Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 1, p. 107 – 117

# On Imbeddings between Weighted Orlicz Spaces

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Es werden Bedingungen für Gewichte $\varrho$ und $\sigma$ untersucht, die notwendig und hinreichend für die Gültigkeit der Einbettung für die Orlicz-Räume $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ sind. Wir benutzen zwei Methoden: die erste ist ein Limitprozeß, der von den $L_p$-Räumen zur Einbettung von gewichteten Zygmund-Klassen in den gewichteten Raum $L_1$ führt, die zweite Methode beruht auf der Dualität in gewichteten Orlicz-Räumen, dem Index der Youngschen Funktion und gibt einen allgemeinen Einbettungssatz.

Найдены необходимые и достаточные условия на весовые функции $\varrho$ и $\sigma$ для того, чтобы существовало вложение пространств Орлича с весом $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$. Применяются два метода: первый является предельным процессом и ведет от $L_p$-пространств к вложению весовых классов Зигмунда в весовое пространство $L_1$, второй подход применяет понятие двойственности в весовых пространствах Орлича, индекса Н-функции и дает обшую теорему вложения.

Necessary and sufficient conditions are given for weight functions $\varrho$ and $\sigma$, which guarantee validity of the imbedding between the Orlicz spaces $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$. We make use of two methods: the first of them is a limit procedure leading to the imbedding of weighted Zygmund classes into weighted $L_1$, the second method employs the concept of duality in weighted Orlicz spaces, index of a Young function and results in a general imbedding theorem.

## 1. Introduction and preliminaries

This paper deals with necessary and sufficient conditions for imbeddings between weighted Orlicz spaces. Actually, it is an amalgam of results for weighted Lebesgue spaces (the particular case of continuous weights was solved by Avantaggiati [1] and the general problem with measures by Kabaila [3]), ideas about limit treatment of necessary and sufficient conditions on weight functions [5, 9], and inequalities linking modulars and norms. We get a general imbedding theorem with no restrictions on the growth of the Young functions involved.

In [5, 9] we have considered two weight weak type inequalities for the maximal operator in Zygmund classes $L(1 + \log^+ L)^K$. It turns out that the necessary and sufficient condition on weights can be obtained by considering a limit case of the Muckenhoupt $A_p$ condition. Rather surprisingly, an analogous procedure applied to the condition

$$\int_{\Omega} \big(\sigma(x)/\varrho(x)\big)^{p/(p-1)} \varrho(x)\, dx < \infty, \quad 1 < p < \infty,$$

which is necessary and sufficient for the imbedding $L_{p,\varrho} \hookrightarrow L_{1,\sigma}$ turns out to provide us with a necessary and sufficient condition for the imbedding $L(1 + \log^+ L)_\varrho^K$ into $L_{1,\sigma}$. (As usual, $\hookrightarrow$ stands for continuous imbedding.) This is the main result of Section 2.

In Section 3 we present a necessary and sufficient condition for the imbedding between weighted Orlicz spaces $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$. Roughly speaking, if $QP^{-1}$ is a Young function and $N$ is the complementary function to $QP^{-1}$, then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ iff

$$\int_\Omega N\big(\mu\sigma(x)/\varrho(x)\big)\,\varrho(x)\,dx < \infty \quad \text{for some } \mu > 0.$$

An alternative technique of indices of Young functions is employed in Section 4. We obtain various relations concerning imbeddings in question under certain restrictive assumptions (as $\Delta_2$, $\Delta_3$, conditions on indices etc.).

Throughout the paper, $\Omega$ will be a measurable subset of $\mathbb{R}^n$. The symbols $\varrho$ and $\sigma$ will denote *weights* in $\Omega$, i.e. measurable and a.e. positive functions in $\Omega$, $L_{p,\varrho}$ will stand for the usual weighted Lebesgue space with norm $\|\cdot\|_{p,\varrho}$. The reader is supposed to be familiar with the basic facts from the theory of modular and Orlicz spaces (we refer, e.g., to [8, 4]). Here, we shall work with (real) weighted Orlicz spaces in the following sense: For any Young function $M$, we consider the *modular*

$$m_\varrho(f, M) = \int_\Omega M\big(|f(x)|\big)\,\varrho(x)\,dx,$$

*the weighted Orlicz space* $L_{M,\varrho}$ is the linear hull of the class of functions with finite modular. This space is endowed with the *Luxemburg norm*

$$\|f\|_{M,\varrho} = \inf\,\{\lambda > 0;\ m_\varrho(f/\lambda, M) \leqq 1\}.$$

Recall that the Young function $M$ satisfies the $\Delta_2$ *condition* (sometimes we write $M \in \Delta_2$) if $M(2t) \leqq CM(t)$ for large $t$. We shall say that $M$ satisfies the $\Delta_2$ condition globally on $(0, \infty)$ if $M(2t) \leqq CM(t)$ for each $t \geqq 0$. If $N$ is the *complementary Young function* with respect to $M$, i.e.

$$N(t) = \sup_{\tau \geqq 0}\big(t\tau - M(\tau)\big), \quad t \geqq 0,$$

then

$$t \leqq M^{-1}(t)\,N^{-1}(t) \leqq 2t, \quad t \geqq 0, \tag{1.1}$$

and the spaces $L_{M,\varrho}$ and $L_{N,\varrho}$ are naturally associated with the duality

$$\langle f, g\rangle = \int_\Omega f(x)\,g(x)\,\varrho(x)\,dx, \quad f \in L_{M,\varrho}, \quad g \in L_{N,\varrho},$$

giving birth to the so-called *Orlicz norm*; on $L_{M,\varrho}$ it is

$$|||f|||_{M,\varrho} = \sup\left\{\int_\Omega f(x)\,g(x)\,\varrho(x)\,dx;\ m_\varrho(g, N) \leqq 1\right\}.$$

It is easy to check (quite analogously as in the case of non-weighted spaces) that

$$\|f\|_{M,\varrho} \leqq |||f|||_{M,\varrho} \leqq 2\|f\|_{M,\varrho}.$$

We shall denote by $E_{M,\varrho}(\Omega)$ the closure in $L_{M,\varrho}(\Omega)$ of the set of all bounded measurable functions with bounded support in $\Omega$. Then $L_{N,\varrho}$ is the dual space of $E_{M,\varrho}$.

Let us still recall that a Young function $M$ satisfies the $\Delta_3$ *condition* ($M \in \Delta_3$) iff there is $K > 0$ such that $tM(t) \leqq M(Kt)$ for large $t$. If $M \in \Delta_3$, then $N \in \Delta_2$ and

$$N(t) \leqq CtM^{-1}(Ct), \quad t \geqq T, \tag{1.2}$$

for some $C, T \geqq 0$.

The basic properties of weighted spaces used here do not go beyond the limits of an easy verification.

We only mention several useful assertions on general modular spaces (for the concept see, e.g., [8]). The modulars in question need not be necessarily strictly convex so that any weighted $L_1$ space is included.

Let $(X_i, m_i)$ be a modular space with the (Luxemburg) norm $\|\cdot\|_i$ $(i = 1, 2)$ and let $T: X_1 \to X_2$ be a sublinear operator. It is clear that the modular inequality $m_2(Tf) \leqq Cm_1(f)$, $f \in X_1$, implies its norm counterpart $\|Tf\|_2 \leqq C\|f\|_1$, $f \in X_1$. The converse is not true, however, we have (the proof is easy and therefore omitted)

Lemma 1.1: *Let $T$, $X_1$, $X_2$ be as above. Then the following two assertions are equivalent:*

(*i*) *There exists $C \geqq 1$ such that $\|Tf\|_2 \leqq C\|f\|_1$.*

(*ii*) *There exists $K \geqq 1$ such that $m_2(Tf/K) \leqq 1$ provided $m_1(f) \leqq 1$.*

*Moreover, the best constants in (i) and (ii) coincide.*

Later, we shall also make use of

Lemma 1.2: *Let $T$, $X_1$, $X_2$ be as above. Let $h$ be a positive function defined on $\langle 0, \infty)$ and bounded in some right neighbourhood of zero. If*

$$\|Tf\|_2 \leqq h(m_1(f)), \quad f \in X_1, \tag{1.3}$$

*then there is $C \geqq 1$ such that for all $f$*

$$\|Tf\|_2 \leqq C\|f\|_1. \tag{1.4}$$

Proof: Suppose (1.4) is not true. Then there is a sequence $\{f_n\} \subset X_1$ such that $\|f_n\|_1 \searrow 0$ and $\|Tf_n\|_2 \nearrow \infty$. For large $n$, $m_1(f_n) \leqq \|f_n\|_1 \leqq 1$, whence $m_1(f_n) \to 0$, which contradicts with (1.3) ∎

As currently adopted, different constants are denoted by the same letters if no confusion can arise.

## 2. The limiting approach

In what follows, $p$ and $p'$ are conjugate exponents, i.e. $p \in (1, \infty)$, $p' = p/(p-1)$. Let us mention the elementary and frequently used formulas

$$p'/p = p' - 1 = 1/(p-1).$$

We start with a special case of a weighted imbedding theorem which appeared in Avantaggiati's paper [1] for continuous weights and generally for measures in Kabaila [3]. For the sake of completeness, we present a proof based on the latter paper, which will, additionally, provide us with a relation between important constants.

Theorem 2.1: *The following statements are equivalent:*

(*i*) $\int_\Omega |f(x)|\,\sigma(x)\,dx \leqq C_p(\sigma, \varrho)\left(\int_\Omega |f(x)|^p\,\varrho(x)\,dx\right)^{1/p}$;

(*ii*) $B_p(\sigma, \varrho) = \int_\Omega \left(\sigma(x)/\varrho(x)\right)^{p'}\varrho(x)\,dx < \infty$.

*Moreover, the best $C_p(\sigma, \varrho)$ in (i) equals $\left(B_p(\sigma, \varrho)\right)^{1/p'}$.*

Proof: Let (i) hold. Putting $f(x) = \big(\sigma(x)/\varrho(x)\big)^{1/(p-1)}$ we get

$$\int_\Omega \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x)\,dx = \int_\Omega |f(x)|\,\sigma(x)\,dx$$

$$\leqq C_p(\sigma,\varrho)\Big(\int_\Omega |f(x)|^p\,\varrho(x)\,dx\Big)^{1/p}$$

$$= C_p(\sigma,\varrho)\left(\int_\Omega \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x)\,dx\right)^{1/p}.$$

The last term is finite as for each $g \in L_p(\varrho)$

$$\left|\int_\Omega g(x)\,\frac{\sigma(x)}{\varrho(x)}\,\varrho(x)\,dx\right| = \left|\int_\Omega g(x)\,\sigma(x)\,dx\right|$$

$$\leqq \int_\Omega |g(x)|\,\sigma(x)\,dx \leqq C_p(\sigma,\varrho)\Big(\int_\Omega |g(x)|^p\,\varrho(x)\,dx\Big)^{1/p},$$

so that the function $\sigma/\varrho$ represents a bounded linear functional on $L_{p,\varrho}$ and belongs to $L_{p',\varrho}$. This yields $\int_\Omega \big(\sigma(x)/\varrho(x)\big)^{p'}\varrho(x)\,dx \leqq C_p(\sigma,\varrho)^{p'}$, hence $B_p(\sigma,\varrho) \leqq C_p(\sigma,\varrho)^{p'}$.

Conversely, if $B_p(\sigma,\varrho)$ is finite, then

$$\int_\Omega |f(x)|\,\sigma(x)\,dx = \int_\Omega |f(x)|\,\sigma(x)\,\varrho^{1/p}(x)\,\varrho^{-1/p}(x)\,dx$$

$$\leqq \|f\|_{p,\varrho}\Big(\int_\Omega \big(\sigma(x)/\varrho(x)\big)^{p'}\varrho(x)\,dx\Big)^{1/p'} = \big(B_p(\sigma,\varrho)\big)^{1/p'}\|f\|_{p,\varrho}.$$

Therefore, (i) holds and $C_p(\sigma,\varrho) \leqq \big(B_p(\sigma,\varrho)\big)^{1/p'}$ ∎

Let $1 \leqq K < \infty$. We introduce the Young functions

$$\Phi_K(t) = \begin{cases} t^2, & 0 \leqq t \leqq 1,\\ t(1+\log t)^K, & t > 1,\end{cases}$$

$$F_K(t) = \sum_{j=2}^{\infty} \frac{t^j}{j!}\left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geqq 0.$$

We shall denote by $\Psi_K$ and $G_K$ the complementary Young functions to $\Phi_K$ and $F_K$, respectively.

The rather lengthy and tedious arithmetical proof of the following lemma is postponed to Appendix.

**Lemma 2.2:** *There exists a constant $\beta_K \geqq 1$ such that*

$$F_K(t) > \exp\left(t^{1/K}/\beta_K\right) - t^{1/K}/\beta_K - 1, \quad t \geqq 1.$$

It is worth to notice that the function $\exp\left(t^{1/K}\right) - t^{1/K} - 1$ is equivalent to $\Psi_K$ for large values of $t$. Indeed, it is easy to verify that $\Phi_K(t)/t < C\Phi_K\big(\sqrt{t}\big)/\sqrt{t}$, $t \geqq 1$, with, say, $C = 3^K$. Thus, by [4, Theorem 6.8], $\Psi_K$ satisfies the $\Delta_3$ condition and, on using (1.2), $\Psi_K^{-1}$ is equivalent to $(1+\log t)^K$ for large $t$.

**Lemma 2.3:** *The functions $F_K, \Phi_K, \Psi_K$ satisfy*

(i) $\Phi_K(t) \geqq t$ *for* $t \geqq 1$;
(ii) $2F_K(t) \geqq t$ *for* $t \geqq 1$;
(iii) $\Psi_K(t) \leqq F_K(t)$ *for* $0 \leqq t \leqq 1$;
(iv) $G_K(t) \leqq C_K \Phi_K(t)$ *for* $t \geqq 1$;
(v) $\Phi_K(t) \leqq C_K t^p (p-1)^{-K}$ *for* $t \geqq 0$ *and* $p \in (1, 2)$.

Proof: The statements (i)–(iii) follow immediately. We prove (iv). Note that $F_K \in \Delta_3$, thus, in virtue of (1.2), $G_K(t) \leqq CtF_K^{-1}(Ct)$ for some $C \geqq 1$ and all large $t$. Making use of Lemma 2.2 we have $F_K(t) \geqq \alpha_K \exp(t^{1/K}\beta_K)$ with some $\alpha_K > 0, \beta_K$ from Lemma 2.2, and for every $t \geqq 1$, so that passing to the inverse functions,

$$G_K(t) \leqq Ct\beta_K^K \log^K (Ct/\alpha_K) \leqq C_K t \log^K t \leqq C_K \Phi_K(t) \quad \text{for } t \geqq 1.$$

As to (v), it is easy to check that $\max_{t>0} \Phi_K(t)\, t^{-p} \leqq K^K(p-1)^{-K}$, so that we can put $C_K = K^K$ ∎

**Lemma 2.4:** *There exists a constant $\delta_K$ such that the (modified) Young inequality $ab \leqq \delta_K(F_K(a) + \Phi_K(b))$ holds for all $a, b \geqq 0$.*

Proof: We shall make use of Lemma 2.3/(i)–(iv). If $a, b \leqq 1$, then $ab \leqq \Phi_K(a) + \Psi_K(b) \leqq \Phi_K(a) + F_K(b)$. If $a < 1, b \geqq 1$, then $ab < b \leqq 2F_K(b) \leqq \Phi_K(a) + 2F_K(b)$. If $a \geqq 1, b < 1$, then $ab < a \leqq \Phi_K(a) + F_K(b)$. Finally, if $a, b > 1$, then, according to the Young inequality, $ab \leqq G_K(a) + F_K(b) \leqq C_K\Phi_K(a) + F_K(b)$. Putting $\delta_K = \max\{2, C_K\}$ concludes the proof ∎

**Lemma 2.5:** *There exists a constant $\theta_K$ such that*

$$\|f\|_{\Phi_K,\varrho} \leqq \frac{\theta_K}{(p-1)^K} \|f\|_{p,\varrho}, \quad f \in L_{p,\varrho}, \quad p \in (1, 2).$$

Proof: The thesis follows directly from Lemma 2.3/(v) and from the definition of the Luxemburg norm ∎

**Theorem 2.6:** *The following conditions on $\sigma, \varrho$ are equivalent:*

(i) $L_{\Phi_K,\varrho}(\Omega) \hookrightarrow L_{1,\sigma}(\Omega)$;
(ii) $C_p(\sigma, \varrho) \leqq C(p-1)^{-K}$, $p \in (1, 2)$;
(iii) $(\sigma/\varrho) \in L_{F_K,\varrho}(\Omega)$.

Proof: We show (i) ⇒ (ii) ⇒ (iii) ⇒ (i): Let (i) hold. Then, according to Lemma 2.5, $\|f\|_{1,\sigma} \leqq C_K(\theta_K/(p-1)^K) \|f\|_{p,\varrho}$, $p \in (1, 2)$.

Now suppose that (ii) is valid. Employing the relation between the best constants from Theorem 2.1 ($j'$ is the conjugate exponent to $j$) we get

$$\begin{aligned}
\int_\Omega F_K\left(\mu \frac{\sigma(x)}{\varrho(x)}\right) \varrho(x)\, dx &= \sum_{j=2}^\infty \frac{\mu^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)} B_{j'}(\sigma, \varrho) \\
&\leqq \sum_{j=2}^\infty \frac{\mu^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)} C^j \left(\frac{1}{j'-1}\right)^{Kj} \\
&\leqq \sum_{j=2}^\infty \frac{\mu^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)} C^j j^j \left(\frac{1}{j'-1}\right)^{j(K-1)} \\
&= \sum_{j=2}^\infty \frac{1}{j!} (C\mu j)^j,
\end{aligned}$$

the last sum being finite provided $0 < \mu < (Ce)^{-1}$.

If (*iii*) is true, then the modified Young inequality (Lemma 2.4) implies

$$\int_\Omega |f(x)|\,\sigma(x)\,dx = \int_\Omega |f(x)|\,\frac{1}{\mu}\,\frac{\mu\sigma(x)}{\varrho(x)}\,\varrho(x)\,dx$$

$$\leqq \frac{\delta_K}{\mu}\int_\Omega \Phi_K(|f(x)|)\,\varrho(x)\,dx + \frac{\delta_K}{\mu}\int_\Omega F_K\left(\mu\,\frac{\sigma(x)}{\varrho(x)}\right)\varrho(x)\,dx$$

$$= C_{K,\mu}\,\Phi_K(|f(x)|)\,\varrho(x)\,dx + C_{K,\mu}B_{\log,K}(\sigma,\varrho).$$

Application of Lemma 1.2 (with $h(t) = C_{K,\mu}(t + B_{\log,K}(\sigma,\varrho))$ leads to the imbedding (*i*) ∎

Remark 2.7: By duality argument, we have also characterized those weights $\varrho$ and $\sigma$ for which $\|f/\varrho\|_{\Psi_K;\varrho} \leqq C\|f/\sigma\|_\infty$.

For $K = 1$ and $\Omega$ bounded, $L_{\Phi_K,\varrho}$ is the *weighted Zygmund class* $L(1 + \log^+ L)_\varrho$ consisting of all functions $f$ with $\int_\Omega |f(x)|\,(1 + \log^+ |f(x)|)\,\varrho(x)\,dx < \infty$. Let us state the following particular

Corollary 2.8: *Let $\Omega$ be bounded and $\varrho \in L_1(\Omega)$. Then the weighted Zygmund class $L(1 + \log^+ L)_\varrho$ is imbedded into $L_{1,\sigma}$ iff $\int_\Omega \exp(\mu\sigma(x)/\varrho(x))\,\varrho(x)\,dx < \infty$ for some $\mu > 0$.*

## 3. General imbeddings

In the rest of the paper, $P$ and $Q$ are Young functions.

Definition 3.1: For a couple of Young functions $P, Q$ we introduce the set $Y(P, Q) = \{K > 0; Q(KP^{-1})$ is equivalent to some Young function$\}$. For $K \in Y(P, Q)$, the Young function equivalent to $Q(KP^{-1})$ will be denoted $Q(KP^{-1})$ again. Recall that two functions $F, G$ are equivalent iff $\alpha^{-1}F(\alpha^{-1}t) \leqq G(t) \leqq \alpha F(\alpha t)$ for all $t$ and some $\alpha \geqq 1$ independent of $t$.

Remark 3.2: Note that if at least one of the functions $P$, $Q$ satisfies the $\Delta_2$ condition globally on $(0, \infty)$, then $Y(P, Q)$ is either empty, or equal to $(0, \infty)$.

Proposition 3.3: *Suppose that $QP^{-1}$ is a Young function. Then the following statements are equivalent:*

(*i*) $\|f\|_{P,\sigma} \leqq \|f\|_{Q,\varrho}$ *for all* $f$,

(*ii*) $\|f\|_{1,\sigma} \leqq \|f\|_{QP^{-1},\varrho}$ *for all* $f$.

Proof: According to Lemma 1.1, (*i*) is subsequently equivalent to the implications $m_\varrho(f, Q) \leqq 1 \Rightarrow m_\sigma(f, P) \leqq 1$, and, on taking $g = P(f)$, $m_\varrho(g, QP^{-1}) \leqq 1 \Rightarrow \int g\sigma \leqq 1$. The repeated use of Lemma 1.1 completes the proof ∎

Realizing that $K\|f\|_{Q,\varrho} = \|f\|_{Q_K,\varrho}$, where $Q_K$ is the Young function given by $Q_K(t) = Q(Kt)$, we easily obtain

Corollary 3.4: *Let $K \in Y(P, Q)$. Then*

$$\|f\|_{P,\sigma} \leqq K\|f\|_{Q,\varrho}\ \forall f \Rightarrow \|f\|_{1,\sigma} \leqq \|f\|_{Q(KP^{-1}),\varrho}\quad \forall f.$$

*Conversely, if $K \in Y(P, Q)$, then for every $C \geqq 1$*

$$\|f\|_{1,\sigma} \leqq C\|f\|_{Q(KP^{-1}),\varrho}\ \forall f \Rightarrow \|f\|_{P,\sigma} \leqq KC\|f\|_{Q,\varrho}\ \forall f.$$

Corollary 3.5: *Let at least one of the functions $P, Q$ satisfy the $\Delta_2$ condition globally on $(0, \infty)$. Then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ iff $L_{QP^{-1},\varrho} \hookrightarrow L_{1,\sigma}$ provided $QP^{-1}$ is a Young function.*

In the sequel, we will assume that $QP^{-1}$ is a Young function. If $K \in Y(P, Q)$, we denote by $N_K$ the function complementary to $Q(KP^{-1})$. We shall study relations between the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ and the condition

$$\int_\Omega N_K\big(\mu\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx < \infty \quad \text{for some} \quad \mu > 0.$$

Proposition 3.6: *Let $\int_\Omega N_K\big(\mu\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx$ be finite for some positive $K$, $\mu$. Then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$.*

Proof: Making use of the Young inequality, we have

$$\begin{aligned} \|f\|_{P,\sigma} = K\|f/K\|_{P,\sigma} &\leqq K\left(1 + \mu^{-1} \int_\Omega P\big(|f(x)|/K\big)\, \big(\sigma(x)/\varrho(x)\big)\, \mu\varrho(x)\, dx\right) \\ &\leqq K + (K/\mu) \int_\Omega Q\big(|f(x)|\big)\, \varrho(x)\, dx + (K/\mu) \int_\Omega N_K\big(\mu\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx \\ &= C + (K/\mu)\, m_\varrho(f, Q), \end{aligned}$$

and in virtue of Lemma 1.2 we arrive at the desired imbedding ∎

Proposition 3.7: *Let the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ hold and suppose $K \in Y(P, Q)$ for some $K \geqq \sup\{\|f\|_{P,\sigma}/\|f\|_{Q,\varrho};\, f \neq 0\}$. Then $\int_\Omega N_K\big(\mu\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx < \infty$ for some $\mu > 0$.*

Proof: Assume $g \in E_{Q(KP^{-1}),\varrho}(\Omega)$. Then, by Corollary 3.4,

$$\left|\int_\Omega g(x)\, \big(\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx\right| \leqq \|g\|_{1,\sigma} \leqq \|g\|_{Q(KP^{-1}),\varrho},$$

in other words, $(\sigma/\varrho) \in L_{N_K,\varrho}$ ∎

Let us summarize the results of this section.

Theorem 3.8: *Let at least one of the functions $P, Q$ satisfy the $\Delta_2$ condition. Then the following statements are equivalent:*

(*i*) $L_{Q,\varrho}(\Omega) \hookrightarrow L_{P,\sigma}(\Omega)$;

(*ii*) $\int_\Omega N(\mu\sigma(x)/\varrho(x))\, \varrho(x)\, dx < \infty$ *for some* $C, \mu > 0$.

Theorem 3.9: *Suppose that $Y(P, Q)$ contains a sequence $K_n \nearrow \infty$. Then the following statements are equivalent:*

(*i*) $L_{Q,\varrho}(\Omega) \hookrightarrow L_{P,\sigma}(\Omega)$;

(*ii*) $\int_\Omega N_C\big(\mu\sigma(x)/\varrho(x)\big)\, \varrho(x)\, dx < \infty$ *for some* $C, \mu > 0$.

Remark 3.10: The duality argument applied to Theorems 3.8 and 3.9 gives the characterization of $\varrho$ and $\sigma$ such that $\|f/\varrho\|_{\tilde{Q},\varrho} \leqq C\, \|f/\sigma\|_{\tilde{P},\sigma}$, where $\tilde{P}$ and $\tilde{Q}$ are the complementary Young functions to $P, Q$, respectively.

## 4. Some remarks on an alternative approach (employing indices)

First, we recall the concept of the lower and upper index of a Young function. Actually, it is a certain refinement of the $\Delta_2$ condition. The existence of finite indices for functions satisfying $\Delta_2$ can be proved e.g. on the base of the theory of submultiplicative functions. We refer to Matuszewska and Orlicz [7], Gustavsson and Peetre [2], and the comprehensive paper by Maligranda [6].

Definition 4.1: Let $\Phi \in \Delta_2$ globally on $(0, \infty)$ and

$$h(\lambda) = h_\Phi(\lambda) = \sup_{t>0} \Phi(\lambda t)/\Phi(t), \quad \lambda \geqq 0.$$

We define the *lower index of* $\Phi$ as $i(\Phi) = \lim_{\lambda\to 0+} \log h(\lambda)/\log \lambda$ and the *upper index of* $\Phi$ as $I(\Phi) = \lim_{\lambda\to\infty} \log h(\lambda)/\log \lambda$.

It follows easily from the definition that for each $\varepsilon > 0$ there exists $C_\varepsilon \geqq 1$ such that

$$\Phi(\lambda t) \leqq C_\varepsilon \max \{\lambda^{i(\Phi)-\varepsilon}, \lambda^{I(\Phi)+\varepsilon}\} \Phi(t), \quad t \geqq 0, \quad \lambda \geqq 0,$$

$$\min \{\mu^{i(\Phi)-\varepsilon}, \mu^{I(\Phi)+\varepsilon}\} \Phi(t) \leqq C_\varepsilon \Phi(\mu t), \quad t \geqq 0, \quad \mu \geqq 0.$$

The following assertion linking modulars and norms in weighted Orlicz spaces is an immediate consequence of the properties of indices and the Luxemburg norm.

Proposition 4.2: *Assume* $\|f\|_{P,\sigma} \leqq C\|f\|_{Q,\varrho}$ *and let* $\varepsilon > 0$.

(*i*) *If* $P \in \Delta_2$ *globally on* $(0, \infty)$, *then*

$$m_\sigma(f, P) \leqq C_\varepsilon \max \{\|f\|_{Q,\varrho}^{i(P)-\varepsilon}, \|f\|_{Q,\varrho}^{I(P)+\varepsilon}\}.$$

(*ii*) *If* $Q \in \Delta_2$ *globally on* $(0, \infty)$, *then*

$$\min \{\|f\|_{P,\sigma}^{i(Q)-\varepsilon}, \|f\|_{P,\sigma}^{I(Q)+\varepsilon}\} \leqq C_\varepsilon m_\varrho(f, Q).$$

(*iii*) *If* $P, Q \in \Delta_2$ *globally on* $(0, \infty)$, *then*

$$m_\sigma(f, P) \leqq C_\varepsilon \max \left\{\big(m_\varrho(f, Q)\big)^{(i(P)/I(Q))-\varepsilon}, \big(m_\varrho(f, Q)\big)^{(I(P)/i(Q))+\varepsilon}\right\}.$$

Proposition 4.3: *The following two statements are true:*

(*i*) *Let* $i(Q) > \alpha > 0$ ($i(Q) = \infty$ *is admissible here*). *If* $\|f\|_{Q,\varrho} > 1$, *then we have* $\|f\|_{Q,\varrho} \leqq C_\alpha\big(m_\varrho(f, Q)\big)^{1/\alpha}$.

(*ii*) *Let* $I(Q) < \beta$ *and* $\|f\|_{Q,\varrho} \leqq 1$. *Then* $\|f\|_{Q,\varrho} \leqq C_\beta\big(m_\varrho(f, Q)\big)^{1/\beta}$.

Proof: (*i*) If $\|f\|_{Q,\varrho} > 1$, then obviously $m_\varrho(f, Q) \geqq \|f\|_{Q,\varrho}$. Therefore suppose that $m_\varrho(f, Q) < \infty$. The definition of $i(Q)$ guarantees the existence of $\lambda_0 \in (0, 1)$ such that $h_Q(\lambda) \leqq \lambda^\alpha$ for $\lambda \in (0, \lambda_0)$. Set $C_\alpha = \lambda_0^{-1}$. We get

$$\int_\Omega Q\left(\frac{|f(x)|}{C_\alpha\big(m_\varrho(f, Q)\big)^{1/\alpha}}\right) \varrho(x)\, dx \leqq \lambda_0^\alpha < 1.$$

(*ii*) Under our assumption we have $Q(\lambda t) \leqq \lambda^\beta Q(t)$, $t \geqq 0$, $\lambda \geqq \lambda_\beta$, for some $\lambda_\beta > 1$. Therefore

$$\int_\Omega Q\left(\lambda_\beta^{1-\beta} \frac{|f(x)|}{\big(m_\varrho(f, Q)\big)^{1/\beta}}\right) \varrho(x)\, dx \leqq \lambda_\beta^{-\beta} \int_\Omega Q\left(\lambda_\beta \frac{|f(x)|}{\big(m_\varrho(f, Q)\big)^{1/\beta}}\right) \varrho(x)\, dx$$

$$\leqq \lambda_\beta^{-\beta} \left(\frac{\lambda_\beta}{\big(m_\varrho(f, Q)\big)^{1/\beta}}\right)^\beta m_\varrho(f, Q) = 1.$$

It suffices to choose $C_\beta = \lambda_\beta^{\beta-1}$. ∎

Remark 4.4: If $\varrho \in L_1(\Omega)$, $Q \in \Delta_2$, and $P \in \Delta_2$ globally on $(0, \infty)$, then the necessity of the condition $(\sigma/\varrho) \in L_{N,\varrho}$ for the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ can be proved directly without use of the representation theorem. Actually, easily $QP^{-1} \in \Delta_2$, thus $N \in \Delta_2$. Combined with (1.2) and Proposition 4.2/(i), this gives

$$\int_\Omega N\left(\frac{\sigma(x)}{\varrho(x)}\right)\varrho(x)\,dx \leqq C\varrho(\Omega) + C\int_\Omega PQ^{-1}\left(\frac{\sigma(x)}{\varrho(x)}\right)\sigma(x)\,dx$$

$$\leqq C\varrho(\Omega) + C_\varepsilon \max\{\|Q^{-1}(\sigma/\varrho)\|_{Q,\varrho}^{i(P)-\varepsilon}, \|Q^{-1}(\sigma/\varrho)\|_{Q,\varrho}^{I(P)+\varepsilon}\}$$

$$\leqq C\varrho(\Omega) + C_\varepsilon\left(1 + \int_\Omega QQ^{-1}(\sigma(x)/\varrho(x))\,\varrho(x)\,dx\right)^{I(P)+\varepsilon}$$

$$= C\varrho(\Omega) + C_\varepsilon(1 + \sigma(\Omega))^{I(P)+\varepsilon}.$$

All above quantities are finite; $\varrho \in L_1(\Omega)$ was an assumption and the integrability of $\sigma$ over $\Omega$ follows from membership of constant functions in $L_{Q,\varrho}(\Omega)$.

Remark 4.5: If $P, Q \in \Delta_2$ and $I(P) < i(Q)$, we can get a quantitative relation for

$$B = \int_\Omega N(\sigma(x)/\varrho(x))\,\varrho(x)\,dx.$$

Indeed, setting $H(t) = P^{-1}(N, (t)/t)$ it is easy to check that $QH(t) \leqq N(t)$, $t \geqq 0$; to clarify this we invoke (1.1). Choose $\varepsilon > 0$ and $\alpha, \beta$ in such a manner that $I(P) + \varepsilon < \alpha < i(Q)$, $I(Q) < \beta$. For $k$ natural put

$$\Omega_k = \{x \in \Omega;\ |x| < k,\ \sigma(x) < k,\ \varrho(x) > k^{-1}\},\quad \Omega_0 = \bigcup_k \Omega_k,$$

and

$$B_k = \int_{\Omega_k} N(\sigma(x)/\varrho(x))\,\varrho(x)\,dx = \int PH(\chi_{\Omega_k}(x)\,\sigma(x)/\varrho(x))\,\sigma(x)\,dx.$$

Applying Proposition 4.2/(i) and Proposition 4.3 we can continue

$$B_k \leqq C_\varepsilon \max\left\{\|H(\chi_{\Omega_k}\sigma/\varrho)\|_{Q,\varrho}^{i(P)-\varepsilon}, \|H(\chi_{\Omega_k}\sigma/\varrho)\|_{Q,\varrho}^{I(P)+\varepsilon}\right\}$$

$$\leqq C_\varepsilon \max\left\{C_\alpha(m_\varrho(\chi_{\Omega_k}\sigma/\varrho, QH))^{(I(P)+\varepsilon)/\alpha}, C_\beta(m_\varrho(\chi_{\Omega_k}\sigma/\varrho, QH))^{(i(P)-\varepsilon)/\beta}\right\}$$

$$\leqq C_\varepsilon \max\left\{C_\alpha B_k^{(I(P)+\varepsilon)/\alpha}, C_\beta B_k^{(i(P)-\varepsilon)/\beta}\right\}.$$

Now, $B_k < \infty$, both the exponents $(I(P) + \varepsilon)/\alpha$ and $(i(P) - \varepsilon)/\beta$ are smaller than 1, and $|\Omega \backslash \Omega_0| = 0$. Thus, $B \leqq \max\left\{(C_\varepsilon C_\alpha)^{\alpha/(\alpha - I(P)-\varepsilon)}, (C_\varepsilon C_\beta)^{\beta/(\beta - i(P)+\varepsilon)}\right\}$. Note that $i(Q) > I(P)$ implies $i(QP^{-1}) > 1$, hence $N \in \Delta_2$ and therefore the constant $\mu$ in the argument of $N$ can simply be omitted.

Remark 4.6: Let us notice that Theorem 2.6 is partly covered by Theorem 3.9. On the other hand, the procedure described in Remark 4.4 cannot be applied in that case.

**Appendix.** This "arithmetic supplement" is added for the sake of completeness.

Proof of Lemma 2.3: We shall show the existence of a constant $\beta$ $(= \beta_K)$ such that

$$\sum_{j=2}^\infty \frac{t^{j/K}}{\beta^j j!} \leqq \sum_{j=2}^\infty \frac{t^j}{j!}\left(\frac{1}{j-1}\right)^{j(K-1)},\quad t \geqq 1,$$

in several steps. As usual, $[\cdot]$ denote the integer part. Let us write

$$\sum_{j=2}^\infty \frac{t^{j/K}}{\beta^j j!} = \sum_{j=2}^{[2K]} + \sum_{l=3}^\infty \sum_{j=[(l-1)K]+1}^{[lK]} = S_2 + \sum_{l=3}^\infty S_l.$$

*Step* 1 (an estimate of $S_2$):

$$S_2 = \sum_{j=2}^{[2K]} \frac{t^{j/K}}{\beta^j j!} \leq \frac{t^2}{[(2K]+1)!} \frac{[2K]+1}{\beta^{[2K]}}$$
$$\times (1 + \beta[2K] + \beta^2[2K]([2K]-1) + \cdots + \beta^{[2K]-2}[2K]!\,2^{-1}).$$

The last sum contains $[2K] - 1$ terms and the largest one equals $2^{-1}[2K]!\,\beta^{[2K]-2}$. Thus

$$S_2 \leq \frac{t^2}{([2K]+1)!} \frac{([2K]-1)([2K]+1)!}{2\beta^2} \leq \frac{t^2}{([2K]+1)!}$$

for sufficiently large $\beta$.

*Step* 2 (an estimate of $S_l$, $l \geq 3$):

$$S_l = \sum_{j=[(l-1)K]+1}^{[lK]} \frac{t^{j/K}}{\beta^j j!} \leq \frac{t^l}{[(lK]+1)!} \frac{[lK]+1}{\beta^{[lK]}}$$
$$\times \left(1 + \beta[lK] + \beta^2[lK]([lK]-1) + \cdots + \beta^{[lK]-[(l-1)K]-1} \frac{[lK]!}{([(l-1)K]+1)!}\right).$$

Now, there are $[lK] - [(l-1)K]$ terms in the last sum and the last is the largest of them. As $[lK] - [(l-1)K] \leq K + 1$ we obtain

$$S_l \leq \frac{t^l}{([lK]+1)!} \frac{K+1}{\beta^{[lK]}} \frac{([lK]+1)!}{([(l-1)K]+1)!} \beta^{[lK]-[(l-1)K]-1}$$
$$\leq \frac{t^l}{([lK]+1)!} \frac{K+1}{\beta^{[(l+1)K]+1}} (lK+1)^{K+1}$$
$$\leq \frac{t^l}{([lK]+1)!} \left(\frac{(lK+1)^{1+1/K}(K+1)^{1/K}}{\beta^{l-1}}\right)^K.$$

The last ratio is smaller than 1 provided $\beta$ is sufficiently large (uniformly with respect to $l$), again. Hence, for such $\beta$, $S_l \leq t^l/([lK]+1)!$

*Step* 3: Combining the estimates obtained we get for $\beta$ large enough

$$\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{([jK]+1)!}, \quad t \geq 1.$$

Easily,

$$\frac{1}{([jK]+1)!} = \frac{1}{j!(j+1)\cdots([jK]+1)} \leq \frac{1}{j!}\left(\frac{1}{j-1}\right)^{[jK]+1-j} \leq \frac{1}{j!}\left(\frac{1}{j-1}\right)^{jK-j},$$

and, finally,

$$\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!}\left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geq 1,$$

the desired inequality ∎

## REFERENCES


[1] Avantaggiati, A.: On compact imbedding theorems in weighted Sobolev spaces. Czechoslovak. Math. J. **29** (104) (1979), 635–648.

[2] Gustavsson, J., and J. Peetre: Interpolation of Orlicz spaces. Studia Math. **60** (1977), 33–59.

[3] Кабайла, В. П.: О вложении пространства $L_p(\mu)$ в $L_r(\nu)$. Лит. Мат. Сб. **21** (1981), 143–148.

[4] Krasnosel'skii, M. A., and Ya. B. Rutitskii: Convex functions and Orlicz spaces. Groningen: Noordhoff 1961.

[5] Krbec, M.: Two weights weak type inequalities for the maximal function in the Zygmund class. Lect. Notes Math. 1302 (1988), 317–320.

[6] Maligranda, L.: Indices and interpolation (Dissertationes Mathematicae No. 234). Warsaw: Polish Sci. Publ. 1985.

[7] Matuszewska, W., and W. Orlicz: On certain properties of $\varphi$-functions. Bull. Acad. Polon. Sci. 8 (1960), 439–443.

[8] Musielak, J.: Orlicz spaces and modular spaces. Lect. Notes Math. 1034 (1983), 1–122.

[9] Pick, L.: Two weights weak type inequality for the maximal function in $L(1 + \log^+ L)^K$. In: Constructive Theory of Functions. Proceedings of the conference held in Varna, May–June 1987. Sofia: Bulg. Acad. Sci. 1988, p. 377–381.

Manuskripteingang: 26. 09. 1988; in revidierter Fassung 13. 02. 1990

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