{1-\gamma^{k+1}}{1-\gamma} d_5 - \gamma^{k+1} d_5 = r_1 - \frac{1-\gamma^{k+2}}{1-\gamma} d_5 \geqq 0.$$

That completes the induction and justifies the claim.

(ii) We shall prove by induction that the iterative algorithm (2) is well-defined and that

$$\|x_n - x_{n+1}\| \leqq t_n - t_{n+1}. \tag{27}$$

Using (2), (19) and (22) we deduce that (27) is true for $n = -1, 0$. Let $k \in \mathbb{N}$ and suppose that (27) holds for all $n \leqq k$. Let $L_{k+1} = \delta F(x_{k+1}, x_k)$. Then by (20) we have

$$\begin{aligned} \|I - L_0^{-1} L_{k+1}\| &= \|L_0^{-1}(L_0 - L_{k+1})\| = \big\|L_0^{-1}\big(\delta F(x_0, x_{-1}) - \delta F(x_{k+1}, x_k)\big)\big\| \\ &\leqq d_6(\|x_0 - x_{k+1}\|^p + \|x_{-1} - x_k\|^p), \\ &\leqq d_6\big((t_0 - t_{k+1})^p + (t_{-1} - t_k)^p\big) < 1 \end{aligned}$$

by the choice of r_1. By the Banach lemma on invertible operators L_{k+1} is invertible and

$$\|L_{k+1}^{-1} L_0\| \leqq \big(1 - d_6(\|x_0 - x_{k+1}\|^p + \|x_{-1} - x_k\|^p)\big)^{-1}. \tag{28}$$

In particular, we have proved that (2) is well-defined for $n = k + 1$. We also have

$$\begin{aligned} \|x_{k+1} - x_{k+2}\| &= \|L_{k+1}^{-1} F(x_{k+1})\| = \big\|L_{k+1}^{-1}\big(F(x_{k+1}) - F(x_k) - L_k(x_{k+1} - x_k)\big)\big\| \\ &\leqq \|L_{k+1}^{-1} L_0\| \, \big\|L_0^{-1}\big(\delta F(x_k, x_{k+1}) - L_k\big)\big\| \, \|x_{k+1} - x_k\|. \end{aligned}$$

By (20) we get

$$\|L_0^{-1}(\delta F(x_k, x_{k+1}) - \delta F(x_k, x_{k-1}))\| \leqq d_6 \|x_{k+1} - x_{k-1}\|^p.$$

From the last three estimates it follows that

$$\|x_{k+1} - x_{k+2}\| \leqq \frac{d_6 \|x_{k+1} - x_{k-1}\|^p}{1 - d_6(\|x_0 - x_{k+1}\|^p + \|x_{-1} - x_k\|^p)} \|x_{k+1} - x_k\|.$$

By (27) and (23) we obtain $\|x_{k+1} + x_{k+2}\| \leqq t_{k+1} - t_{k+2}$.

We have proved that the iterative algorithm (2) is well-defined and that (27) holds for all n. Therefore,

$$\|x_n - x_k\| \leqq t_n - t_k, \qquad -1 \leqq n \leqq k. \tag{29}$$

That is, $\{x_n\}$ is a Cauchy sequence in a Banach space and as such it converges to some $x^* \in E$. Letting $k \to \infty$ in (29) we obtain (24). The element $x^* \in E$ is a root of the equation $F(x) = 0$. Indeed, we have by (2) and (20)

$$\begin{aligned} \|L_0^{-1}F(x_{k+1})\| &= \|L_0^{-1}(\delta F(x_k, x_{k+1}) - L_k)(x_{k+1} - x_k)\| \\ &\leqq d_6(\|x_k - x_k\|^p + \|x_{k+1} - x_{k-1}\|^p) \|x_{k+1} - x_k\| \to 0 \end{aligned}$$

as $k \to \infty$. That is $F(x^*) = 0$. We will now show (25). By (20) it follows that

$$\begin{aligned} \|I - L_0^{-1}\delta F(x_n, x^*)\| &= \|L_0^{-1}((L_0 - \delta F(x_0, x_0)) + (\delta F(x_0, x_0) - \delta F(x_n, x^*)))\| \\ &\leqq d_6(\|x_0 - x_{-1}\|^p + \|x_0 - x_n\|^p + \|x_0 - x^*\|^p) \\ &\leqq d_6((t_{-1} - t_0)^p + (t_0 - t_n)^p + (t_0 - t^*)^p) < 1 \end{aligned}$$

by the choice of r_1. By the Banach lemma on invertible operators it follows that the linear operator $\delta F(x_n, x^*)$ is invertible and

$$\|\delta F(x_n, x^*)^{-1} L_0\| \leqq (1 - d_6(\|x_0 - x_{-1}\|^p + \|x_0 - x_n\|^p + \|x_0 - x^*\|^p))^{-1}.$$

Using the identity

$$x_n - x^* = \delta F(x_n, x^*)^{-1} (F(x_n) - F(x^*)) = ((\delta F(x_n, x^*))^{-1} L_0) L_0^{-1}F(x_n)$$

we obtain (25). ∎

We can now show a uniqueness result.

Proposition: *Let F be a nonlinear operator defined on an open convex subset D of a Banach space E with values in a Banach space $\hat{E}$. Assume that*

(a) *the hypotheses of Theorem 2 are true,*

(b) *the r_1 from Theorem 2/(c) satisfies*

$$2d_6(r_1 + d_4)^p + (c/(p+1) + d_6) r_1^p < 1. \tag{30}$$

Then the iterative algorithm (2) is well-defined, remains in $\bar{U}(x_0, r_1)$ and converges to a unique solution x^* of the equation $F(x) = 0$ in $\bar{U}(x_0, r_1)$.

Proof: The existence of a solution x^* of the equation $F(x) = 0$ was proved in Theorem 2. Let us assume that there exists a second solution y^* of this equation in $\bar{U}(x_0, r_1)$, with r_1 satisfying (21) and (30). By (2) and Lemma 1 we have

$$\begin{aligned} x_{n+1} - y^* = L_n^{-1}L_0[&L_0^{-1}(\delta F(x_n, x_{n-1}) - \delta F(x_n, x_n))(x_n - y^*) \\ &+ L_0^{-1}(F'(x_n)(x_n - y^*) - (F(x_n) - F(y^*)))]. \end{aligned}$$

Taking norms above and using (4), (20), (27) and (28) we get

$$\|x_{n+1} - y^*\| \leqq \frac{d_6(p+1)(t_{n-1} - t_n)^p + c\|x_n - y^*\|^r}{(p+1)\left[1 - d_6\left((t_0 - t_n)^p + (t_{-1} - t_{n-1})^p\right)\right]} \|x_n - y^*\|$$
$$\leqq \ldots \leqq \alpha^{n+1} \|x_0 - y^*\|,$$

where α denotes an upper bound on the fraction and $0 < \alpha < 1$ by the choice of r_1. The above inequality gives $y^* = \lim x_n = x^*$ ∎

Remark: The estimates (24) and (25) are called a-priori error estimates, since the iteration $\{t_n\}_{n \geq -1}$ can be computed in advance, provided that t_{-1}, t_0 and t_1 are known.

3. Applications

We now complete this paper with two possible applications whose computational details are left for the motivated reader.

Example 1: Theorem 1 can be realized for operators F which satisfy an autonomous differential equation of the form $F'(x) = G(F(x))$, for some given operator G. As $F'(x^*) = G(0)$, the inverse $F'(x^*)^{-1}$ can be evaluated without knowing the solution x^*. Consider for example the scalar equation $F(x) = 0$, where F is given by $F(x) = e^x - q$. Note that $F'(x) = F(x) + q$. That is, $F'(x^*) = q$. Let us define the divided difference operator $\delta F(v, w)$ by $\delta F(v, w) = (F(v) - F(w))/(v - w)$, $v \neq w$. The linear operator $\delta F(v, w)$ is now a function of two variables v and w. By expanding $\delta F(v, w)$ about (v, w) and using Taylor's theorem in two variables, a number $d_3 \geqq 0$ satisfying (12) can easily be found. By Theorem 1, if $x_0, x_{-1} \in B$, then the iterative algorithm (2) can be used to approximate the solution $x^* = \ln q$ of the equation $F(x) = 0$.

A more interesting application is given by the following example.

Example 2. Consider the differential equation

$$y'' + y^{1+p} = 0, \quad p \in (0, 1], \quad y(0) = y(1) = 0.$$

We divide the interval $[0, 1]$ into n subintervals and we set $h = 1/n$. Let $\{v_k\}$ be the points of subdivision with $0 \leqq v_0 < v_1 < \ldots < v_n = 1$. A standard approximation for the second derivative is given by

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad y_i = y(v_i), \quad i = 1, \ldots, n-1.$$

Take $y_0 = y_n = 0$ and define the operator $F: \mathbb{R}_+^{n-1}$ by

$$F(y) = H(y) + h^2\varphi(y),$$

$$H = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}, \quad \varphi(y) = \begin{bmatrix} y_1^{1+p} \\ y_2^{1+p} \\ \vdots \\ y_{n-1}^{1+p} \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Then

$$F'(y) = H + h^2(p+1) \begin{bmatrix} y_1^p & & & 0 \\ & y_2^p & & \\ & & \ddots & \\ 0 & & & y_{n-1}^p \end{bmatrix} \tag{31}$$

The Newton-Kantorovich hypotheses on which the work in [1, 3, 5, 6, 8—12] is based for the solution of the equation $F(y) = 0$ may not be satisfied.

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities y_i^{-p} and they may not exist.

Let $y \in \mathbb{R}^{n-1}$, $M \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and define the norms of y and M by $\|y\| = \max_j |y_j|$ and $\|M\| = \max_j \sum_i |m_{ij}|$. For all $y, z \in \mathbb{R}_+^{n-1}$, with $y_j, z_j > 0$ for all j, we obtain for $p = 1/2$, say

$$\|F'(y) - F'(z)\| = \left\| \operatorname{diag} \left\{ \frac{3}{2} h^2 (y_j^{1/2} - z_j^{1/2}) \right\} \right\| = \frac{3}{2} h^2 \max_j |y_j^{1/2} - z_j^{1/2}|$$

$$\leqq \frac{3}{2} h^2 \left(\max_j |y_j - z_j| \right)^{1/2} = \frac{3}{2} h^2 \|y - z\|^{1/2}.$$

That is, $c = 3/2h^2$. Therefore, the results in [3, 5—12] cannot be applied here.

Let us define the divided difference $\delta F(v, w)$ as in (9). The functions $P_j F_i$ given by (8) and the numbers b_{jk}^i, d_1 and d_2 given by (10) and (11) can easily be evaluated using (31). However, we do not need to do that. We can choose $n = 10$ which gives nine equations for iteration (2). Since a solution would vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be $130 \sin \pi x$. This gives us the vector

$$z_{-1} = \begin{bmatrix} 4.01524\text{ E} + 01 \\ 7.63785\text{ E} + 01 \\ 1.05135\text{ E} + 02 \\ 1.23611\text{ E} + 02 \\ 1.29999\text{ E} + 02 \\ 1.23675\text{ E} + 02 \\ 1.05257\text{ E} + 02 \\ 7.65462\text{ E} + 01 \\ 4.03495\text{ E} + 01 \end{bmatrix}.$$

Choose z_0 by setting $z_0(v_i) = z_{-1}(v_i) - 10^{-4}$, $i = 1, 2, \ldots, n$. Using the iterative algorithm (2), after seven iterations we get

$$z_6 = \begin{bmatrix} 3.35745\text{ E} + 01 \\ 6.52029\text{ E} + 01 \\ 9.15666\text{ E} + 01 \\ 1.09168\text{ E} + 02 \\ 1.15363\text{ E} + 02 \\ 1.09168\text{ E} + 02 \\ 9.15666\text{ E} + 01 \\ 6.52029\text{ E} + 01 \\ 3.35745\text{ E} + 01 \end{bmatrix} \quad \text{and} \quad z_7 = \begin{bmatrix} 3.35740\text{ E} + 01 \\ 6.52027\text{ E} + 01 \\ 9.15664\text{ E} + 01 \\ 1.09168\text{ E} + 02 \\ 1.15363\text{ E} + 02 \\ 1.09168\text{ E} + 02 \\ 9.15664\text{ E} + 01 \\ 6.52027\text{ E} + 01 \\ 3.35740\text{ E} + 01 \end{bmatrix}.$$

We choose $z_6 = x_{-1}$ and $z_7 = x_0$ for our Theorem 2. We get the following results:

$$d_4 \leqq 5\text{E} - 04, \qquad d_5 \leqq 9.15311\text{E} - 05,$$

$$d_6 \leqq .767646, \qquad c = 3h^2/2 = .015, p = 1/2.$$

Let us choose $r_1 = .01$. Then (22) and (23) give

$t_{-1} = 1.05\text{ E} - 02$, $\quad t_0 = 1.\text{ E} - 02$, $\quad t_1 = 9.908469\text{ E} - 03$,

$t_2 = 9.906717159\text{ E} - 03$, $\quad t_3 = 9.90670366\text{ E} - 03$, $\quad \ldots$, $\quad t^* = 9.9066\text{ E} - 03$.

$A_1 = 1.913932\text{ E} - 02$, $\quad A_2 = 7.61273767\text{ E} - 03$, $\quad \gamma = 9.313595\text{ E} - 02$.

It can easily be seen that with the above values both the hypotheses of Theorem 2 and those of the Proposition are satisfied. Hence by Theorem 2, the iterative algorithm (2) is well-defined, remains in $\overline{U}(x_0, r_1)$ and converges to a unique solution x^* of equation $F(y) = 0$ in $\overline{U}(x_0, r_1)$.

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VERFASSER:

Prof. Dr. Ioannis K. Argyros
Department of Mathematical Sciences
College of Arts and Sciences
New Mexico State University
Box 30001
Las Cruces, New Mexico 88003-0001
USA