Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 2, 123-148

Transmutations and Ascent

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We treat the method of ascent for partial differential equations by means of transmutations. We construct a kernel function for the radial generalized axially symmetric potential theory Dirichlet problem that involves n radial variables. We also construct ascent type formulas for hypergeometric functions of operators and apply one of them to construct a solution of an ill posed Cauchy problem. Generating functions of special multivariable solution sets of partial differential equations are also considered.

Key words: *Transmutation, Laplace transform, ascent, kernel function, hypergeometric operators, quasi inner product, generating functions*

AMS subject classification: Primary 35A22, secondary 35E05, 33A35

§1. Introduction

The basic idea behind the method of ascent in partial differential equations is to construct solutions of problems involving several variables from associated component problems that involve fewer variables. The origins of this approach may be found in the technique of separation of variables. A number of systematic studies on ascent were initiated soon after J. Hadamard's development of the method of descent for the wave and other hyperbolic type problems [19]. In [10], for example, F. J. Bureau used an ascent approach to treat problems related to wave propagation. His technique involved introducing additional variables into solution forms for associated lower dimensional problems and carrying out suitable manipulations on these variables. Investigations over the past twenty years indicate that a very promising approach to ascent can be tied to the construction of solution operators for higher dimensional problems from component solution operators for lower dimensional ones by means of transmutations. This permits bringing in a wide variety of tools such as semigroups and continuous groups, distributions, operator theory– and in the case of complex equations, function theoretic tools such as the Bergman kernel. R. P. Gilbert employed this method to treat boundary value problems in [15, 16]. It was further used by L. R. Bragg and J. W. Dettman in [8] to determine convergence regions for multinomial representations for a class of singular initial value problems in several space variables from a knowledge of the bounds on polynomials associated with corresponding one space variable problems [7] and by L. R. Bragg [3] to construct solutions of "higher dimensional" abstract wave problems. The ascent method has also been called upon in [14, 21, 24].

In order to illustrate some key ideas of the approach of this paper, we summarize some of the notions from [3]. Let X be a Banach space and let $A_i = B_i^2, i = 1, 2$, in which the B_i are generators of bounded continuous groups in X. If $B_1 \cdot B_2 = B_2 \cdot B_1$ and $\varphi \in \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$, then a solution of the problem

$$\frac{d^2W(t)}{dt^2} = (A_1 + A_2)W(t),\ t > 0;\ W(0+) = 0,\ W_t(0+) = \varphi \tag{1.1}$$

can be expressed by means of the transmutation formula

$$W(t) = \Gamma(\frac{3}{2})\mathcal{L}_s^{-1}\left\{s^{-\frac{3}{2}}U_{A_1}(\frac{1}{4s})\left[U_{A_2}(\frac{1}{4s})\varphi\right]\right\}_{s\to t^2} \tag{1.2}$$

in which the $U_{A_i}(t)$ denote the semigroups of operators generated by the A_i and $\mathcal{L}_s^{-1}\{\ \}_{s\to\tau}$ denotes an inverse Laplace transform with s the variable of the transform and τ the variable of inversion. Upon rewriting the right member of (1.2) in the form $\Gamma(\frac{3}{2})\mathcal{L}_S^{-1}\{s^{\frac{3}{2}}\left[(s^{-\frac{3}{2}}U_{A_1}(\frac{1}{4s})\right]\left[s^{-\frac{3}{2}}U_{A_2}(\frac{1}{4s})\varphi\right]\}_{s\to t^2}$ and carrying out the inversion by calling upon the connection between the group of operators generated by the B_i and the corresponding semigroups of operators generated by the A_i, it was shown that the function $W(t)$ could be expressed as a real convolution of wave type solution operators $\mathcal{V}_{B_1}(\sqrt{t})$ and $\mathcal{V}_{B_2}(\sqrt{t})$, corresponding to B_1 and B_2 acting on the data φ. This convolution served to define the cosine of a sum of semigroup generators [17]. Thus, the transmutation in (1.2) transformed the product property for semigroup operators into a real convolution property for continuous groups.

From this description, we can give at least one plausable formulation for constructing ascent formulas that will serve as a model for later developments.

Theorem 1. *Let $S_{A_i}(t), i = 1, 2$, denote solution operators for a pair of well posed abstract problems that involve the semigroup generators A_i. Further, assume that there exists an invertible integral transformation T such that $S_{A_i}(t)\varphi_i = TU_{A_i}(t)\varphi_i$, $i = 1, 2$, in which the $U_{A_i}(t)$ are the semigroups of operators generated by the A_i and $\varphi_i \in \mathcal{D}(A_i)$. If $\varphi \in \mathcal{D}(A_i) \cap \mathcal{D}(A_2)$, then*

$$S_{A_1+A_2}(t)\varphi = T\left[(T^{-1}S_{A_1}(t))(T^{-1}S_{A_2}(t)\varphi)\right].$$

Proof. This follows by the straightforward calculation

$$\begin{aligned} S_{A_1+A_2}(t)\varphi &= T\left[U_{A_1+A_2}(t)\varphi\right] = T\left[U_{A_1}(t)(U_{A_2}(t)\varphi)\right] \\ &= T\left[(T^{-1}S_{A_1}(t))(T^{-1}S_{A_2}(t)\varphi)\right]. \end{aligned}$$

If we further assume that the properties of T^{-1} permit rewriting the last member of this relation in the form $T\left[\tilde{T}\left\{S_{A_1}(t)(S_{A_2}(t)\varphi)\right\}\right]$ where $\tilde{T}$ is some appropriate integral operator, then we have

$$S_{A_1+A_2}(t)\varphi = T\left[\tilde{T}\left\{S_{A_1}(t)(S_{A_2}(t)\varphi)\right\}\right]. \tag{1.3}$$

This gives a decomposition of $S_{A_1+A_2}(t)$ in terms of component operators $S_{A_1}(t)$ and $S_{A_2}(t)$. ◻

The types of assumptions needed to obtain (1.3) are fairly restrictive and it is a rare situation in which one can obtain the precise form (1.3). One can, however, obtain formulas that are slight departures from (1.3) that have the ascent flavor and which are convenient for application to a wide variety of well posed and ill posed problems in partial differential equations. Of particular interest are those cases in which the $S_{A_i}(t)$ is in some restricted class of hypergeometric operators. The operator T can then be taken to be one of the following: the Laplace transform, the inverse Laplace transform, or a convolution [2]. For other classes of solution operators $S_{A_i}(t)$, the Fourier transform or the Stieltjes transform may well serve as possible choices for T (see [11, 12]).

In this paper, we exploit the use of the scheme suggested by (1.3) to construct a number of ascent type formulas and to construct multivariable Green's functions for Dirichlet problems. For the case of ill posed problems in partial differential equations, we make use of a number of results from [4]. The approach suggested by (1.3) also appears to offer a convenient means for developing special function representations of solutions of well posed and ill posed problems. While we will construct some generating functions for some of these functions, we defer the treatment of expansion theories to a later paper.

In Section 2, we introduce some notation and provide evaluations for some complex convolution integrals that will be used for Green's function representations. We also recall some facts about the kernel for the initial value radial heat problem and summarize formulas on transmutations as well as formulas from [4]. In Section 3, we give a complex convolution formula for the Green's function for a standard abstract Dirichlet problem in terms of lower dimensional component Green's functions and in Section 4, we construct a Green's function for a generalized axially symmetric potential theory type Dirichlet problem in which the underlying equation involves a sum of several radial Laplacian operators. The convolution decompositions of these Green's functions will be stated as theorems. In Section 5, we deduce a number of theorems, somewhat analogous to (1.3) when the $S_{A_i}(t)$ is one of the hypergeometric operators ${}_0F_1(\_;\beta;tA_i)$, ${}_1F_0(\alpha;\_;tA_i)$, ${}_0F_2(\_;\beta_1,\beta_2;tA_2)$ and ${}_1F_1(\alpha;\beta;tA_i)$. Finally, in Section 6, we couple the ascent method with complex transformation notions to obtain a solution representation for an ill posed generalization of the wave equation and to construct convolution versions of multivariable generating functions and the functions they generate.

§2. Some Preliminaries

(a) *Notation:* Throughout Sections 3 and 4, we call upon the Laplace transform and its inverse ([13, 22, 26]). If $F(s)$ is the Laplace transform of $f(t)$, we write $F(s) = \mathcal{L}(f(t))_{t\to s}$ and $f(t) = \mathcal{L}^{-1}(F(s))_{s\to t}$. If $F_1(s)$ and $F_2(s)$ are the respective transforms of $f_1(t)$ and $f_2(t)$, then we write

$$f_1(t) * f_2(t) = \mathcal{L}^{-1}(F_1(s)F_2(s))_{s\to t} = \int_0^t f_1(t-\sigma)f_2(\sigma)\,d\sigma. \tag{2.1}$$

To avoid confusion, we use the symbol $\wedge$ to denote the complex convolution and write

$$F_1(s) \wedge F_2(s) = \mathcal{L}(f_1(t)f_2(t))_{t\to s} = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} F_1(s-\zeta)F_2(\zeta)\,d\zeta. \tag{2.2}$$

(b) *Some Laplace Transform Formulas.* Since

$$\mathcal{L}(t^{\alpha-1})_{t\to s} = \Gamma(\alpha)/s^{\alpha},\ \alpha > 0, \text{ and } \mathcal{L}(t^{\beta-1})_{t\to s} = \Gamma(\beta)/s^{\beta},\ \beta > 0,$$

it follows by (2.2) that $\mathcal{L}(t^{\alpha+\beta-2})_{t\to s} = \Gamma(\alpha)\Gamma(\beta)(s^{-\alpha}\wedge s^{-\beta})$ for $\alpha,\beta > 0$ and $\alpha+\beta > 1$. But $\mathcal{L}(t^{\alpha+\beta-2})_{t\to s} = \Gamma(\alpha+\beta-1)/s^{\alpha+\beta-1}$. A comparison of these gives the formula

$$s^{-\alpha}\wedge s^{-\beta} = \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}s^{-(\alpha+\beta-1)}, \alpha,\beta > 0,\ \alpha+\beta > 1. \tag{2.3}$$

By carrying out a calculation similar to the one above, one can show, using formulas 4 and 5 on page 73 of [22] and the duplication formula associated with $\Gamma(2z)$, that if $\alpha > 0$, $\beta > 0$, $\alpha + 2\beta - 1 > 0$, then

$$\begin{aligned} s^{-\alpha}\wedge(s^2-\lambda^2)^{-\beta} &= \frac{\Gamma(\alpha+2\beta-1)}{\Gamma(\alpha)\Gamma(\beta)2^{\alpha-1}}\frac{1}{s^{\alpha+2\beta-1}} \\ &\quad {}_2F_1\left(\frac{\alpha+2\beta-1}{2},\frac{\alpha+2\beta+1}{2};\beta+\frac{1}{2};\frac{\lambda^2}{s^2}\right). \end{aligned} \tag{2.4}$$

From (2.3), one can, with shifts in contours and the use of analyticity, obtain the familiar integral evaluation

$$(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{d\zeta}{(a+\zeta)^{\mu}(b-\zeta)^{v}} = \frac{\Gamma(\mu+v-1)}{\Gamma(\mu)\Gamma(v)(a+b)^{\mu+v-1}} \tag{2.5}$$

if $\mu, v > 0, \mu + v > 1$, $a, b > 0$ with $0 < c < b$.

(c) *The Radial Heat Kernel.* The initial value problem for the radial heat equation is given by

$$(2.6) \qquad u_t(r,t) = \left(D_r^2 + \frac{\mu-1}{r}D_r\right)u(r,t),\ u(r,0+) = \varphi(r), \mu > 1,$$

and its solution is defined by the formula

$$(2.7) \qquad u(r,t) = \int_0^\infty K_\mu(r,\xi,t)\varphi(\xi)\,d\xi$$

in which the heat kernel $K_\mu(r,\xi,t)$ is defined by the formula (see [1]).

$$(2.8) \qquad K_\mu(r,\xi,t) = \frac{1}{2t} r^{1-\frac{\mu}{2}} \xi^{\frac{\mu}{2}} e^{-\frac{(r^2+\xi^2)}{4t}} I_{\frac{\mu}{2}-1}\left(\frac{r\xi}{2t}\right)$$

In this, $I_v(z)$ is a modified Bessel function of index v. If we replace (2.6) by the multivariable heat problem

$$(2.9) \qquad \begin{aligned} U_t(r_1,\ldots,r_n,t) &= \sum_{j=1}^n \Delta_{\mu_j} U(r_1,\ldots,r_n,t),\ t > 0 \\ U(r_1,\ldots,r_n,0+) &= \varphi(r_1,\ldots,r_n) \end{aligned}$$

where $\Delta_{\mu_j} = D_{r_j}^2 + (\frac{\mu_j-1}{r_j})D_{r_j}$, then its solution is given by

$$(2.10) \qquad U(r_1,\ldots,r_n,t) = \int_{\mathbf{E}_n^+} \left(\prod_{j=1}^n K_{\mu_j}(r_j,\xi_j,t)\right)\varphi(\xi_1,\ldots,\xi_n)\,d\xi_1\cdots d\xi_n$$

in which E_n^+ denotes the portion of n-dimensional Euclidean space E_n in which all of the coordinates are non-negative.

(d) *Some Transmutations and Related Formulas.* The formula used in (1.2) expresses the solution of the wave problem (1.1) in terms of semi group operators acting on the data φ. We can express this formula in an alternative form, namely, if $u(t)$ is a bounded solution of the abstract heat problem

$$(2.11) \qquad u_t(t) = Au(t),\ t > 0;\ u(0+) = \varphi,\ \varphi \in \mathcal{D}(A),$$

then the function

$$W(t) = \Gamma(\frac{3}{2})\mathcal{L}^{-1}\left\{s^{-\frac{3}{2}}u(\frac{1}{4s})\right\}_{s\to t^2} \tag{2.12}$$

satisfies the initial value wave problem

$$W_{tt}(t) = AW(t),\ W(0+) = 0,\ W_t(0+) = \varphi. \tag{2.13}$$

The formula for the solution of (2.13) with initial conditions interchanged is (see [5])

$$W(t) = \Gamma(1/2)t\mathcal{L}^{-1}\{s^{-\frac{1}{2}}u(1/4s)\}_{s\to t^2}$$

Similarly, the function

$$E(t) = t^{1-a}\Gamma\left((a+1)/2\right)\mathcal{L}^{-1}\{s^{-(\frac{a+1}{2})}u(1/4s)\}_{s\to t^2}$$

is a solution of the Euler-Poisson-Darboux problem

$$E_{tt}(t) + \frac{a}{t}E_t(t) = AE(t),\ E(0+) = \varphi,\ E_t(0+) = 0$$

where $a \geq 0$ [5].

Finally, the function

$$v(t) = \frac{t^{1-a}}{\Gamma(\frac{1-a}{2})}\int_0^\infty e^{-\sigma t^2}\sigma^{-(\frac{a+1}{2})}u(\frac{1}{4\sigma})\,d\sigma,\ a < 1, \tag{2.14}$$

defines a solution of the Dirichlet problem (see [2])

$$v_{tt}(t) + \frac{a}{t}v_t(t) + Av(t) = 0,\ t > 0;\ v(0+) = \phi \tag{2.15}$$

provided that $A = B^2$ where B generates a bounded group of operators. If A is the standard Laplacian operator $D^2_{x_1} + \cdots + D^2_{x_n}$ and $a = 0$, then (2.15) reduces to the classical Dirichlet problem.

For a number of problems we will consider, the operator A in the equation (2.11) will be replaced by a general polynomial partial differential operator $P(D) = P(D_{x_1}, \ldots, D_{x_n})$.

In this case, the problem (2.11) with $\varphi = \varphi(x_1, \ldots, x_n)$ is generally ill posed. However, it may have a solution if we restrict φ to lie in some suitable class of functions. In [4], this class was selected to be the set $\mathfrak{A}(x)$ of entire functions $\varphi(x) = \varphi(x_1, \ldots, x_n)$ satisfying the condition

$$|\varphi(x)| \leq M \exp\left(\sum_{j=1}^{n} \tau_j |x_j|^{\rho_j}\right) \tag{2.16}$$

in which M is a generic constant and $0 < \rho_j < 1, \tau_j > 0$ for $j = 1, 2, \ldots, n$. Under these conditions, it was shown that we could construct a solution of

$$u_t(x,t) = P(D)u(x,t), u(x,0) = \varphi.$$

Moreover, it followed that

$$|u(x,t)| \leq M^*(\rho,\tau)e^{k|t|} \exp\left(\sum_{j=1}^{n} \tau_j |x_j|^{\rho_j}\right) \tag{2.17}$$

in which $M^*(\rho,\tau)$ is a constant depending on the ρ_j and τ_j and in which K is a positive constant determined by the coefficients in the polynomial operator $P(D)$. This bound on $u(x,t)$ suffices for the existence of the inverse Laplace transform in such formulas as (2.12). On the other hand, this growth is too large for an integral of the form (2.14) to be assigned an evaluation.

Finally, we note that if $f(z)$ is an entire function of z and $\varphi(x) \in \mathfrak{A}(x)$, then we can define the symbolic function $f(tP(D))\varphi(x)$ by the formula

$$f(tP(D))\varphi = \int_0^\infty e^{-\sigma}(f(\underline{t}) \circ u(x,\underline{\sigma}))\, d\sigma \tag{2.18}$$

in which $f(\underline{t}) \circ u(x,\underline{\sigma})$ denotes the quasi inner product $(2\pi)^{-1}\int_0^{2\pi} f(te^{i\theta})u(x,\sigma e^{-i\theta})\, d\theta$. One can establish a growth bound on $|f(tP(D))\varphi(x)|$ similar to the bound on $u(x,t)$ in (2.17).

§3. A Green's Function Convolution Formula

In view of the fact that the solution of the wave problem (1.1) can be expressed as a real convolution of solution operators acting on φ, it is useful to see how ascent is exhibited in connection with the abstract Dirichlet problem

$$(3.1) \qquad v_{tt}(t) + \frac{a}{t} v_t(t) + \sum_{i=1}^{n} A_i v(t) = 0, \ t > 0; \ v(0+) = \varphi$$

where $A_i = B_i^2$ with B_i the generator of a bounded continuous group in the Banach space X and $\varphi \in \bigcap_{i=1}^{n} D(A_i)$. While this formula will offer little specifically for the problem (3.1), it will illustrate a useful means for tackling more complicated problems in which the Green's function is not known.

Now, it is a relatively easy task to show that

$$(3.2) \qquad v(t) = \int_{E_n} K(\xi_1, \ldots, \xi_n, t) \left(\prod_{j=1}^{n} G_{B_i}(\xi_i \right) \varphi \, d\xi_1 \cdots d\xi_n$$

in which the $G_{B_i}(\xi_i)$ are the groups of operators generated by the B_i and in which $K(\xi_1, \ldots, \xi_n, t)$ is the Green's function for the classical version of (3.1), namely

$$(3.3) \qquad \overline{V}_{tt}(x,t) + \frac{a}{t} \overline{V}_t(x,t) + \Delta_n \overline{V}(x,t) = 0, \ t > 0, \ \overline{V}(x,0) = \varphi(x)$$

where $x = (x_1, \ldots, x_n)$ and $\Delta_n = D_{x_1}^2 + \cdots + D_{x_n}^2$.

On the other hand, we have (using the kernel function for the heat equation $u_t(x,t) = \Delta_n u(x,t)$ in (2.14))

$$(3.4) \qquad K(\xi_1, \ldots, \xi_n, t) = \frac{t^{1-a}}{\pi^{n/2} \Gamma(\frac{1-a}{2})} \int_0^\infty e^{-\sigma t^2} \sigma^{\frac{n-a-1}{2}} \left(\prod_{i=1}^{n} e^{-\xi_i^2 \sigma} \right) d\sigma.$$

If we select $s = t^2$ and transform all of the individual factors in the second member of (3.4) ($n + 1$ of them), then we obtain

Theorem 2. *The Green's function* $K(\xi_1,\ldots,\xi_n,t)$ *in* (3.2) *is given by*

$$K(\xi_1,\ldots,\xi_n,t)=\frac{t^{1-a}}{\pi^{n/2}\Gamma(\frac{1-a}{2})}\left(\int_0^\infty e^{-s\sigma}\sigma^{\frac{n-a-1}{2}}\,d\sigma\right)\wedge\left(\int_0^\infty e^{-s\sigma}e^{-\xi_1^2\sigma}\,d\sigma\right)\wedge\cdots$$
$$\wedge\left(\int_0^\infty e^{-s\sigma}e^{-\xi_n^2\sigma}\,d\sigma\right)$$
$$=\frac{\Gamma(\frac{n-a+1}{2})t^{1-a}}{\pi^{n/2}\Gamma(\frac{1-a}{2})}\left(\frac{1}{s^{\frac{n-a+1}{2}}}\wedge\frac{1}{s+\xi_1^2}\wedge\cdots\wedge\frac{1}{s+\xi_n^2}\right),\quad s=t^2.$$

The last member provides the desired complex convolution formula for obtaining the required Green's function by ascent.

One can construct the closed formula for $K(\xi_1,\ldots,\xi_n,t)$ from this theorem. For example, suppose we take $a=0$ and $n=2$. Then the last member in Theorem 2 becomes

$$K(\xi_1,\xi_2,t)=(2\pi)^{-1}t\cdot(s^{-\frac{3}{2}}\wedge(s+\xi_1^2)^{-1}\wedge(s+\xi_2^2)^{-1}). \tag{3.5}$$

But, by (2.3), $(s+\xi_1^2)^{-1}\wedge(s+\xi_2^2)^{-1}=(s+\xi_1^2+\xi_2^2)^{-1}$ and, by (2.5), $s^{-\frac{3}{2}}\wedge(s+\xi_1^2+\xi_2^2)^{-1}=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(s-\zeta)^{-\frac{3}{2}}(\zeta+\xi_1^2+\xi_2^2)^{-1}\,d\zeta=(s+\xi_1^2+\xi_2^2)^{-\frac{3}{2}}$. Upon replacing s by t^2 and inserting this into the last member of (3.5), we obtained the desired classical two space variable kernel $K(\xi_1,\xi_2,t)=t/\left[2\pi(t^2+\xi_1^2+\xi_2^2)^{\frac{3}{2}}\right]$. It is left to the reader to show that the last member of Theorem 2 permits one to obtain the well-known formula

$$K(\xi_1,\ldots,\xi_n,t)=\frac{\Gamma(\frac{n-a+1}{2})}{\pi^{n/2}\Gamma(\frac{1-a}{2})}\frac{t^{1-a}}{[t^2+\sum_{i=1}^n\xi_i^2]^{\frac{n-a+1}{2}}}.$$

§4. The Radial Generalized Axially Symmetric Potential Theory Kernel

We now follow up the construction of the formula (3.5) with the construction of a corresponding Green's function $\mathcal{F}_n(\underline{r},\underline{\xi},t)$ involved in the integral representation of the solution of the generalized radial generalized axially symmetric potential theory Dirichlet problem

$$\begin{cases} W_{tt}(\underline{r},t)+\frac{a}{t}W_t(\underline{r},t)+\sum_{j=1}^n\Delta_{\mu_j}W(r,t)=0,\ t>0 \\ W(\underline{r},0+)=\varphi(\underline{r}) \end{cases} \tag{4.1}$$

where $\underline{r} = (r_1, \ldots, r_n)$ and the Δ_{μ_j} are as in (2.9), $j = 1, 2, \ldots, n$ (see [25] for a general background on this equation). In the work to follow, we assume that $\varphi(\underline{r})$ has compact support. To obtain this kernel, we call upon the transmutation (2.14) as well as the formula (2.8) for the one space variable radial heat kernel. Because of the special Laplace transforming properties of the modified Bessel functions I_v that appear in (2.8), it is useful to first treat the case of (4.1) when $n = 2$. We then use ascent to handle the general case of (4.1).

From [2] and in analogy with (3.4), the solution of (4.1) for $n = 2$ is defined by

$$W(r_1, r_2, t) = \int_{E_2^+} \frac{t^{1-a}}{\Gamma(\frac{1-a}{2})} \varphi(\xi_1, \xi_2) \times \left\{ \int_0^\infty e^{-\sigma t^2} \sigma^{-\frac{(a+1)}{2}} K_{\mu_1}(r_1, \xi_1, \frac{1}{4\sigma}) K_{\mu_2}(r_2, \xi_2, \frac{1}{4\sigma})\, d\sigma \right\} d\xi_1 d\xi_2 \tag{4.2}$$

in which the K_{μ_i} are as in (2.8). Upon inserting the expressions for the K_{μ_i} into this, we find that

$$W(r_1, r_2, t) = \int_{E_2^+} \mathcal{F}_2(r_1, r_2, \xi_1, \xi_2, t) \varphi(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \tag{4.3}$$

where

$$\mathcal{F}_2(r_1, r_2, \xi_1, \xi_2, t) = \frac{4t^{1-a}}{\Gamma(\frac{1-a}{2})} r_1^{1-\frac{\mu_1}{2}} r_2^{1-\frac{\mu_2}{2}} \xi_1^{\frac{\mu_1}{2}} \xi_2^{\frac{\mu_2}{2}} \times \int_0^\infty e^{-s\sigma} \sigma^{\frac{3-a}{2}} I_{\frac{\mu_1}{2}-1}(A\sigma) I_{\frac{\mu_2}{2}-1}(B\sigma)\, d\sigma \tag{4.4}$$

with

$$s = t^2 + r_1^2 + r_2^2 + \xi_1^2 + \xi_2^2,\ A = 2r_1\xi_1, \text{ and } B = 2r_2\xi_2. \tag{4.5}$$

But, for $c + v_1 + v_2 > -1$ (see [22]),

$$\int_0^\infty e^{-s\sigma} \sigma^c I_{v_1}(A\sigma) I_{v_2}(B\sigma)\, d\sigma = \frac{2^c A^{v_1} B^{v_2} \Gamma((c + v_1 + v_2 + 1)/2) \Gamma((c + v_1 + v_2 + 2)/2)}{\pi^{\frac{1}{2}} \Gamma(v_1 + 1) \Gamma(v_2 + 1) s^{c+v_1+v_2+1}}. \tag{4.6}$$

With the identifications $v_i = \mu_i/2 - 1$ and $c = (3-a)/2$, it follows from this and (4.3), after simplification, that

$$W(r_1, r_2, t) = \lambda \int_{E_2^+} \frac{t^{1-a}\xi_1^{\mu_1-1}\xi_2^{\mu_2-1}\varphi(\xi_1,\xi_2)}{[t^2+r_1^2+r_2^2+\xi_1^2+\xi_2^2]^{(1+\mu_1+\mu_2-a)/2}}\, d\xi_1 d\xi_2 \tag{4.7}$$

where

$$\lambda = 2^{(3+\mu_1+\mu_2-a)/2}\frac{\Gamma((1+\mu_1+\mu_2-a)/4)\Gamma((3+\mu_1+\mu_2-a)/4)}{\pi^{\frac{1}{2}}\Gamma(\mu_1/2)\Gamma(\mu_2/2)}. \tag{4.8}$$

By using the duplication formula associated with $\Gamma(2z)$ by selecting $z = (1+\mu_1+\mu_2-a)/4$ [20], the expression for λ can be simplified to

$$\lambda = \frac{2^2\Gamma((1+\mu_1+\mu_2-a)/2)}{\Gamma(\mu_1/2)\Gamma(\mu_2/2)} \tag{4.9}$$

Let us next consider the case of problem (4.1) when $n = 2m$. Corresponding to (4.2), we have

$$W(\underline{r}, t) = \int_{E_{2m}^+} \varphi(\underline{\xi})\mathcal{F}_{2m}(\underline{r}, \underline{\xi}, t)\, d\xi, \quad d\xi = d\xi_1 \cdots d\xi_{2m} \tag{4.10}$$

in which

$$\mathcal{F}_{2m}(\underline{r}, \underline{\xi}, t) = \frac{t^{1-a}}{\Gamma(\frac{1-a}{2})}\int_0^\infty e^{-\sigma t^2}\sigma^{-\frac{(a+1)}{2}}\left(\prod_{j=1}^{2m} K_{\mu_j}(r_j, \xi_j, 1/4\sigma)\right) d\sigma. \tag{4.11}$$

Using the expressions for the K_{μ_j}, this can be written in the form

$$\mathcal{F}_{2m}(\underline{r}, \underline{\xi}, t) = \frac{2^{2m}}{\Gamma(\frac{1-a}{2})}\left(\prod_{j=1}^{2m} r_j^{1-\frac{\mu_j}{2}}\xi^{\frac{\mu_j}{2}}\right) t^{1-a} I^{2m} \tag{4.12}$$

where

$$I^{2m} = \int_0^\infty e^{-s\sigma}\sigma^{2m-\frac{a}{2}-\frac{1}{2}}\left(\prod_{j=1}^{2m} I_{\frac{\mu_j}{2}-1}(2r_j\xi_j\sigma)\right) d\sigma \tag{4.13}$$

with $s = \underline{r}^2 + \underline{\xi}^2 + t^2, \underline{r}^2 = r_1^2 + \cdots + r_{2m}^2$, and $\underline{\xi}^2 = \xi_1^2 + \cdots + \xi_{2m}^2$. The integral I^{2m} in this is simply a Laplace transform of the $m+1$ factors

$$\sigma^{2m-\frac{a}{2}-\frac{1}{2}} \text{ and } I_{\frac{\mu_{2j-1}}{2}-1}(2r_{2j-1}\xi_{2j-1}\sigma)I_{\frac{\mu_{2j}}{2}-1}(2r_{2j}\xi_{2j}\sigma),\ j = 1, \ldots, m.$$

Since $(\mu_{2j-1}+\mu_{2j})/2 > 1$, we find that

$$I^{2m} = \left(\frac{\Gamma(2m-\frac{a}{2}+\frac{1}{2})}{s^{(2m-\frac{a}{2}+\frac{1}{2})}}\right) \wedge \left(\prod_{j=1}^{m} \wedge \int_0^\infty e^{-s\sigma^2} I_{\frac{\mu_{2j-1}}{2}-1}(2r_{2j-1}\xi_{2j-1}\sigma) I_{\frac{\mu_{2j}}{2}-1}(2r_{2j}\xi_{2j}\sigma)\, d\sigma\right). \tag{4.14}$$

If we make use of the evaluation (4.6) for each of the transforms appearing under the product sign in (4.14) and use (4.14) and (4.12) in (4.11), we obtain the required ascent relation as stated in

Theorem 4.1. *The kernel function $\mathcal{F}_{2m}(\underline{r},\underline{\xi},t)$ is given by*

$$\begin{aligned}\mathcal{F}_{2m}(\underline{r},\underline{\xi},t) &= 2^{(\sum_{j=1}^{2m}\mu_j)/2} t^{1-a} \left(\prod_{j=1}^{2m} \xi_j^{\mu_j-1}\right) \lambda^* \\ &\times s^{-(2m+\frac{1-a}{2})} \wedge \left(\prod_{j=1}^{m} \wedge s^{-(\mu_{2j-1}+\mu_{2j}-2)/2}\right)\end{aligned} \tag{4.15}$$

in which

$$\lambda^* = \frac{\Gamma(2m+\frac{1-a}{2})}{\pi^{m/2}\Gamma(\frac{1-a}{2})} \prod_{j=1}^{m} \left(\frac{\Gamma((\mu_{2j-1}+\mu_{2j}-2)/4))\Gamma((\mu_{2j-1}+\mu_{2j})/4))}{\prod_{j=1}^{2m}\Gamma(\frac{\mu_j}{2})}\right)$$

with s replaced by $\underline{r}^2+\underline{\xi}^2+t^2$ after carrying out the convolution where $\underline{r}^2 = r_1^2+\cdots+r_{2m}^2$, $\underline{\xi}^2 = \xi_1^2+\cdots+\xi_{2m}^2$.

Applying the duplication formula for the products of the gamma functions in the numerator of the expression for λ^*, inserting this evaluation in (4.15) and evaluating the convolutions in (4.15) via (2.3), we finally obtain, after replacing s by $t^2+\underline{r}^2+\underline{\xi}^2$:

$$\begin{aligned}\mathcal{F}_{2m}(\underline{r},\underline{\xi},t) &= \left\{\lambda^{**} t^{1-a}\left(\prod_{j=1}^{2m}\xi_j^{\mu_j-1}\right)\right\} (t^2+\underline{r}^2+\underline{\xi}^2)^{-(\sum_{j=1}^{2m}\frac{\mu_j}{2}+\frac{1-a}{2})}, \\ \lambda^{**} &= \left(2^{2m}\Gamma\left(\left(\sum_{j=1}^{2m}\frac{\mu_j}{2}\right)+\frac{1-a}{2}\right)\right) \Big/ \left[\Gamma(\frac{1-a}{2})\prod_{j=1}^{2m}\Gamma(\frac{\mu_j}{2})\right].\end{aligned} \tag{4.16}$$

Finally, we consider the case of problem (4.1) when $n = 2m+1$. Following the development for the case $n = 2m$, we have

$$\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t) = \frac{2^{2m+1}t^{1-a}}{\Gamma(\frac{1-a}{2})}\left(\prod_{j=1}^{2m+1} r_j^{1-\mu_j/2}\xi_j^{\mu_j/2}\right)$$

$$\times \int_0^\infty e^{-s\sigma}\sigma^{2m+1-\frac{a}{2}-\frac{1}{2}}\left(\prod_{j=1}^{2m+1} I_{\frac{\mu_j}{2}-1}(2r_j\xi_j\sigma)\right)d\sigma$$

with $s = t^2+\underline{r}^2+\underline{\xi}^2, \underline{r}^2 = r_1^2+\cdots+r_{2m+1}^2$, and $\underline{\xi}^2 = \xi_1^2+\cdots+\xi_{2m+1}^2$. By regarding s as a transforming variable in this, we can express that relation as

Theorem 4.2. *The kernel function $\mathcal{F}_{2m+1}(\underline{r},\underline{\xi}t)$ is given by*

$$\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t) = \frac{2^{2m+1}t^{1-a}}{\Gamma(\frac{1-a}{2})}\left(\prod_{j=1}^{2m+1} r_j^{1-\mu_j/2}\xi_j^{\mu_j/2}\right)$$

$$\times\left\{\int_0^\infty e^{-s\sigma}\sigma^{\frac{\mu_{2m+1}}{2}-1}I_{\frac{\mu_{2m+1}}{2}-1}(2r_{2m+1}\xi_{2m+1}\sigma)\,d\sigma\right\}$$

$$\wedge\left\{\int_0^\infty e^{-s\sigma}\sigma^{2m+1-\frac{a}{2}-\frac{1}{2}-\frac{\mu_{2m+1}}{2}+1}\left(\prod_{j=1}^{2m} I_{\frac{\mu_j}{2}-1}(2r_j\xi_j\sigma)\right)d\sigma\right\}.$$

We must restrict μ_{2m+1} so that $2m+\frac{3}{2}-\frac{a}{2}-\mu_{\frac{2m+1}{2}} > -1$ or $\mu_{2m+1} < 4m-(a-5)$.

In order to make use of the formula (4.16) in the second bracketed term in the right member of Theorem 4.2, we let $a_1 = a+\mu_{2m+1}-4$. Then with some rearranging of terms, $\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t)$ can be written as

$$\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t) = \left\{\frac{t^{1-a_1}}{\Gamma(\frac{1-a_1}{2})}2^{2m}\left(\prod_{j=1}^{2m} r_j^{1-\frac{\mu_j}{2}}\xi_j^{\mu_j/2}\right)\right.$$

$$(4.17)\qquad \left.\times\int_0^\infty e^{-s\sigma}\sigma^{2m-\frac{a_1}{2}-\frac{1}{2}}\left(\prod_{j=1}^{2m} I_{\frac{\mu_j}{2}-1}(2r_j\xi_j\sigma)\right)d\sigma\right\}$$

$$\wedge\left\{\frac{\Gamma(\frac{1-a_1}{2})}{\Gamma(\frac{1-a}{2})}2t^{4-\mu_{2m+1}}r_{2m+1}^{1-\frac{\mu_{2m+1}}{2}}\xi_{2m+1}^{\frac{\mu_{2m+1}}{2}}\right.$$

$$\left.\times\int_0^\infty e^{-s\sigma}\sigma^{\frac{\mu_{2m+1}}{2}-1}I_{\frac{\mu_{2m+1}}{2}-1}(2r_{2m+1}\xi_{2m+1}\sigma)\,d\sigma\right\}.$$

Now, the first bracketed term in this, by (4.16), has the evaluation

$$\frac{2^{2m}\Gamma(\alpha_1)}{\Gamma(\frac{1-a_1}{2})\prod_{j=1}^{2m}\Gamma(\frac{\mu_j}{2})}\frac{t^{1-a_1}(\prod_{j=1}^{2m}\xi_j^{\mu_j-1})}{s^{\alpha_1}} \tag{4.18}$$

where $\alpha_1=\sum_{j=1}^{2m}(\frac{\mu_j}{2})+\frac{1-a_1}{2}$. The second bracketed term in the right member of (4.17) can be computed by using formula 5 on page 73 of [22] and has the evaluation

$$\frac{\Gamma(\frac{1-a_1}{2})}{\Gamma(\frac{1-a}{2})}\frac{2^{\mu_{2m+1}-1}}{\pi^{\frac{1}{2}}}\frac{t^{4-\mu_{2m+1}}\Gamma(\beta_1)\xi_{2m+1}^{\mu_{2m+1}-1}}{\left[s^2-4r_{2m+1}^2\xi_{2m+1}^2\right]^{\beta_1}}$$

with $\beta_1=(\mu_{2m+1}-1)/2$. Introducing this and (4.18) back into (4.17) and simplifying, we find that

$$\begin{aligned}\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t)&=\frac{2^{2m+\mu_{2m+1}-1}}{\Gamma(\frac{1-a}{2})}\frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\pi^{\frac{1}{2}}(\prod_{j=1}^{2m}\Gamma(\frac{\mu_j}{2}))}t^{1-a}\\&\times\prod_{j=1}^{2m+1}\xi_j^{\mu_j-1}\times\frac{1}{s^{\alpha_1}}\wedge\frac{1}{(s^2-4r_{2m+1}^2\xi_{2m+1}^2)^{\beta_1}}.\end{aligned} \tag{4.19}$$

These calculations are valid provided $a\neq 2k+5-\mu_{2m+1}, k=0,1,2,\ldots$ The complex convolution in (4.19) can now be carried out by using the formula (2.4). If we do this, replace s by $t^2+\underline{r}^2+\underline{\xi}^2$, and simplify, we get

$$\begin{aligned}\mathcal{F}_{2m+1}(\underline{r},\underline{\xi},t)&=\tilde{\tilde{\lambda}}t^{1-a}\left(\prod_{j=1}^{2m+1}\xi_j^{\mu_j-1}\right)\times\frac{1}{\left[t^2+\underline{r}^2+\underline{\xi}^2\right]^{\alpha_1+2\beta_1-1}}\\&\times{}_2F_1\left(\frac{\alpha_1+2\beta_1-1}{2},\frac{\alpha_1+2\beta_1+1}{2};\beta_1+\frac{1}{2};\frac{4r_{2m+1}^2\xi_{2m+1}^2}{\left[t^2+\underline{r}^2+\underline{\xi}^2\right]^2}\right)\end{aligned}$$

in which

$$\tilde{\tilde{\lambda}}=\frac{2^{2m+\mu_{2m+1}+\alpha_1}\,\Gamma(\alpha_1+2\beta_1-1)}{\pi^{\frac{1}{2}}\Gamma(\frac{1-a}{2})(\prod_{j=1}^{2m}\Gamma(\frac{\mu_j}{2}))}.$$

Note that we employed ascent to compute $\mathcal{F}_{2m+1}$ by complex convolution from a knowledge of the form for $\mathcal{F}_{2m}$. Thus, we have a reversal of the descent procedure of Hadamard who first solved the wave problem in $2m+1$ space variables and then obtained the solution of the corresponding problem in $2m$ space variables by projecting out the additional space variable.

§5. Some Hypergeometric Ascent Formulas

The transmutation formulas given in Section 2 and employed in Sections 3 and 4 are basically disguised versions of hypergeometric type operators [2]. From these formulas, we were able to develop ascent relations for the initial value wave problem and Dirichlet Green's functions. Hypergeometric solution operators appear rather regularly in connection with well posed and ill posed initial value problems. As a further example of this, let us observe that the solution of the ill posed problem

$$D_t^3 Z(x,t) = P_1(D_x)Z(x,t), Z(x,0) = \varphi(x), Z_t(x,0) = Z_{tt}(x,0) = 0,$$

in which $P_1(D_x)$ is a polynomial in the derivative operator D_x, can be symbolically expressed by the formula

$$Z(x,t) = {}_0F_2\left(\text{—}; \frac{1}{3}, \frac{2}{3}; \frac{1}{27}t^3 P_1(D_x)\right) \cdot \varphi(x).$$

If $\varphi(x)$ is entire in x of growth ρ, $0 < \rho < 1$, and type $\tau > 0$, then the right member of this can be assigned a meaning by employing the results of [4]. If the polynomial operator $P_1(D_x)$ were to be replaced in this problem by $P_1(D_x) + P_2(D_y)$ ($P_2(D_y)$ a polynomial in D_y), $\varphi(x)$ were to be replaced by a data function $\psi(x,y)$, and if $Z(x,t)$ were to be replaced by $\tilde{Z}(x,y,t)$, then the formula for $\tilde{Z}(x,t)$ could be replaced by

$$\tilde{Z}(x,y,t) = {}_0F_2\left(\text{—}; \frac{1}{3}, \frac{2}{3}; \frac{1}{27}t^3(P_1(D_x) + P_2(D_y))\right) \cdot \psi(x,y). \tag{5.1}$$

The question then arises: can we replace the formal solution operator in (5.1) by some ascent formula of a type similar to the once suggested by (1.3)? If yes, then we can employ the techniques of [4] to assign a meaning to (5.1). In this section, we exhibit ascent formulas explicitly for operators involving the hypergeometric functions ${}_0F_1$, ${}_1F_0$, ${}_0F_2$ and ${}_1F_1$. Some of these will have a limited applicability because of restrictions imposed on the data function by the particular form of the resulting ascent formula. We return to (5.1) in Section 6.

(i) *The ${}_0F_1$ Operator.* The canonical problem that leads to this operator is given by

$$[tD_t(tD_t+\beta-1)-tA_1]\,u(t)=0,\;\; u(0+)=\varphi,\;\; \beta>0, \tag{5.2}$$

in which A_1 is a semigroup generator in the Banach space X and $\varphi\in\mathcal{D}(A_1)$. The solution of this can be given in the symbolic form $u(t)={}_0F_1(\_\_;\beta;tA_1)\varphi$ and this can be expressed as

$${}_0F_1(\_\_;\beta;tA_1)\varphi=t^{1-\beta}\Gamma(\beta)\mathcal{L}^{-1}\{s^{-\beta}U_{A_1}(1/s)\varphi\}_{s\to t}.$$

We give two types of ascent formulas for this ${}_0F_1$ operator, a simple version and a more complicated one that involves derivatives. The form of the resulting formula will be determined by the selection of the parameters β_j (see below).

For the simpler version, we write

$$\begin{aligned}&{}_0F_1(\_\_;\beta;t(A_1+A_2))\varphi\\ &\quad=t^{1-\beta}\mathcal{L}^{-1}\{s^{-\beta}U_{A_1+A_2}(1/s)\varphi\}_{s\to t}\\ &\quad=\Gamma(\beta)t^{1-\beta}\mathcal{L}^{-1}\{(s^{-\beta_1}U_{A_1}(1/s))(s^{-\beta_2}U_{A_2}(1/s)\varphi)\}_{s\to t}\end{aligned} \tag{5.3}$$

in which $\varphi\in\mathcal{D}(A_1)\cap\mathcal{D}(A_2)$ and where β_1 and β_2 are selected to be positive with $\beta_1+\beta_2=\beta$. From the real convolution theorem for the Laplace transform, we deduce

Theorem 5.1 *An operator addition formula for ${}_0F_1(\_\_;\beta;t(A_1+A_2))\varphi$ is given by*

$$\begin{aligned}&{}_0F_1(\_\_;\beta;t(A_1+A_2))\varphi\\ &=t^{1-\beta}\Gamma(\beta)\left[\left(\frac{t^{\beta_1-1}}{\Gamma(\beta_1)}\,{}_0F_1(\_\_;\beta_1;tA_1)\right)*\left(\frac{t^{\beta_2-1}}{\Gamma(\beta_2)}\,{}_0F_1(\_\_;\beta_2;tA_2)\varphi\right)\right]\\ &=\frac{\Gamma(\beta)t^{1-\beta}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^t(t-\xi)^{\beta_1-1}\xi^{\beta_2-1}\left[{}_0F_1(\_\_;\beta_1;(t-\xi)A_1)\{{}_0F_1(\_\_;\beta_2;\xi A_2)\varphi\}\right]d\xi\end{aligned}$$

We note that neither of the parameters β_1 or β_2 in this can be assigned the value β. Then strictly speaking, the formula in Theorem 5.1 is not of the form (1.3) but is a slight departure from it. Nevertheless, it is of the ascent type in the sense that the operator ${}_0F_1(\_\_;\beta_2;\xi A_2)$ acts on φ prior to applying the operator ${}_0F_1(\_\_;\beta_1;(t-\xi)A_1)$.

We can select the parameters β_1 and β_2 to be the same and equal β by rewriting (5.3) in the form

$$\begin{aligned} &{}_0F_1(\_\_;\beta;t(A_1+A_2))\varphi \\ &= \Gamma(\beta)t^{1-\beta}\mathcal{L}^{-1}\{s^{\beta}(s^{-\beta}U_{A_1}(1/s))(s^{-\beta}U_{A_2}(1/s)\varphi)\}_{s\to t} \\ (5.4)\qquad &= \frac{t^{1-\beta}}{\Gamma(\beta)}\Big\{\mathcal{L}_s^{-1}(s^{\beta})_{s\to t} * \int_0^t (t-\xi)^{\beta-1}\xi^{\beta-1} \\ &\times [{}_0F_1(\_\_;\beta;(t-\xi)A_1)\{{}_0F_1(\_\_;\beta;\xi A_2)\varphi\}]\Big\}\,d\xi \end{aligned}$$

If β in this is a positive integer, then $\mathcal{L}_s^{-1}\{s^{\beta}\}_{s\to t} = \delta^{(\beta)}(t)$. On the other hand, if $\beta = n-\nu$ with n a positive integer and $0<v<1$, then $\mathcal{L}_s^{-1}\{s^{\beta}\}_{s\to t} = (\Gamma(\nu))^{-1}\delta^{(n)}(t) * t^{v-1}$.

Although Theorem 5.1 and the formula (5.4) were obtained by using semigroup notions, the right members of these can be assigned a meaning if A_1 and A_2 are replaced by a variety of differential operators and φ is replaced by some appropriate entire function.

(ii) *The ${}_1F_0$ Operator.* The solution operator ${}_1F_0(\alpha;\_\_;tA_1)$, $\alpha>0$, arises in connection with the canonical problem

$$[tD_t - tA_1(tD_t+\alpha)]\,u(t) = 0,\ t>0;\ u(0+)=\varphi. \tag{5.5}$$

This has the solution

$$\begin{aligned} u(t) = {}_1F_0(\alpha;\_\_;tA_1)\varphi &= (\Gamma(\alpha))^{-1}\int_0^{\infty} e^{-\sigma}\sigma^{\alpha-1}\,[U_{A_1}(\sigma t)\varphi]\,d\sigma \\ &= (\Gamma(\alpha))^{-1}t^{-\alpha}\int_0^{\infty} e^{-\sigma/t}\,\sigma^{\alpha-1}\,[U_{A_1}(\sigma)\varphi]\,d\sigma. \end{aligned} \tag{5.6}$$

Suppose we select $\alpha_1>0, \alpha_2>0$ such that $\alpha-1=(\alpha_1-1)+(\alpha_2-1)$ or $\alpha_1+\alpha_2=\alpha+1$. Then the ascent relation for this $u(t)$ can be stated as

Theorem 5.2 *An operator addition relation for* ${}_1F_0(\alpha;\_\_;(A_1+A_2)t)\varphi$ *is given by the formula*

$$\begin{aligned} u(t) &= {}_1F_0(\alpha;\_\_;(A_1+A_2)t)\varphi \\ &= \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-\sigma/t}\left\{(\sigma^{\alpha_1-1}U_{A_1}(\sigma))(\sigma^{\alpha_2-1}U_{A_2}(\sigma)\varphi)\right\}\,d\sigma \\ &= \frac{t^{\alpha_2-\alpha_1}\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-\alpha_1}(1-t\xi)^{-\alpha_2} \\ &\times\ {}_1F_0\left(\alpha_1;\_\_;\xi^{-1}A_1\right)\left({}_0F_1\left(\alpha_2;\_\_;\frac{tA_2}{1-t\xi}\right)\varphi\right)\,d\xi. \end{aligned}$$

Next, suppose that $0<\alpha<2$ and take $s=1/t$. From (5.6),

$$\int_0^\infty e^{-\sigma s}\sigma^{\alpha-1}U_{A_1}(\sigma)\varphi\,d\sigma = \frac{\Gamma(\alpha)}{s^\alpha}\ {}_1F_0\ (\alpha;\_\_;\frac{1}{s}A_1)\varphi.$$

From this, we obtain

Theorem 5.3. *An alternative operator addition relation for* ${}_1F_0(\alpha;\_\_;(A_1+A_2)t)\varphi$ *is given by the formula*

$$\begin{aligned} u(t) &= {}_1F_0(\alpha;\_\_;(A_1+A_2)t)\varphi \\ &= \frac{s^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-s\sigma}\left[\sigma^{1-\alpha}(\sigma^{\alpha-1}U_{A_1}(\sigma))(\sigma^{\alpha-1}U_{A_2}(\sigma)\varphi)\right]\,d\sigma\bigg|_{s=1/t} \\ &= \Gamma(\alpha)\Gamma(2-\alpha)t^{-\alpha}\left\{\frac{1}{s^{2-\alpha}}\wedge(s^{-\alpha}\ {}_1F_0(\alpha;\_\_;\frac{1}{s}A_1))\right. \\ &\qquad \left.\wedge(s^{-\alpha}\ {}_1F_0(\alpha;\_\_;\frac{1}{s}A_2)\varphi)\right\}\bigg|_{s=1/t}. \end{aligned}$$

This is simply a version of the ascent relation for the operator ${}_1F_0(\alpha;\_\_;At)$ *in which both of the parameters* α_1 *and* α_2 *are equal to* α.

(iii) *Other Hypergeometric Formulas.* Ascent type formulas for the ${}_0F_n$ type hypergeometric operators can be constructed successively from the corresponding ascent formulas for the ${}_0F_{n-1}$ operators by calling upon the classical formula

$$(5.7)\qquad {}_0F_n(\_\_;\gamma_1,\ldots,\gamma_n;at) = \Gamma(\gamma_n)\mathcal{L}^{-1}\left\{s^{-\gamma_n}\ {}_0F_{n-1}(\_\_;\gamma_1,\ldots,\gamma_{n-1};\frac{a}{s})\right\}_{s\to t}$$

for the $\gamma_i > 0$ and $a > 0$. Let us specifically consider the case when $n = 2$ with $\gamma_1 = \beta$ and $\gamma_2 = \delta$, $\beta > 0$, $\delta > 0$.

If a and b are positive constants, then the formula in Theorem 5.1 can be expressed as a classical relation for hypergeometric functions, namely

$$\begin{aligned}&{}_0F_1(\_\_;\beta;t(a+b))\\&=\frac{\Gamma(\beta)t^{1-\beta}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^t(t-\xi)^{\beta_1-1}\xi^{\beta_2-1}\ {}_0F_1(\_\_;\beta_1;(t-\xi)a){}_0F_1(\_\_;\beta_2;\xi b)\,d\xi.\end{aligned}$$

Suppose we insert this, with appropriate changes of variables, in the right member of (5.7) with $\gamma_1 = \beta$, $\gamma_2 = \delta$. Let $\delta_1, \delta_2 > 0$ such that $\delta_1 + \delta_2 = \delta$. By carrying out the required inversion in the right member of (5.7) by a procedure similar to the one used with (5.6) and Theorem 5.1, we can establish

Theorem 5.4 *An alternative operator addition relation for* ${}_1F_0(\alpha;\_\_;\beta,\delta;(a+b)t)$ *is given by*

$$\begin{aligned}&{}_0F_2(\_\_;\beta,\delta;(a+b)t)\\&=\frac{\Gamma(\beta)\Gamma(\delta)t^{1-\delta}}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\delta_1)\Gamma(\delta_2)}\int_0^1(1-\lambda)^{\beta_1-1}\lambda^{\beta_2-1}\\&\times\left\{\int_0^t(t-\eta)^{\delta_1-1}\eta^{\delta_2-1}\ {}_0F_2(\_\_;\beta_1,\delta_1;a(1-\lambda)(t-\eta)){}_0F_2(\_\_;\beta_2,\delta_2;b\lambda\eta)\,d\eta\right\}d\lambda\end{aligned}$$

This can, of course, be expressed in an operator form by replacing the constants a and b by appropriate differential operators and operating on suitable data functions (entire). We will call upon this resulting operator formula in Section 6.

With a somewhat more tedious calculation, we can also establish, for appropriate operators A and B and data φ, that for $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = \alpha + 1$ and $\beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = \beta$, we have

$$\begin{aligned}{}_1F_1(\alpha;\beta;t(A+\beta))\varphi&=\frac{\Gamma(\beta)}{\Gamma(\beta_1)\Gamma(\beta_2)}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha)}t^{\frac{1-\alpha}{2}}\int_0^1(1-\lambda)^{\beta_1-1}\lambda^{\beta_2-1}\\&\times\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(1-t\zeta)^{-\alpha_1}\zeta^{-\alpha_2}\ {}_1F_1\left(\alpha_1;\beta_1;\frac{(1-\lambda)tA}{1-t\zeta}\right)\right.\\&\times\left.\left\{{}_1F_1\left(\alpha_2;\beta_2;\frac{\lambda\beta}{\zeta}\right)\varphi\right\}d\zeta\right\}d\lambda.\end{aligned}\tag{5.8}$$

§6. Some Further Problems

In this final section, we provide some applications of ascent to ill posed problems by employing the hypergeometric ascent theorems along with the formula (2.18). We show, for example, how we can assign a meaning to the $\tilde{Z}(x,y,t)$ given formally in (5.1) and then consider some applications to generating functions of special polynomials and other special functions. Throughout, we make use of the class $\mathcal{O}\!\mathcal{L}$ of entire functions and the existence of solutions of generalized heat problems of the form $u_t(x,t) = P(D_x)u(x,t)$, $u(x,0) \in \mathcal{O}\!\mathcal{L}(x)$, in which $P(D_x)$ is a polynomial in the derivative operator D_x.

(i) *The Function* $\tilde{Z}(x,y,t)$. Symbolically, the formula (5.1) defines a function $\tilde{Z}(x,y,t)$ which satisfies the ill posed initial value problem

$$\text{(6.1)} \qquad \begin{cases} D_t^3\tilde{Z}(x,y,t) = (P_1(D_x)+P_2(D_y))\tilde{Z}(x,y,t), \;\; \tilde{Z}(x,y,0) = \psi(x,y) \\ \\ \tilde{Z}_t(x,y,0) = \tilde{Z}_{tt}(x,y,0) = 0, \quad \psi(x,y) \in \mathcal{O}\!\mathcal{L}(x,y). \end{cases}$$

With the change of variables $\tau = \frac{1}{27}t^3$ in (5.1), the operator obtained from Theorem 5.4, by replacing a by $P_1(D_x)$, b by $P_2(D_y)$, and t by τ, when acting on $\psi(x,y)$ leads to the relation

$$\begin{aligned} \tilde{Z}(x,y,3\tau^{\frac{1}{3}}) &= \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\beta_1)\Gamma(\frac{1}{3}-\beta_1)\Gamma(\delta_1)\Gamma(\frac{2}{3}-\delta_1)}\int_0^1 (1-\lambda)^{\beta_1-1}\lambda^{-\beta_1-\frac{2}{3}} \\ \text{(6.2)} \qquad &\times\left\{\int_0^\tau (\tau-\eta)^{\delta_1-1}\eta^{-\frac{1}{3}-\delta_1}\; {}_0F_2(\_\_;\beta_1,\delta_1;P_1(D_x)(1-\lambda)(\tau-\eta))\right. \\ &\times\left.\left[{}_0F_2\left(\_\_;\frac{1}{3}-\beta_1,\frac{2}{3}-\delta_1;\lambda\eta P_2(D_y)\right)\psi(x,y)\right]d\eta\right\}d\lambda \end{aligned}$$

in which $0<\beta_1<\frac{1}{3}, 0<\delta_1<\frac{2}{3}$.

Using the results of [4], let $\tilde{u}(x,y,\tau)$ satisfy the equation $\tilde{u}_\tau(x,y,\tau) = P_2(D_y)\tilde{u}(x,y,\tau)$, $\tilde{u}(x,y,0) = \psi(x,y)$. The variable x is carried along as a parameter in this. Then from this $\tilde{u}(x,y,\tau)$, we can assign a meaning to the bracketed term in (6.2) by calling upon the formula (2.18). Suppose we denote the function thus defined by $\tilde{\psi}(x,y,\lambda\eta)$. Then we have

$$\text{(6.3)} \qquad \tilde{\psi}(x,y,\lambda\eta) = \int_0^\infty e^{-\sigma}\left({}_0F_2\left(\_\_;\frac{1}{3}-\beta_1,\frac{2}{3}-\delta_1;\underline{\lambda\eta}\right)\circ\tilde{u}(x,y,\underline{\sigma})\right)d\sigma.$$

We can apply the operator ${}_0F_2(\_\_;\beta_1,\delta_1;(1-\lambda)(\tau-\eta)P_1(D_x))$ to $\tilde{\psi}(x,y,t)$ in the same way, namely let $\tilde{\tilde{u}}(x,y,\tau)$ denote a solution of $\tilde{\tilde{u}}_\tau(x,y,\tau)=P_1(D_{x_1})\tilde{\tilde{u}}(x,y,\tau)$, $\tilde{\tilde{u}}(x,y,0)=\tilde{\psi}(x,y,\lambda\eta)$ (here, y is carried along as a parameter). Then let $\hat{\psi}(x,y,\lambda,\eta,\tau)$ denote the value of ${}_0F_2(\_\_;\beta_1,\delta_1;P_1(D_x)(1-\lambda)(\tau-\eta))\tilde{\psi}(x,y,\lambda\eta)$. Again, by (2.18) we get

$$\hat{\psi}(x,y,\lambda,\eta,\tau)=\int_0^\infty e^{-\sigma}({}_0F_2(\_\_;\beta_1,\delta_1;\underline{(1-\lambda)(\tau-\eta)})\circ\tilde{\tilde{u}}(x,y,\underline{\sigma}))\,d\sigma \tag{6.4}$$

The entireness properties of the ${}_0F_2$ hypergeometric functions and the data guarantee that the improper integrals defining $\tilde{\psi}$ and $\hat{\psi}$ exist. Finally, we get

$$\tilde{Z}(x,y,3\tau^{\frac{1}{3}})=\frac{\Gamma(+\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\beta_1)\Gamma(\frac{1}{3}-\beta_1)\Gamma(\delta_1)\Gamma(\frac{2}{3}-\delta_1)}\int_0^1(1-\lambda)^{\beta_1-1}\,\lambda^{-\beta_1-\frac{2}{3}}$$
$$\times\left(\int_0^\tau(\tau-\eta)^{\delta_1-1}\eta^{-\frac{1}{3}-\delta_1}\hat{\psi}(x,y,\lambda,\eta,\tau)\,d\eta\right)d\lambda.$$

Upon replacing τ in this by $t^3/27$, we obtain the solution of (6.1).

(ii) ***Multivariable Generating Functions.*** Special functions and polynomial sets play a central role in the various representation theories of partial differential equations. This is clearly the case for the separation of variables technique. However, it shows up in other ways such as in the study of heat solution representations ([1, 23]), the treatment of various hyperbolic and elliptic problems [7], and in the development of function theories for partial differential equations. In most of the cases considered, the entering special functions or polynomials involve two variables. It would be useful to have available multivariable extensions of these to treat higher dimensional representations and function theories. In the following, we construct two generating functions, in convolution form, that define special solution sets for evolution type partial differential equations that involve several space variables. The first of these is tied to a generalization of the wave equation while the second is connected with the Laplace equation. The terms entering into these convolution forms will themselves be generating functions associated with lower dimensional problems. Applications to representations of solutions will be deferred to a later paper.

(A) *A Generalized "Wave Equation"*. Suppose we consider the problem

$$w_{tt}(\underline{x},t) = (D_{x_1}^{p_1} + \cdots + D_{x_n}^{p_n})w(\underline{x},t),\ w(\underline{x},0) = 0,\ w_t(\underline{x},0) = x_1^{m_1}\cdots x_n^{m_n} \tag{6.5}$$

in which $\underline{x} = (x_1, \ldots, x_n)$ and the p_i are positive even integers. Now the solution of the heat problem

$$u_t(\underline{x},t) = (D_{x_1}^{p_1} + \cdots + D_{x_n}^{p_n})u(\underline{x},t),\ u(x,0) = \exp(a_1x_1 + \cdots + a_nx_n)$$

in which the a_i are parameters is given by

$$u(\underline{x},t) = \exp\left(\sum_{i=1}^{n} a_ix_i + t\sum_{i=1}^{n} a_i^{p_i}\right). \tag{6.6}$$

By making use of the transmutation (2.12) , we find that a solution $G(\underline{x}, t, a_1, \ldots, a_n)$ of (6.6) with $x_1^{m_1}\cdots x_n^{m_n}$ replaced by $\exp(a_1x_1 + \cdots + a_nx_n)$ is given by

$$\begin{aligned} &G(\underline{x}, t, a_1, \ldots, a_n) \\ &= \Gamma(\tfrac{3}{2})\exp(a_1x_1 + \cdots + a_nx_n)\mathcal{L}^{-1}\left\{s^{-\frac{3}{2}}e^{(\sum_{i=1}^n a_i^{p_i})/4s}\right\}_{s\to t^2} \\ &= \exp(a_1x_1 + \cdots + a_nx_n)\sinh\left(t\left(\sum_{i=1}^{n} a_i^{p_i}\right)^{\frac{1}{2}}\right)\Big/\left(\sum_{i=1}^{n} a_i^{p_i}\right)^{\frac{1}{2}}. \end{aligned} \tag{6.7}$$

The coefficient of $a_1^{m_1}\cdots a_n^{m_n}$ in the expansion of this generating function will yield the desired solution of (6.6). However, the selecting out of these coefficients is difficult. It is more convenient to express the G in terms of two variable polynomial generators $G_i(x_i, t, a_i)$ in which $G_i(x_i, t, a_i) = \Gamma(\frac{3}{2})\mathcal{L}^{-1}\left\{x^{-\frac{3}{2}}e^{a_ix_i + a_i^{p_i}/4s}\right\}_{s\to t^2}$. To do this we rewrite (6.7) in the form

$$\begin{aligned} &G(\underline{x}, t, a_1, \ldots, a_n) \\ &= e^{\sum a_ix_i}\Gamma(\tfrac{3}{2})\left[\mathcal{L}^{-1}\left\{s^{\frac{3n-3}{2}}\right\}_{s\to\tau} * \left(\prod_{i=1}^{n} * \mathcal{L}^{-1}\left\{s^{-\frac{3}{2}}e^{a_i^{p_1}/4s}\right\}_{s\to\tau}\right)\right]_{\tau=t^2} \\ &= \frac{1}{[\Gamma(\frac{3}{2})]^{n-1}}\left[\mathcal{L}^{-1}\left\{s^{\frac{3n-3}{2}}\right\}_{s\to\tau} * \left(\prod_{i=1}^{n} * G_i(x_i, \tau^{\frac{1}{2}}, a_i)\right)\right]_{\tau=t^2} \\ &= \frac{2^{n-1}}{\pi^{\frac{n-1}{2}}}\left[\mathcal{L}^{-1}\left\{s^{\frac{3n-3}{2}}\right\}_{s\to\tau} * \left(\prod_{i=1}^{n} * G_i(x_i, \tau^{\frac{1}{2}}, a_i)\right)\right]_{\tau=t^2}. \end{aligned}$$

If $n = 2m+1$, then $\mathcal{L}^{-1}(s^{(3n-3)/2})_{s\to\tau} = \delta^{(3m)}(\tau)$ while if $n = 2m$, then $\mathcal{L}^{-1}(s^{(3n-3)/2})_{s\to\tau}$ $= \pi^{-\frac{1}{2}}\delta^{(3m-1)} * \tau^{-\frac{1}{2}}$. In either case, it follows from this that the polynomial $P_{m_1,\ldots,m_n}(\underline{x},t)$ of the problem (6.5) can be given by

$$\begin{aligned} &P_{m_1,\ldots,m_n}(\underline{x},t) \\ &= \frac{1}{m_1!\cdots m_n!}\frac{\partial^m}{\partial a_1^{m_1}\cdots\partial a_n^{m_n}}G(\underline{x},t,a_1,\ldots,a_n)\big|_{a_1=\cdots=a_n=0} \\ &= \frac{2^{n-1}}{\pi^{\frac{n-1}{2}}}\left[\mathcal{L}^{-1}\left\{s^{(3n-3)/2}\right\}_{s\to\tau} * \left(\prod_{i=1}^{n} *P_{m_i}(x_i,\tau^{\frac{1}{2}})\right)\right]_{\tau=t^2} \end{aligned}$$

where $P_{m_i}(x_i,t)$ is the coefficient of $a_i^{m_1}$ in the expansion of $G_i(x_i,t,a_i)$.

(B) *Laplace Potential Functions.* The associated heat functions $w_n(x,t)$ were studied in [23] and their transmutants were employed in [9] to treat representations of solutions of Dirichlet problems in exterior regions. These transmuted functions involved the real and imaginary parts of reciprocal powers of $z = x+iy$. It would be useful to have multivariable versions of these to discuss higher dimensional Dirichlet problems. To obtain these, we call upon the generating functions for $w_n(x,t)$ and the transmutation (2.14).

The generator for the $w_n(x,t)$ is given by $(4\pi t)^{-\frac{1}{2}}e^{-(x-2a)^2/4t} = \sum_{i=0}^{\infty} w_i(x,t)a^i$. Hence, the function $(4\pi t)^{\frac{-n}{2}}e^{-\sum_{i=1}^{n}(x_i-2a_i)^2/4t}$ generates products of associated heat functions which satisfy the heat equation $u_t = \sum_{i=1}^{n} D_{x_i}^2 u$. If we introduce this generating function into (2.14) and denote its transmutant by $L(\underline{x},t,a_i,\ldots,a_n)$, we obtain directly that

$$L(\underline{x},t,a_1,\ldots,a_n) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}\left[t^2+\sum_{i=1}^{n}(x_i-2a_i)^2\right]^{\frac{n+1}{2}}}. \tag{6.8}$$

Again, it is useful to connect this L up with some set of known special functions. To do this, we first note that

$$L(\underline{x}, \tau^{\frac{1}{2}}, a_1, \ldots, a_n)$$

(6.9)
$$= \frac{\tau^{\frac{1}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-\sigma\tau} \sigma^{\frac{n-1}{2}} \left(\prod_{i=1}^n \sigma^{-\frac{1}{2}} e^{-(x_i - 2a_i)^2 \sigma} \right) d\sigma$$
$$= \frac{\tau^{\frac{1}{2}} \Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \left\{ \tau^{-\frac{(n+1)}{2}} \wedge \left(\prod_{i=1}^n \wedge \frac{1}{(\tau + (x_i - 2a_i)^2)} \right) \right\} \Bigg|_{\tau = t^2}.$$

Now $\left[t^2 + (x - 2a)^2\right]^{-1} = (t^2 + x^2)^{-1}(1 - 2\lambda z + z^2)^{-1}$ in which $\lambda = x(t^2 + x^2)^{-\frac{1}{2}}$ and $z = 2a(t^2 + x^2)^{-\frac{1}{2}}$. Moreover, by [20] we have that $(1 - 2\lambda z + z^2)^{-1} = \sum_{k=0}^\infty U_k(\lambda) z^k$ in which the $U_k(\lambda)$ denote Tchebichev polynomials of the second kind. From this, it follows that

$$(t^2 + (x - 2a)^2)^{-1} = \sum_{k=1}^\infty U_k \left(\frac{x}{\sqrt{t^2 + x^2}} \right) \frac{2^k a^k}{(t^2 + x^2)^{\frac{k+2}{2}}}.$$

Using this in (6.9), it readily follows that the coefficient of $a_1^{m_1} \cdots a_n^{m_n}$ in the expansion of (6.9) is given by the complex convolution

$$2^{m_1 + \cdots + m_n} \frac{\Gamma(\frac{n+1}{2}) \tau^{\frac{1}{2}}}{\sqrt{\pi}} \left\{ \tau^{-(\frac{n+1}{2})} \wedge \left(\prod_{i=1}^n \wedge U_{m_i} \left(\frac{x_i}{\sqrt{\tau + x_i^2}} \right) (\tau + x_i^2)^{-(\frac{m_i+2}{2})} \right) \right\}$$

with $\tau = t^2$.

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Received 02.03.1990, in revised form 05.12.1990

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