On Imbeddings between Weighted Orlicz Spaces

M. KRBEC and L. PICK

Es, werden Bedingungen für Gewichte ϱ und σ untersucht, die notwendig und hinreichend für die Gültigkeit der Einbettung für die Orlicz-Räume $L_{Q,q} \hookrightarrow L_{P,q}$ sind. Wir benutzen zwei Methoden: die erste ist ein Limitprozeß, der von den L_p -Räumen zur Einbettung von gewichteten Zygmund-Klassen in den gewichteten Raum L₁ führt, die zweite Methode beruht auf der Dualität in gewichteten Orlicz-Räumen, dem Index der Youngschen Funktion und gibt einen allgemeinen Einbettungssatz.

Найдены необходимые и достаточные условия на весовые функции е и о для того, чтобы существовало вложение пространств Орлича с весом $L_{Q,q} \hookrightarrow L_{P,q}$. Применяются два метода: первый является предельным процессом и ведет от L_p -пространств к вложению весовых классов Зигмунда в весовое пространство L_1 , второй подход применяет понятие двойственности в весовых пространствах Орлича, индекса Н-функции и дает обшую теорему вложения.

Necessary and sufficient conditions are given for weight functions ρ and σ , which guarantee validity of the imbedding between the Orlicz spaces $L_{Q,q} \leftarrow L_{P,q}$. We make use of two methods: the first of them is a limit procedure leading to the imbedding of weighted Zygmund classes into weighted L_1 , the second method employs the concept of duality in weighted Orlicz spaces, index of a Young function and results in a general imbedding theorem.

1. Introduction and preliminaries

This paper deals with necessary and sufficient conditions for imbeddings between weighted Orlicz spaces. Actually, it is an amalgam of results for weighted Lebesgue spaces (the particular case of continuous weights was solved by AVANTAGGIATI [1] and the general problem with measures by KABAILA [3]), ideas about limit treatment of necessary and sufficient conditions on weight functions [5, 9], and inequalities linking modulars and norms. We get a general imbedding theorem with no restrictions on the growth of the Young functions involved.

In [5, 9] we have considered two weight weak type inequalities for the maximal operator in Zygmund classes $L(1 + \log^4 L)^K$. It turns out that the necessary and sufficient condition on weights can be obtained by considering a limit case of the Muckenhoupt A_p condition. Rather surprisingly, an analogous procedure applied to the condition

$$
\int\limits_{\Omega} \big(\sigma(x)/\varrho(x) \big)^{p/(p-1)} \varrho(x) \, dx < \infty, \quad 1 < p < \infty, \quad \ldots \quad \ldots
$$

which is necessary and sufficient for the imbedding $L_{p,q} \hookrightarrow L_{1,q}$ turns out to provide us with a necessary and sufficient condition for the imbedding $L(1 + \log^+ L)_{\rho}^K$ into $L_{1,\sigma}$. (As usual, \hookrightarrow stands for continuous imbedding.) This is the main result of Section 2.

In Section 3 we present a necessary and 'sufficient éondition for the imbedding 108 M. KRBEC and L. Prox

In Section 3 we present a necessary and sufficient condition for the imbedding

between weighted Orlicz spaces $L_{Q,\rho} \hookrightarrow L_{P,\sigma}$. Roughly speaking, if QP^{-1} is a Young function and *N* is the complementary function to $\tilde{Q}P^{-1}$, then $\tilde{L}_{Q,q} \hookrightarrow L_{P,q}$ iff

$$
\int_{\Omega} N(\mu \sigma(x)/\varrho(x)) \varrho(x) dx < \infty \quad \text{for some } \mu > 0.
$$

An alternative technique of indices of Young functions is employed in Section 4. We obtain various relations concerning imbeddings in question under certain restrictive assumptions (as A_2 , A_3 , conditions on indices etc.).

Throughout the paper, Ω will be a measurable subset of \mathbb{R}^n . The symbols ρ and σ will denote *weights* in Ω , i.e. measurable and a.e. positive functions in Ω , $L_{p,q}$ will stand for the usual weighted Lebesgue space with norm $\|\cdot\|_{p,o}$. The reader is supposed to be familiar with the basic facts from the theory of modular and Orlicz spaces (è refer, e.g., to [8,4]). Here, we shall work with (real) weighted Orlicz spaces in the following sense: For any Young function *M,* we consider the *mdutar* $\int N(\mu\sigma(x)/\rho(x)) \rho(x) dx < \infty$ for some $\mu > 0$.
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$$
m_o(f, M) = \int_{\Omega} M(|f(x)|) \, \varrho(x) \, dx,
$$

ted *Orlicz space* L_M , *e* is the linear
This space is endowed with the l
 $||f||_{M \cdot e} = \inf \{ \lambda > 0; \, m_e(f/\lambda, M) \le$
that the Young function *M* satisfies

modular. This space is endowed with the *Luxemburg norm*

$$
||f||_{M,\varrho} = \inf \{ \lambda > 0; \ m_{\varrho}(f/\lambda, M) \leq 1 \}.
$$

Recall that the Young function M satisfies the $A₂$ *condition* (sometimes we write $M \in \Lambda_2$ if $M(2t) \leq CM(t)$ for large *t*. We shall say that *M* satisfies the Λ_2 condition Recall that the Young function *In* satisfies the Δ_2 condution (sometimes we write $M \in \Delta_2$) if $M(2t) \leq CM(t)$ for large *t*. We shall say that *M* satisfies the Δ_2 condition globally on $(0, \infty)$ if $M(2t) \leq CM(t)$ *function in the Laxemburg norm*
function function \mathcal{U}_M *,* \mathcal{U}_P *is the initial function of the class of function m
<i>function function M* satisfies the Λ_2 condition (somet $M \in \Lambda_2$) if $M(2t) \leq CM(t)$ f *f (trial) (trial) (trial) (trial) (trial) (trial) (i) (t) (method) (method) (method) M* **assisting** the Λ *₂ come* $\mathbb{Z}M(t)$ *for large t.* We shall say that $M(2t) \leq CM(t)$ for each $t \geq 0$. I

globally on
$$
(0, \infty)
$$
 if $M(2t) \leq CM(t)$ for each $t \geq 0$. If N is the complementary Y oung
\nfunction with respect to M₁, i.e.
\n
$$
N(t) = \sup_{t \geq 0} (t - M(\tau)), \quad t \geq 0,
$$
\nthen
\n
$$
t \leq M^{-1}(t) N^{-1}(t) \leq 2t, \quad t \geq 0,
$$
\nand the spaces $L_{M,e}$ and $L_{N,e}$ are naturally associated with the duality,
\n
$$
\langle f, g \rangle = \int_{Q} f(x) g(x) \, \varrho(x) \, dx, \quad f \in L_{M,e}, \quad g \in L_{N,e},
$$
\n
$$
\text{giving birth to the so-called Orlicz norm: on $L_{M,e}$, it is.
$$

then

$$
t \leq M^{-1}(t) N^{-1}(t) \leq 2t, \quad t \geq 0, \tag{1.1}
$$

$$
\langle f,g\rangle=\int\limits_{Q}f(x)\,g(x)\,\varrho(x)\,dx,\quad f\in L_{M,\varrho}\,,\quad g\in L_{N,\varrho}\,,
$$

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th to the so-called *Orlicz norm*, on
$$
L_{M,e}
$$
 it is
\n
$$
|||f|||_{M,e} = \sup \left\{ \int f(x) g(x) \, \varrho(x) \, dx; \, m_e(g, N) \leq 1 \right\}.
$$

 $\frac{1}{2}$ in the set of It is easy to check (quite analogously as in the case of non weighted spaces) that Fing birth to the so-called Orlicz norm; on $L_{M,e}$ it is $|||f|||_{M,e} = \sup_{\substack{i,j,k=1 \ i,j=1,\dots,k}} \left\{ \int_{\Omega} f(x) g(x) \, \varrho(x) \, dx; \, m_e(g,N) \right\}$
is easy to check (quite analogously as in the case of $||f||_{M,e} \leq |||f|||_{M,e} \leq 2||f||_{M,e}$.
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, The total is so-called *Orlicz norm*; on $L_{M,e}$ it is $|||f|||_{M,e} = \sup \left\{ \int f(x) g(x) \varrho(x) dx; m_e(g, N) \leq 1 \right\}$.
 $||f||_{M,e} = \sup \left\{ \int f(x) g(x) \varrho(x) dx; m_e(g, N) \leq 1 \right\}$.
 \downarrow to check (quite analogously as in the case of non-weighted s

We shall denote by $E_{M,\rho}(\Omega)$ the closure in $L_{M,\rho}(\Omega)$ of the set of all bounded measurable functions with bounded support in Ω . Then $L_{N,q}$ is the dual space of $E_{M,q}$.

Let us still recall that a Young function *M* satisfies the Λ_3 *condition* ($M \in \Lambda_3$) iff there is $K > 0$ such that $tM(t) \leq M(Kt)$ for large t. If $M \in \Lambda_3$, then $N \in \Lambda_2$ and

$$
N(t) \leq C t M^{-1}(C t), \quad t \geq T, \tag{1.2}
$$

for some $C, T \geq 0$.

The basic properties of weighted spaces used here do not go beyond the limits of an easy verification.

We only mention several useful assertions **on general modular spaces (for the concept see, e.g., [8]). The modulars in question need not be necessarily** strictly **convex** so that any weighted L_1 space is included.
Let (X_i, m_i) be a modular space with the (Luxemburg) norm $\|\cdot\|_i$ ($i = 1, 2$) and let

increase 1 The basic properties of weighted spaces used here
an easy verification.
We only mention several useful assertions on gener
cept see, e.g., [8]). The modulars in question need no
so that any weighted L_1 spa $T: X_1 \to X_2$ be a sublinear operator. It is clear that the modular inequality $m_2(T)$ $\leq Cm_1(f), f \in X_1$, implies its norm counterpart $||Tf||_2 \leq C||f||_1, f \in X_1$. The converse **is not true, however, we have (the proof is easy and therefore omitted)** The basic properties of weighted spaces used here do not go be
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so th X_2 be a sublinear operator. It is clean $f \in X_1$, implies its norm counterpare, however, we have (the proof is easy a 1.1: Let T , X_1 , X_2 be as above. The
re exists $C \geq 1$ such that $||Tf||_2 \leq C||f||$
ere exists

Lemma 1.1: Let T, X_1 , X_2 be as above. Then the following two assertions are equi*valent:*

(*i*) There exists $C \geq 1$ such that $||Tf||_2 \leq C||f||_1$, we can also added the contract of f

(ii) There exists $K \geq 1$ *such that* $m_2(Tf|K) \leq 1$ *provided* $m_1(f) \leq 1$.

Later, **we shall** also **make use of**

Later, we shall also make use of
 Lemma 1.2: Let T, X₁, X₂ <i>be as above. Let h be a positive function defined on $(0, \infty)$ *and bounded in some right neighbourhood of zero. If* counterpart $||T||_2 \leq C||f||_1$, $f \in X_1$,
proof is easy and therefore omitted
above. Then the following two asset
 $Tf||_2 \leq C||f||_1$.
 $m_2(Tf|K) \leq 1$ provided $m_1(f) \leq 1$.
ad (ii) coincide.
above. Let *h* be a positive f

$$
||Tf||_2 \leq h(m_1(f)), \quad f \in X_1, \tag{1.3}
$$

then there is $C \geq 1$ *such that for all f*

$$
||Tf||_2 \leq C||f||_1. \tag{1.4}
$$

For exists $C \geq 1$ such that
 r e exists $K \geq 1$ such that

the best constants in (i)
 α such that
 α shall also make use o

a 1.2: Let T , X_1 , X_2 be a
 α
 α is α and α is $Tf\|_{2} \leq h(m_1(f))$, **Proof:** Suppose (1.4) is not true. Then there is a sequence ${f_n} \subset X_1$ such that then there is $C \ge 1$ such that for all f
 $||Tf||_2 \le C||f||_1$.

Proof: Suppose (1.4) is not true. Then there is a set
 $||f_n||_1 \searrow 0$ and $||Tf_n||_2 \nearrow \infty$. For large *n*, $m_1(f_n) \le ||f_n||_1 \le$

contradicts with (1.3) $||f_n||_1 \searrow 0$ and $||Tf_n||_2 \nearrow \infty$. For large *n*, $m_1(f_n) \leq ||f_n||_1 \leq 1$, whence $m_1(f_n) \to 0$, which (ii) There exists $K \geq 1$ such that

Moreover, the best constants in (i) an

Later, we shall also make use of

Lemma 1.2: Let T , X_1 , X_2 be as and

and bounded in some right neighbour
 $||T||_2 \leq h(m_1(f))$, $f \in X_1$,
 Later, we shall also make use of

Lemma 1.2: Let T , \overline{X}_1 , \overline{X}_2 be as above. Let h be a positive

and bounded in some right neighbourhood of zero. If
 $||T||_2 \leq h(m_1(f))$, $f \in \overline{X}_1$,

then there is $C \geq 1$ s

As currently adopted, different constants are **denoted by the same letters if no** المحمد المحمد
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$$
x_1, x_2, \ldots, x_n \in \mathbb{R}^n, \quad x_1, x_2, \ldots, x_n \in \mathbb{R}^n
$$

Y.

2. The limiting approach

In what follows, p and p' are conjugate exponents, i.e. $p \in (1, \infty)$, $p' = p/(p - 1)$. Let **us mention the elementary and frequently** used **formulas**

$$
p'/p = p' - 1 = 1/(p - 1).
$$

We start with a special case of a weighted imbedding theorem w**hich appeared** in AVANTAGGIATI's paper [1] for continuous weights and generally for measures in KABAILA [3]. For the sake of completeness, we present a proof based on the latter **paper, which will, additionally, provide us with a relation between important constants.**

Theorem 2.1: *The following 8taiement8 are equivalent:*

Thus,
$$
P(x) = \sum_{n=0}^{\infty} P(x)
$$
 is a set of $P(x)$.

\nTheorem 2.1: The following statements are equivalent to:

\n(i) $\int_{\Omega} |f(x)| \sigma(x) \, dx \leq C_p(\sigma, \rho) \left(\int_{\Omega} |f(x)|^p \rho(x) \, dx \right)^{1/p};$

\n(ii) $B_p(\sigma, \rho) = \int_{\Omega} \left(\sigma(x)/\rho(x) \right)^{p'} \rho(x) \, dx < \infty.$

Moreover, the best $C_p(\sigma, \rho)$ *in (i) equals* $(B_p(\sigma, \rho))^{1/p'}$ *.*

Proof: Let *(i)* hold. Putting $f(x) = (\sigma(x)/\rho(x))^{1/(p-1)}$ we get *d* **L.** Prck
 d. Putting $f(x) = (\sigma(x)/\varrho(x))$
 $\varrho(x) dx = \int_a^b |f(x)| \sigma(x) dx$ RBEC and L. PICK

(*i*) hold. Putting $f(x) =$
 $\frac{\sigma(x)}{\varrho(x)}$ $\rho'(x) dx = \int_{\Omega} |f(x)|$
 $\leq C_p(\sigma,$ $f(x) = (\sigma(x)/\rho(x))^{1/(p-1)}$ we get
 $\int_{\sigma} |f(x)| \sigma(x) dx$
 $C_p(\sigma, \rho) \left(\int_a^a |f(x)|^p \rho(x) dx \right)^{1/p}$
 $C_p(\sigma, \rho) \left(\int_{a}^a \left(\frac{\sigma(x)}{2} \right)^{p^s} \rho(x) dx \right)^{1/p}$. *a*
 a(*x*) $|P \varrho(x) dx$ $\bigg\}^{1/p}$
 a(*x*) $\bigg\}^{p'} \varrho(x) dx$ $\bigg\}^{1/p}$ $\begin{aligned} \log f(x) &= \big(\sigma(x)/\varrho(x)\big)^{1/(p-1)} \\ \equiv & \int_{\mathcal{Q}} |f(x)| \; \sigma(x) \; dx \\ \leq & C_p(\sigma, \varrho) \left(\int_{\mathcal{Q}} |f(x)|^p \; \varrho(x) \right) \\ &= C_p(\sigma, \varrho) \left(\int_{\mathcal{Q}} \left(\frac{\sigma(x)}{\varrho(x)} \right)^{p'} \right) \\ \text{each } & g \in L_p(\varrho) \end{aligned}$ *Q(x)dX)* Proof: Let (*i*) hold. Putting $f(x) = (\sigma(x)/\varrho(x))^{1/p-1}$. w
 $\int_{2}^{x} \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x) dx = \int_{2}^{x} |f(x)| \sigma(x) dx$
 $\leq C_p(\sigma, \varrho) \left(\int_{\Omega} |f(x)|^p \varrho(x) dx\right)$
 $= C_p(\sigma, \varrho) \left(\int_{\Omega} \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x)\right)$

The last term is finit $= C_p(\sigma, \rho)$
 IFFRA is finite as for each $g \in R$
 $\left| \int_a^b g(x) \frac{\sigma(x)}{\rho(x)} \rho(x) dx \right| = \left| \int_a^b f(x) \right|$ $\int_{\Omega} g(x) \sigma(x) dx$
 f $|g(x)| \sigma(x) dx \leq C_p(\sigma, \rho) \left(\int_{\Omega} |g(x)|^p \rho(x) dx \right)^{1/p}$ $g(\sigma, \varrho)$ $\left(\int_{\mathcal{B}} \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x) dx\right)^{1/p}$
 $g \in L_p(\varrho)$
 $\left|\int_{\mathcal{B}} g(x) \sigma(x) dx\right|$
 $\left|\int_{\mathcal{B}} g(x) \sigma(x) dx\right|$
 $\left|\int_{\mathcal{B}} |g(x)| \sigma(x) dx \leq C_p(\sigma, \varrho) \left(\int_{\mathcal{B}} |g(x)|^p \varrho(x) dx\right)\right|$

ts a bounded linear functional on so that the function a/ρ represents a bounded linear functional on $L_{p,q}$ and belongs Fig. 10 if inite as for each $g \in L_p(q)$
 $\int_a^b g(x) \frac{\sigma(x)}{\varrho(x)} \varrho(x) dx$ = $\left| \int_a^b g(x) \sigma(x) dx \right|$
 $\leq \int_a^b |g(x)| \sigma(x) dx \leq C_p(\sigma, \varrho) \left(\int_a^b |g(x)|^p \varrho(x) dx \right)$

is function σ/ϱ represents a bounded linear functional on L_p , and
 $\leq \int\limits_{\Omega} |g(x)| \sigma(x) dx \leq C_p(\sigma, \varrho) \left(\int\limits_{\Omega} |g(x)|^p \varrho(x) dx \right)^{1/p},$

to
$$
L_{p',e}
$$
. This yields $\int_{a}^{b} (\sigma(x)/\rho(x))^{p'} \rho(x) dx \leq C_p(\sigma, \rho)^{p'}$, hence $B_p(\sigma, \rho) \leq C_p(\sigma, \rho)^{p'}$.
\nConversely, if $B_p(\sigma, \rho)$ is finite, then
\n
$$
\int_{a}^{b} |f(x)| \sigma(x) dx = \int_{a}^{b} |f(x)| \sigma(x) \rho^{1/p}(x) \rho^{-1/p}(x) dx
$$
\n
$$
\leq ||f||_{p,e} \left(\int_{a}^{b} (\sigma(x)/\rho(x))^{p'} \rho(x) dx\right)^{1/p'} = (B_p(\sigma, \rho))^{1/p'} ||f||_{p,e}.
$$
\nTherefore, (i) holds and $C_p(\sigma, \rho) \leq (B_p(\sigma, \rho))^{1/p'}$

 $(B_p(\sigma,\varrho))^{1/p'}$ iii

Therefore, (i) holds and
$$
C_p(\sigma, \rho) \leq (B_p(\sigma, \rho))^{1/p^r}
$$

\nLet $1 \leq K < \infty$. We introduce the Young functions
\n
$$
\Phi_K(t) = \begin{cases} t^2, & 0 \leq t \leq 1, \\ t(1 + \log t)^K, & t > 1, \end{cases}
$$
\n
$$
F_K(t) = \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geq 0.
$$

We shall denote by Ψ_K and G_K the complementary Young functions to Φ_K and F_K , respectively. $F_K(t) = \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{n-j}$, $t \ge 0$.

is easill denote by Ψ_K and G_K the complementary You spectively.

The rather lengthy and tedious arithmetical proof

stponed to Appendix.

Lemma 2.2: There exi $F_K(t) = \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{k-1}$, $t \ge 0$.

denote by Ψ_K and G_K the complementary Young function

ely.

ther lengthy and tedious arithmetical proof of the follend

d to Appendix.

a 2.2: There exists a

The rather lengthy and tedious arithmetical proof of the following lemma is

$$
F_K(t) > \exp(t^{1/K}|\beta_K) - t^{1/K}|\beta_K - 1, t \geq 1.
$$

It is worth to notice that the function $\exp (t^{1/K}) - t^{1/K} - 1$ is equivalent to Ψ_K for large values of t. Indeed, it is easy to verify that. $\Phi_K(t)/t < C\Phi_K(\sqrt{t})/\sqrt{t}$, $t \ge 1$, with, say, $C = 3^K$. Thus, by [4, Theorem 6.8], Ψ_K satisfies the Λ_3 condition and, on using (1.2), Ψ_K^{-1} is equivalent to $(1 + \log t)^K$ for large t. We shall denote by Ψ_K and G_K the complementary Young functions to Φ_I
respectively.
The rather lengthy and tedious arithmetical proof of the following
postponed to Appendix.
Lemma 2.2: There exists a constant $\beta_K \$

Lemma 2.3: The functions F_K , Φ_K , Ψ_K satisfy

(*i*) $\Phi_K(t) \geq t$ for $t \geq 1$; for $t \ge 1$; $\lambda_i(i)$ $2F_K(t) \geq t$ (*iii*) $\Psi_K(t) \leq F_K(t)$ for $0 \leq t \leq 1$; $for t \geq 1;$ (iv) $G_K(t) \leq C_K \Phi_K(t)$ (v) $\Phi_K(t) \leq C_K t^p (p-1)^{-K}$ for $t \geq 0$ and $p \in (1, 2)$.

Proof: The statements $(i) - (iii)$ follow immediately. We prove (iv) . Note that $F_K \in \Delta_3$, thus, in virtue of (1.2), $G_K(t) \leq C t F_K^{-1}(Ct)$ for some $C \geq 1$ and all large t.
Making use of Lemma 2.2 we have $F_K(t) \geq \alpha_K \exp(t^{1/K} \beta_K)$ with some $\alpha_K > 0$, β_K from Lemma 2.2, and for every $t \geq 1$, so that passing to the inverse functions, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$
G_K(t) \leq C t \beta_K^{-K} \log^{K} (C t | \alpha_K) \leq C_K t \log^{K} t \leq C_K \Phi_K(t) \text{ for } t \geq 1.
$$

As to (v), it is easy to check that max $\Phi_K(t)$ $t^{-p} \leq K^{K}(p-1)^{-K}$, so that we can put $C_K = K^K$
Lemma 2.4: There exists a constant δ_K such that the (modified) Young inequality

 $ab \leq \delta_K(F_K(a) + \Phi_K(b))$ holds for all $a, b \geq 0$. of an over degree was called \$2

Proof: We shall make use of Lemma 2.3/(i)-(iv). If $a, b \le 1$, then ab $\leq \Phi_K(a) + \Psi_K(b) \leq \Phi_K(a) + F_K(b)$. If $a < 1, b \geq 1$, then $ab < b \leq 2F_K(b) \leq \Phi_K(a) + 2F_K(b)$. If $a \geq 1, b < 1$, then $ab < a \leq \Phi_K(a) + F_K(b)$. Finally, if $a, b > 1$, then, according to the Young inequality, $ab \leq G_K(a) + F_K(b) \leq C_K \Phi_K(a) + F_K(b)$. Putting $\delta_K = \max\{2, C_K\}$ concludes the proof \blacksquare

Lemma 2.5: There exists a constant θ_K such that and Annual Library American

$$
||f||_{\phi_{\mathbf{X}},\rho} \leq \frac{\theta_K}{(p-1)^K} ||f||_{p,\rho}, \quad f \in L_{p,\rho}, \quad p \in (1,2), \quad \text{for every } \rho \in \mathbb{R}^n, \quad \text{for all } \rho \in \mathbb{R}^n.
$$

Proof: The thesis follows directly from Lemma $2.3/(v)$ and from the definition of the Luxemburg norm \blacksquare . We have a set of the set $\mathcal{O}(\mathcal{O}_\mathcal{O})$. The $\mathcal{O}(\mathcal{O}_\mathcal{O})$

Theorem 2.6: The following conditions on σ , ρ are equivalent: $\mathcal{L}(\mathcal{L}^{\text{max}})$ and $\mathcal{L}(\mathcal{L}^{\text{max}})$ $\sim 10^{-1}$ (i) $L_{\Phi_{K},\rho}(\Omega) \hookrightarrow L_{1,\sigma}(\Omega)$; (i) $L\varphi_{R,\theta}(\infty)$

(ii) $C_p(\sigma,\varrho) \leq C(p-1)^{-K}$, $p \in (1,2)$;
 \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots

Proof: We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Let (i) hold. Then, according to Lemma 2.5, $||f||_{1,\sigma} \leq C_K (\theta_K/(p-1)^K) ||f||_{p,\sigma}$, $p \in (1,2)$.
Now suppose that (ii) is valid. Employing the relation between the best constants

from Theorem 2.1 (j' is the conjugate exponent to j) we get

$$
\int\limits_{\Omega} F_K \left(\mu \, \frac{\sigma(x)}{\varrho(x)} \right) \varrho(x) \, dx = \sum\limits_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j-1} \right)^{j(K-1)} B_j(\sigma, \varrho)
$$
\n
$$
\leq \sum\limits_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j-1} \right)^{j(K-1)} C^j \left(\frac{1}{j'-1} \right)^{Kj}
$$
\n
$$
\leq \sum\limits_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j-1} \right)^{j(K-1)} C^j j! \left(\frac{1}{j'-1} \right)^{j(K-1)}
$$
\n
$$
= \sum\limits_{j=2}^{\infty} \frac{1}{j!} (C\mu j)^j,
$$

the last sum being finite provided $0 < \mu < (Ce)^{-1}$.

 $\left\{ \cdots \right\}$

If (iii) is true, then the modified Young inequality (Lemma 2.4) implies

$$
\int_{\Omega} |f(x)| \sigma(x) dx = \int_{\Omega} |f(x)| \frac{1}{\mu} \frac{\mu \sigma(x)}{\varrho(x)} \varrho(x) dx
$$
\n
$$
\leq \frac{\delta_K}{\mu} \int_{\Omega} \Phi_K(|f(x)|) \varrho(x) dx + \frac{\delta_K}{\mu} \int_{\Omega} F_K \left(\mu \frac{\sigma(x)}{\varrho(x)}\right) \varrho(x) dx
$$
\n
$$
= C_{K,\mu} \Phi_K(|f(x)|) \varrho(x) dx + C_{K,\mu} B_{\log,K}(\sigma, \varrho).
$$

Application of Lemma 1.2 (with $h(t) = C_{K,\mu}^{\Lambda}(t) + B_{\log,K}(\sigma,\varrho)$) leads to the imbedding (i) **I**

Remark 2.7: By duality argument, we have also characterized those weights ρ and σ for which $||f/e||\psi_{\kappa,e} \leq C||f/\sigma||_{\infty}$.

For $K = 1$ and Ω bounded, $L_{\phi_{K},\rho}$ is the weighted Zygmund class $L(1 + \log^+ L)_{\rho}$ consisting of all functions f with $\int |f(x)| (1 + \log^+ |f(x)|) \varrho(x) dx < \infty$. Let us state the following particular

Corollary 2.8: Let Ω be bounded and $\rho \in L_1(\Omega)$. Then the weighted Zygmund class $L(1 + \log^+ L)_e$ is imbedded into $L_{1,e}$ iff $\int \exp \left(\mu \sigma(x) / \varrho(x) \right) \varrho(x) dx < \infty$ for some $\mu > 0$.

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3. General imbeddings

- 1

In the rest of the paper, P and Q are Young functions.

Definition 3.1: For a couple of Young functions P, Q we introduce the set $Y(P,Q) = {K > 0; Q(KP^{-1})}$ is equivalent to some Young function}. For $K \in Y(P,Q)$, the Young function equivalent to $Q(KP^{-1})$ will be denoted $Q(KP^{-1})$ again. Recall that two functions F, G are equivalent iff $\alpha^{-1}F(\alpha^{-1}t) \leq G(t) \leq \alpha F(\alpha t)$ for all t and some $\alpha \geq 1$ independent of t. 化原子 人名

Remark 3.2: Note that if at least one of the functions P, Q satisfies the Δ_2 condition globally on $(0, \infty)$, then $Y(P, Q)$ is either empty, or equal to $(0, \infty)$.

Proposition 3.3: Suppose that QP^{-1} is a Young function. Then the following statements are equivalent:

 $||f||_{P,\sigma} \leq ||f||_{Q,\rho}$ for all f, \cdot (i) $||f||_{1,\sigma} \leq ||f||_{QP^{-1},\rho}$ for all f. (ii)

Proof: According to Lemma 1.1, (i) is subsequently equivalent to the implications $m_o(f, Q) \leq 1 \Rightarrow m_o(f, P) \leq 1$, and, on taking $g = P(f), m_o(g, QP^{-1}) \leq 1 \Rightarrow \int g \sigma$ \leq 1. The repeated use of Lemma 1.1 completes the proof \blacksquare

Realizing that $K||f||_{Q,q} = ||f||_{Q_{K},q}$, where Q_K is the Young function given by $Q_K(t)$ $= Q(Kt)$, we easily obtain

Corollary 3.4: Let $K \in Y(P, Q)$. Then

 $||f||_{P,\sigma} \leq K||f||_{Q,\rho} \,\forall f \Rightarrow ||f||_{1,\sigma} \leq ||f||_{Q(KP^{-1}),\rho} \,\forall f.$

Conversely, if $K \in Y(P, Q)$, then for every $C \geq 1$

 $||f||_{1,\sigma} \leq C||f||_{Q(KP^{-1}),\rho} \,\forall f \Rightarrow ||f||_{P,\sigma} \leq KC||f||_{Q,\rho} \,\forall f.$

 \sim 4.4

Corollary 3.5: Let at least one of the functions P , Q satisfy the Λ ₂ condition globally *Imbeddings between Weighted Orlicz Spaces* 113
 Corollary 3.5: Let at least one of the functions P, Q satisfy the Λ_2 *condition globally*
 on $(0, \infty)$. Then $L_{Q,g} \hookrightarrow L_{P,g}$ *iff* $L_{Q_{P^{-1},g}} \hookrightarrow L_{1,g}$ *provided Q*

In the sequel, we will assume that QP^{-1} is a Young function. If $K \in Y(P, Q)$, we denote by N_K the function complementary to $Q(KP^{-1})$. We shall study relations between the imbedding $L_{Q,\rho} \hookrightarrow L_{P,\sigma}$ and the condition Imbeddings between
 t one of the functions P, Q

iff $L_{QP^{-1},e} \hookrightarrow L_{1,e}$ provide

sume that QP^{-1} is a You

on complementary to $Q(K \hookrightarrow L_{P,e}$ and the condition
 $x) dx < \infty$ for some μ Imbeddings betw

Corollary 3.5: Let at least one of the functions P

on $(0, \infty)$. Then $L_{Q,q} \hookrightarrow L_{P,\sigma}$ iff $L_{QP^{-1},q} \hookrightarrow L_{1,\sigma}$ prot

In the sequel, we will assume that QP^{-1} is a

we denote by N_K the function comple

$$
\int_{\Omega} N_K\big(\mu\sigma(x)/\varrho(x)\big)\,\varrho(x)\,dx<\infty\quad\text{for some}\quad\mu>0\,.
$$

Proposition 3.6: Let $\int N_K(\mu \sigma(x)/\varrho(x)) \varrho(x) dx$ be finite for some positive K, μ .

Proof: Making use of the Young inequality, we have

Corollary 3.5. Let at least one of the functions P, Q satisfy the
$$
\Delta_t
$$
 condition global on $(0, \infty)$. Then $L_{Q,\rho} \hookrightarrow L_{P,\sigma}$ iff $L_{Q,P-1,\rho} \hookrightarrow L_{1,\sigma}$ provided QP^{-1} is a Young function. In the sequel, we will assume that QP^{-1} is a Young function. If $K \in Y(P,Q)$, we denote by N_K the function complementary to $Q(KP^{-1})$. We shall study relation between the imbedding $L_{Q,\rho} \hookrightarrow L_{P,\sigma}$ and the condition $\int_{\Omega} N_K(\mu \sigma(x)/\rho(x)) \rho(x) dx < \infty$ for some $\mu > 0$. Proposition 3.6: Let $\int N_K(\mu \sigma(x)/\rho(x)) \rho(x) dx$ be finite for some positive K, μ Proposition 3.6: Let $\int N_K(\mu \sigma(x)/\rho(x)) \rho(x) dx$ be finite for some positive K, μ Proposition 3.7: Let $f \in \{1 + \mu^{-1} \int P(|f(x)|/K) \left(\sigma(x)/\rho(x)\right) \rho(x) dx\}$ and in virtue of Lemma 1.2 we arrive at the desired imbedding 1. For some $K \geq \sup \{||f||_{P,\sigma}||f||_{Q,\sigma}, \{1 + \mu\}||_{Q,\sigma} \}$ and in virtue of Lemma 1.2 we arrive at the desired imbedding 1. For some $K \geq \sup \{||f||_{P,\sigma}||f||_{Q,\sigma}; f \neq 0\}$. The $\int_{\Omega} N_K(\mu \sigma(x)/\rho(x)) \rho(x) dx < \int_{\mu} \rho(x) dx \neq 0$. Proposition 3.7: Let the imbedding $L_{Q,\sigma} \hookrightarrow L_{P,\sigma}$ hold and suppose $K \in Y(P,Q)$ for some $K \geq \sup \{||f||_{P,\sigma}||f||_{Q,\sigma}; f \neq 0\}$. Then, by Corollary 3.4, $\int_{\Omega} g(x) (\sigma(x)/\rho(x)) \rho(x) dx \leq \sup \{|g$

and in virtue of Lemma 1.2 we arrive at the desired imbedding \mathbf{U}

Proposition 3.7: Let the imbedding $L_{Q,q} \hookrightarrow L_{P,q}$ *hold and suppose* $K \in Y(P,Q)$ *for some* $K \geq \sup \{||f||_{P,\sigma}/||f||_{Q,\rho}; f \neq 0\}$ *. Then* $\int N_K(\mu \sigma(x)/\rho(x)) \rho(x) dx < \infty$ *for some* $\mu>0$. *d* in virtue of Lemma 1.2 we arrive at

Proposition 3.7: Let the imbedding
 $\begin{aligned}\n &\cdot \text{some } K \geq \sup \{ ||f||_{P,\sigma} / ||f||_{Q,\sigma} ; f \neq 0 \}. \\
 &> 0.\n \end{aligned}$ Proof: Assume $g \in E_{Q(KP^{-1}),\rho}(\Omega)$. Then
 $\begin{vmatrix}\n &\int g(x) \left(\sigma(x)/\rho(x) \right) \rho(x) dx \\
 &\int \rho(x$

Proof: Assume $g \in E_{Q(KP^{-1}),q}(\Omega)$. Then, by Corollary 3.4,

$$
A \subseteq \sup \{ ||f||P,\sigma/||f||q,\rho,f \neq 0\}.
$$

As
sume $g \in E_{Q(KP^{-1}),\rho}(\Omega)$. Then, by Corollary 3.

$$
\left| \int_{\Omega} g(x) \left(\sigma(x)/\rho(x) \right) \rho(x) dx \right| \leq ||g||_{1,\sigma} \leq ||g||_{Q(KP^{-1}),\rho},
$$

Let us summarize the results of this **section.**

Theorem 3.8: Let at least one of the functions P **,** Q **satisfy the** Λ_2 **condition. Then** *the following statements are equivalent:*

(*ii*) $\int N(\mu\sigma(x)/\rho(x)) \rho(x) dx < \infty$ *for some C,* $\mu > 0$.

Theorem 3.9: Suppose that $Y(P, Q)$ contains a sequence $K_n \nearrow \infty$. Then the following *statements are equivalent:*

 (t) $L_{Q,\varrho}(\Omega) \hookrightarrow L_{P,\sigma}(\Omega);$ (ii) $\int N_c(\mu\sigma(x)/\varrho(x))\varrho(x) dx < \infty$ for some $C, \mu > 0$.

Rem ark' 3.10: The duality argument applied to Theorems 3.8 and 3.9 gives the characterization of ϱ and σ such that $||f||\varrho||\tilde{\varrho}_{\sigma} \leq C ||f||\sigma||\tilde{\varrho}_{\sigma}$, where \tilde{P} and \tilde{Q} are the complementary **Young functions to** *P. Q,* **respectively.**

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4. Some **remarks on an alternative approach (employing indices)**

First, **we** recall **the concept of the lower and upper** index **of a Young function. Actu**ally, it is a certain refinement of the Λ_2 condition. The existence of finite indices for functions satisfying Λ_2 can be proved e.g. on the base of the theory of submulti-

 $1 - 12 - 12 = 12$

plicative functions. We refer to MATUSZEWSKA and ORLICZ [7], GUSTAVSSON and **PEETRE [2], and the comprehensive paper by MALIGRANDA [6]:**
 PEETRE [2], and the comprehensive paper by MALIGRANDA [6]:
 h(λ) = $h_{\phi}(\lambda)$ = sup $\Phi(\lambda i)/\Phi(i)$, $\lambda \ge 0$.

Definition 4.1: Let
$$
\Phi \in A_2
$$
 globally on $(0, \infty)$ and

$$
b(\lambda)=h_{\phi}(\lambda)=\sup_{t>0}\Phi(\lambda t)/\Phi(t),\quad \lambda\geq 0.
$$

We define the *lower index of* Φ as $i(\Phi) = \lim_{n \to \infty} \log h(\lambda) / \log \lambda$ and the *upper index of* Φ as $I(\Phi) = \lim_{h \to 0} \log h(\lambda) / \log \lambda$. $=$ $\lim_{\lambda \to 0+} \log h(\lambda)$ 114 M. KRBEG.an

plicative functions.

PEETRE [2], and the

Definition 4.1:
 $h(\lambda) = h_{\phi}(\lambda)$

We define the *lower*

as $I(\Phi) = \lim_{\lambda \to \infty} h(\lambda)$

It follows easily fit

that
 $\Phi(\lambda t) \leq C_{\epsilon}$,
 $\min_{i} {\mu}^{i(\Phi) - \epsilon}$ $h(\lambda) = h_{\phi}(\lambda) = \sup_{t>0} \Phi(\lambda t)/\Phi(t), \quad \lambda \geq 0.$

e define the lower index of Φ as $i(\Phi) = \lim_{t \to 0} \log h(\lambda)/\log \lambda$ and the upper $I(\Phi) = \lim_{t \to \infty} \log h(\lambda)/\log \lambda$.

It follows easily from the definition that for each $\varepsilon > 0$ there ex

It follows easily from the definition that for each $\epsilon > 0$ there exists $C_{\epsilon} \ge 1$ such that and the state of the

e define the *lower index of*
$$
\Phi
$$
 as $i(\Phi) = \lim_{\lambda \to 0^+} \log h(\lambda)/\log \lambda$ and
\n $I(\Phi) = \lim_{\lambda \to \infty} \log h(\lambda)/\log \lambda$.
\nIt follows easily from the definition that for each $\varepsilon > 0$ then
\nat
\n $\Phi(\lambda t) \leq C_{\varepsilon} \max \{\lambda^{i(\Phi)-\varepsilon}, \lambda^{I(\Phi)+\varepsilon}\} \Phi(t), \quad t \geq 0, \quad \lambda \geq 0,$
\n $\min \{\mu^{i(\Phi)-\varepsilon}, \mu^{I(\Phi)+\varepsilon}\} \Phi(t) \leq C_{\varepsilon} \Phi(\mu t), \quad t \geq 0, \quad \mu \geq 0.$
\nne following assertion linking modulus and norms in weight
\nimmediate consequence of the properties of indices and the I
\nProposition 4.2: Assume $||f||_{P,\sigma} \leq C||f||_{Q,\rho}$ and let $\varepsilon > 0$.
\n(i) If $P \in \Lambda_2$ globally on $(0, \infty)$, then
\n $m_{\sigma}(f, P) \leq C_{\varepsilon} \max \{||f||_{Q,\rho}^{(P)-\varepsilon}, ||f||_{Q,\rho}^{(P)+\varepsilon}\}$.
\n(ii) If $Q \in \Lambda_2$ globally on $(0, \infty)$, then
\n $\min \{||f||_{P,\sigma}^{Q-\varepsilon}, ||f||_{Q,\sigma}^{LQ+\varepsilon}\} \leq C_{\varepsilon} m_{\varepsilon}(f, Q)$.
\n(iii) If $P, Q \in \Lambda_2$ globally on $(0, \infty)$, then

The following assertion linking modulars and norms in weighted Orlicz spaces is an immediate consequence of the properties of indices and the Luxemburg norm.

-
-
- **Matte consequence of the properties of sittion 4.2: Assume** $||f||_{P,\sigma} \leq C||f||_{Q,\rho}$ **of** $P \in \Lambda_2$ **globally on** $(0, \infty)$ **, then** $m_{\sigma}(f, P) \leq C_{\epsilon}$ **max** $\{||f||_{Q,\rho}^{i(P)+\epsilon}, ||f||_{Q,\rho}^{i(P)+\epsilon}\}$
-

(*ii*) If $Q \in \Delta_2$ globally on $(0, \infty)$, then
 $\min \{||f||_{P, \sigma}^{Q, -\epsilon}, ||f||_{P, \sigma}^{I Q, +\epsilon} \} \leq C_{\epsilon} m_o(f, Q).$

(*iii*) If $P, Q \in \Delta_2$ globally on $(0, \infty)$, then *(iii) If P, Q* $\in A_2$ *globally on* $(0, \infty)$ *, then*

$$
m_{\sigma}(f, P) \leq C_{\epsilon} \max \{||f||_{Q,\varrho}^{(P)}\cdot, ||f||_{Q,\varrho}^{(P)+\epsilon}\}.
$$
\n
$$
(ii) \quad If \quad Q \in \Lambda_{2} \text{ globally on } (0, \infty), \text{ then}
$$
\n
$$
\min \{||f||_{P,\sigma}^{(Q)-\epsilon}, ||f||_{P,\sigma}^{(Q)+\epsilon}\} \leq C_{\epsilon} m_{\varrho}(f, Q).
$$
\n
$$
(iii) \quad If \quad P, \quad Q \in \Lambda_{2} \text{ globally on } (0, \infty), \text{ then}
$$
\n
$$
m_{\sigma}(f, P) \leq C_{\epsilon} \max \{ (m_{\varrho}(f, Q))^{(i(P)/I(Q))-\epsilon}, (m_{\varrho}(f, Q))^{(I(P)/I(Q))+\epsilon} \}.
$$
\n
$$
\text{Proposition 4.3: The following two statements are true:}
$$
\n
$$
(i) \quad \text{Let} \quad i(Q) > \alpha > 0 \quad (i(Q) = \infty \quad \text{is admissible here}). \quad \text{If } ||f||_{Q,\varrho} > 1, \text{ then we have}
$$

. .. (i) Let $i(Q) > \alpha > 0$ $(i(Q) = \infty$ is admissible here). If $||f||_{Q,e} > 1$, then we have Prope

(i) Let
 $||f||_{Q,\varrho}$ \leq

(ii) Let $C_a(m_e(f, Q))^{1/2}$ *(i.e.)* $1/2$, $Q \in \mathbb{Z}_2$ yields $m_o(f, Q)$ (i.e.), then
 $m_o(f, P) \leq C_e$ max $\{(m_e(f, Q))^{(i(P)/I(Q)) - \epsilon}, (m_e(f, Q))^{(I(P)/I(Q))}\}$

Proposition 4.3: The following two statements are true:

(i) Let $i(Q) > \alpha > 0$ $(i(Q) = \infty$ is admissible he

 $\begin{aligned} \text{(a)}_e &\triangleq \text{C}_a(m_e(f, \mathcal{Q})) \text{ for } f(i) \text{ } \text{ } Let \text{ } I(Q) < \beta \text{ and } ||f||_{Q,e} \leq 1. \text{ Then } ||f||_{Q,e} \leq C_\beta(m_e(f, Q))^{1/\beta}. \end{aligned}$

Proof: (i) If $||f||_{Q,e} > 1$, then obviously $m_e(f, Q) \geq ||f||_{Q,e}$. Therefore suppose that

 $m_o(f, P) \leq C_e \max \left\{ (m_o(f, Q))^{(i(P)/1(Q)) - \epsilon}, (m_o(f, Q))^{(I(P)/i(Q)) + \epsilon} \right\}.$
 Proposition 4.3: The following two statements are true:

(i) Let $i(Q) > \alpha > 0$ $(i(Q) = \infty$ is admissible here. If $||f||_{Q, \rho} > 1$, then we have
 $|\delta_e \geq C_a |m_o(f, Q)|$ $m_{\varrho}(f, Q) < \infty$. The definition of $i(Q)$ guarantees the existence of $\lambda_0 \in (0, 1)$ such that *m_s*(*f, P*) $\leq C_{\epsilon}$ max $\{||f||_{Q}^{(P)-\epsilon}, ||f||_{Q}^{(P)-\epsilon}\}$.

(*ti*) *If* $Q \in \Delta_2$ globally on $(0, \infty)$, then

min $\{||f||_{P,Q}^{(Q)-\epsilon}, ||f||_{P,Q}^{(Q)+\epsilon}\} \leq C_{\epsilon} m_e(f,Q)$.

(*tii*) *If* $P, Q \in \Delta_2$ globally on $(0, \infty)$, then
 $h_0(\lambda) \leq \lambda^{\alpha}$ for $\lambda \in (0, \lambda_0)$. Set $C_{\alpha} = \lambda_0^{-1}$. We get

$$
\int\limits_{\Omega} Q\left(\frac{|f(x)|}{C_{\mathfrak{a}}(m_{\varrho}(f,Q))^{1/\mathfrak{a}}}\right)\varrho(x)\,dx\leq \lambda_0^{\mathfrak{a}}<1.
$$

(ii) **Under our assumption we have** $Q(\lambda t) \leq \lambda^{\beta} Q(t), t \geq 0, \lambda \geq \lambda_{\beta}$ **, for some** $\lambda_{\beta} > 1$ **.** $\int_{\Omega} Q\left(\frac{|f(x)|}{C_{a}(m_{e}(f,Q))^{1/a}}\right) \varrho(x) dx \leq \lambda_{0}^{a} < 1.$
(*ii*) Under our assumption we have $Q(\lambda t) \leq \lambda^{\beta} Q(t), t \geq 0, \lambda \geq \lambda_{\beta}$, for sor
Therefore

$$
\int_{\Omega} Q\left(\frac{|f(x)|}{C_a(m_e(f,Q))^{1/a}}\right) \varrho(x) dx \leq \lambda_0^{\alpha} < 1.
$$
\n(ii) Under our assumption we have $Q(\lambda t) \leq \lambda^{\beta} Q(t), t \geq 0, \lambda \geq \lambda_{\beta}$, for some $\lambda_{\beta} > 1$.

\nTherefore

\n
$$
\int_{\Omega} Q\left(\lambda_{\beta}^{1-\beta} \frac{|f(x)|}{(m_e(f,Q))^{1/\beta}}\right) \varrho(x) dx \leq \lambda_{\beta}^{-\beta} \int_{\Omega} Q\left(\lambda_{\beta} \frac{|f(x)|}{(m_e(f,Q))^{1/\beta}}\right) \varrho(x) dx
$$
\n
$$
\leq \lambda_{\beta}^{-\beta} \left(\frac{\lambda_{\beta}}{(m_e(f,Q))^{1/\beta}}\right)^{\beta} m_e(f,Q) = 1.
$$
\nIt suffices to choose $C_{\beta} = \lambda_{\beta}^{\beta-1} \cdot \mathbb{I}$

 ~ 200 ks

Remark 4.4: If $\rho \in L_1(\Omega)$, $Q \in \Delta_3$, and $P \in \Delta_2$ globally on $(0, \infty)$, then the necessity of the condition $(\sigma|g) \in L_{N,\rho}$ for the imbedding $L_{Q,\rho} \hookrightarrow L_{P,\sigma}$ can be proved directly without use of the
representation theorem. Actually, easily $QP^{-1} \in \Delta_n$, thus $N \in \Delta_2$. Combined with (1.2) and Pro-Imbeddings between Weighted Orlicz

Remark 4.4: If $\varrho \in L_1(\Omega)$, $Q \in \Delta_s$, and $P \in \Delta_s$ globally on $(0, \infty)$, then the

condition $(\sigma|\varrho) \in L_{N,\varrho}$ for the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\varrho}$ can be proved directly representa

(z)\ C(Q) + *cf PQ1* (*e(X) a(x)dx* ^a *j a a* (f) + C1 maxjIIQ_1(o/e)IIt, !Q1eI!} *^5 C(Q)* + *C.* **(1** *+f QQ(a(x)/Q(x))e(x)dx)Ht* = *C(Q) +* **C7(1 + a(Q))1(i.** *B=fN(a(x)/e(x)) e(x) dx.* we invoke (1.1). Choose r> 0 and a, *fi* in such a manner that *1(P) + r <* **tx** *<1(Q), 1(Q) <fi.* For *^k*natural put ^S

All above quantities are finite; $\rho \in L_1(\Omega)$ was an assumption and the integrability of σ over Ω follows from membership of constant functions in $L_{Q,\rho}(\Omega)$.

Remark 4.5: If $P, Q \in \Delta_2$ and $I(P) < i(Q)$, we can get a quantitative relation for

$$
B=\int N(\sigma(x)/\varrho(x))\,\varrho(x)\;dx.
$$

Remark 4.5: If $P, Q \in \Delta_2$ and $I(P) < i(Q)$, we can get a quantitative relation for
 $B = \int N(\sigma(x)/\rho(x)) \rho(x) dx$.

Indeed, setting $H(t) = P^{-1}(N, (t)/t)$ it is easy to check that $QH(t) \leq N(t)$, $t \geq 0$; to clarify this

we invoke (1.1). $\begin{aligned} \mathfrak{e} &\mathfrak{e} \mathfrak{e} \mathfrak$

$$
\Omega_k = \{x \in \Omega; |x| < k, \sigma(x) < k, \varrho(x) > k^{-1}\}, \quad \Omega_0 = \bigcup_{k} \Omega_k,
$$

and

$$
B_k = \int_{\Omega_k} N(\sigma(x)/\varrho(x)) \varrho(x) dx = \int P H(\chi_{\Omega_k}(x) \sigma(x)/\varrho(x)) \sigma(x) dx.
$$

Applying Proposition 4.2/(i) and Proposition 4.3 we can continue

Applying Proposition 4.2/(i) and Proposition 4.3 we can continue
\n
$$
B_k \leq C_{\epsilon} \max \{|H(\chi_{D_k} \sigma|\varrho)||_{Q,\varrho}^{i(P)-\epsilon}, \|H(\chi_{D_k} \sigma|\varrho)||_{Q,\varrho}^{i(P)+\epsilon}\}\
$$
\n
$$
\leq C_{\epsilon} \max \left\{|G_{\alpha}(m_{\varrho}(\chi_{D_k} \sigma|\varrho))||_{Q,\varrho}^{i(P)-\epsilon}, \|H(\chi_{D_k} \sigma|\varrho)||_{Q,\varrho}^{i(P)+\epsilon}\right\}
$$
\n
$$
\leq C_{\epsilon} \max \left\{C_{\alpha}(m_{\varrho}(\chi_{D_k} \sigma|\varrho, QH))^{(I(P)+\epsilon)/\alpha}, C_{\beta}(m_{\varrho}(\chi_{D_k} \sigma|\varrho, QH))^{(i(P)-\epsilon)/\beta}\right\}
$$
\n
$$
\leq C_{\epsilon} \max \left\{C_{\alpha}B_{k}^{(I(P)+\epsilon)/\alpha}, C_{\beta}B_{k}^{(i(P)-\epsilon)/\beta}\right\}.
$$
\nNow, $B_k < \infty$, both the exponents $(I(P) + \varepsilon)/\alpha$ and $(i(P) - \varepsilon)/\beta$ are smaller than 1, an $|\mathcal{Q}|\mathcal{Q}_0| = 0$. Thus, $B \leq \max \left\{(C_{\epsilon}C_{\alpha})^{a/(a-I(P)-\epsilon)}, (C_{\epsilon}C_{\beta})^{\beta/(\beta-i(P)+\epsilon)}\right\}$. Note that $i(Q) > I(P)$

are smaller than 1, and $D\setminus D_0 = 0$. Thus, $B \leq \max \left\{ (C_{\epsilon}C_{\epsilon})^{a/(a-1(P)-\epsilon)}, (C_{\epsilon}C_{\rho})^{\beta/(b-1(P)+\epsilon)} \right\}$. Note that $i(Q) > I(P)$ implies $i(QP^{-1}) > 1$, hence $\dot{N} \in \Delta_2$ and therefore the constant μ in the argument of N 'can simply be omitted. and
 $B_k = \int_R N(\sigma(x)/\varrho(x)) \varrho(x) dx$

Applying Proposition 4.2/(i) and
 $B_k \leq C_t \max \{ ||H(\chi_{D_k})||$
 $\leq C_t \max \{ C_a (m_\varrho(x)) \}$
 $\leq C_t \max \{ C_a B_t \}$

Now, $B_k < \infty$, both the exponsity
 $|\Omega \backslash \Omega_0| = 0$. Thus, $B \leq \max \{ ($ ((implies $i(QP^{-1}) > 1$ **•** \leftarrow ∞ , $\frac{1}{2}$, ∞ , $\frac{1}{2}$, ∞ , $\frac{1}{2}$, ∞ , $\frac{1}{2}$, $\frac{1}{2$ Now, $B_k < \infty$, both the exponents $(I(P) + \varepsilon)/\alpha$ and $(i(P) - \varepsilon)/\beta$ are smalle $|\Omega|Q_0| = 0$. Thus, $B \le \max \{(C_e C_s)^{4/(e - I(P) - \varepsilon)}, (C_e C_\beta)^{\beta/(\beta - i(P) + \varepsilon)}\}$. Note that implies $i(QP^{-1}) > 1$, hence $N \in \Delta_2$ and therefore the const

Remark 4.6: Let us notice that Theorem 2.6 is partly covered by Theorem 3.9.. On the other hand, the procedure described in Remark 4.4 cannot be applied in that case.

Appendix. This "arithmetic supplement" is added for the sake of completeness.

Proof of Lemma 2.3: We shall show the existence of a constant β (= β_K) such that

endix 4.0: Let us notice that Theorem 2.0 is part, the procedure described in Remark 4.4 cannot be **andix. This "arithmetic supplement" is added for root of Lemma 2.3: We shall show the existence of
$$
\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{i(K-1)}, \quad t \geq 1,
$$**

Appendix. This "arithmetic supplement" is added for the sake of completeness.
\nProof of Lemma 2.3: We shall show the existence of a constant
$$
\beta
$$
 (= β_K) such that\n
$$
\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geq 1,
$$
\nin several steps. As usual, [-] denote the integer part. Let us write\n
$$
\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^j j!} = \sum_{j=2}^{\lfloor 2K \rfloor} \sum_{l=3}^{\infty} \sum_{j=\lfloor (l-1)K \rfloor+1}^{\lfloor lK \rfloor} = S_2 + \sum_{l=3}^{\infty} S_l.
$$

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\nStep 1 (an estimate of
$$
S_2
$$
):
\n
$$
S_3 = \sum_{j=2}^{\lfloor 2K \rfloor} \frac{t^{j/K}}{\beta^j j!} \le \frac{t^2}{\lfloor (2K) + 1 \rfloor!} \frac{\lfloor 2K \rfloor + 1}{\beta^{(2K)}}
$$
\n
$$
\times (1 + \beta[2K] + \beta^2[2K] \left([2K] - 1 \right) + \dots + \beta^{(2K)-2} [2K]! \ 2^{-1}).
$$
\nThe last sum contains $[2K] - 1$ terms and the largest one equals $2^{-1}[2K]! \ \beta^{(2K)-2}$. Thus
\n
$$
S_3 \le \frac{t^2}{\left([2K] + 1 \right)!} \frac{\left([2K] - 1 \right) \left([2K] + 1 \right)!}{2\beta^2} \le \frac{t^2}{\left([2K] + 1 \right)!}
$$
\nfor sufficiently large β .
\nStep 2 (an estimate of S_1 , $l \ge 3$):

$$
S_2 \leq \frac{t^2}{([2K]+1)!} \cdot \frac{([2K]+1)([2K]+1)!}{2\beta^2} \leq \frac{t^2}{([2K]+1)!}
$$

for sufficiently large β .

Step 2 (an estimate of $S_l, l \geq 3$):

$$
S_1 = \sum_{j=2}^{n} \frac{P}{\beta^j j!} \leq \frac{1}{[(2K]+1)!} \frac{[2K]+1}{\beta^{(2K)}}
$$

\n
$$
\times (1 + \beta[2K] + \beta^2[2K] \{(2K] - 1) + \cdots + \beta^{(2K)-2}[2K] \cdot (2^{-1}).
$$

\nThe last sum contains $[2K] - 1$ terms and the largest one equals $2^{-1}[2K] \cdot \beta^{(2K)-2}$. Thus
\n
$$
S_2 \leq \frac{t^2}{((2K)+1)!} \frac{((2K)-1) \cdot ((2K)+1)!}{2\beta^2} \leq \frac{t^4}{((2K)+1)!}
$$

\nfor sufficiently large β .
\nStep 2 (an estimate of S_i , $i \geq 3$):
\n
$$
S_i = \frac{[iK]}{i-1} \cdot \frac{p^{i/K}}{2} \leq \frac{t^i}{[(iK]+1)!} \cdot \frac{[iK]+1}{\beta^{(iK]}}
$$

\n
$$
\times \left(1 + \beta[iK] + \beta^2[iK] \cdot ([iK] - 1) + \cdots + \beta^{(iK)-((i-1)K)-1} \cdot \frac{[iK]!}{((i-1)K]+1)!}\right).
$$

\nNow, there are $[iK] - [(i-1)K] \leq K + 1$ we obtain
\n
$$
S_i \leq \frac{t^i}{((iK]+1)!} \cdot \frac{K+1}{\beta^{(iK)!}} \cdot \frac{((iK)+1)!}{(((i-1)K)+1)!} \cdot \beta^{(iK)+((i-1)K)+1)}
$$

\n t^i $K+1$ \therefore $K+1$ \therefore $K+1$

Now, there are $\left[lK\right] - \left[\left(l-1\right)K\right]$ terms in the last sum and the last is the largest of them. **V -**

$$
\times \left(1 + \beta[lK] + \beta^2[lK] ([lK] - 1) + \cdots + \beta^{l(K)-\{l-1\}K\} - i \frac{[lK]!}{([l-1)K] + 1]!}\right)
$$

\nNow, there are $[lK] - [(l-1)K]$ terms in the last sum and the last is the largest
\nAs $[lK] - [(l-1)K] \leq K + 1$ we obtain
\n
$$
S_i \leq \frac{t^i}{([lK]+1)!} \frac{K+1}{\beta^{l(K)}} \frac{([lK]+1)!}{([(l-1)K]+1)!} \beta^{l(K)-\{l-1\}K\} - i
$$
\n
$$
\leq \frac{t^i}{([lK]+1)!} \frac{K+1}{\beta^{l(K+1)K+1}} (lK+1)^{K+1}
$$
\n
$$
\leq \frac{t^i}{([lK]+1)!} \left(\frac{lK+1)^{1+1/K}(K+1)^{1/K}}{\beta^{l-1}}\right)^K.
$$
\nThe last ratio is smaller than 1 provided β is sufficiently large (uniformly with res
\nagain. Hence, for such β , $S_t \leq t^l/([lK]+1)!$
\nStep 3: Combining the estimates obtained we get for β large enough
\n
$$
\sum_{j=2}^{\infty} \frac{t^{jK}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{([jK]+1)!}, \quad t \geq 1.
$$
\nEasily,
\n
$$
\frac{1}{([jK]+1)!} = \frac{1}{j!(j+1)\cdots((jK]+1)} \leq \frac{1}{j!} \left(\frac{1}{j-1}\right)^{\frac{1}{j(K+1)-1}} \leq \frac{1}{j!} \left(\frac{1}{j-1}\right)^{\frac{1}{j(K+1)-1}}
$$
and, finally,

The last ratio is smaller than 1 provided β is sufficiently large (uniformly with respect to *l*), again. Hence, for such β , $S_l \leq t^l/([lK] + 1)!$ $\sqrt{2}$

Step 3: Combining the estimates obtained we get for β large enough

$$
\sum_{j=2}^{\infty} \frac{t^{j/K}}{\beta^{j}j!} \leq \sum_{j=2}^{\infty} \frac{t^{j}}{(|jK|+1)!}, \quad t \geq 1.
$$

$$
\leq \frac{t^l}{((lKj+1)!} \left(\frac{(lK+1)^{1+1/K}(K+1)^{1/K}}{\beta^{l-1}} \right)^K.
$$
\nThe last ratio is smaller than 1 provided β is sufficiently large (uniformly with respect to *l*),
again. Hence, for such β , $S_l \leq t^l/((lKj+1)!)$
Step 3: Combining the estimates obtained we get for β large enough

$$
\sum_{j=2}^{\infty} \frac{t^{jK}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{((jKj+1)!}, t \geq 1.
$$

Easy,

$$
\frac{1}{((jKj+1)!)} = \frac{1}{j!(j+1)\cdots((jKj+1))} \leq \frac{1}{j!} \left(\frac{1}{j-1}\right)^{ijk+1-j} \leq \frac{1}{j!} \left(\frac{1}{j-1}\right)^{jK-j},
$$
and, finally,

$$
\sum_{j=2}^{\infty} \frac{t^{jK}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{i(K-1)}, t \geq 1,
$$
the desired inequality **I**
REFERENCES

$$
\frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{d\mathbf{r}}{d\mathbf{r}} \right|^2 d\mathbf{r} \, d\mathbf{r}
$$

$$
\sum_{j=2}^{\infty} \frac{t^{j/K}}{t^{\beta}j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)}, \qquad t \geq 1,
$$

inequality \blacksquare

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