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On the Linearization of the Satsuma-Mimura Diffusion Equation in the Spatial Periodic Case

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Dedicated to Prof. R. Klötzler on the occasion of his 60th birthday

For the solution of the diffusion equation of Satsuma and Mimura in the spatially periodic case general formulae are given. Furthermore some simple criteria for existence and non-existence of the solution for all times are derived and illustrated by examples.

Key words: *Nonlinear diffusion equations, Cauchy problem*
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Introduction

Some years ago J. SATSUMA and M. MIMURA [2-4] introduced a class of non-linear diffusion equations involving singular integral terms for describing some diffusion phenomena with non-local aggregation. They developed an exact linearization method for these equations and derived some interesting particular solutions in explicit form by this method. Recently in their paper [5] they investigated the stability behaviour of the steady state solutions in the spatially periodic case.

The steady state solutions of Satsuma and Mimura are rederived in systematic way in our paper [6] by reducing the stationary equations to complex differential equations of first order. Methods of complex analysis are also used in our joint paper with W. GERLACH [1] to reduce the general initial-value problem for the equation of Satsuma-Mimura in the spatially periodic case to a Hammerstein integral equation for which by Schauder's and Banach's fixed point theorems the existence of a solution in some time interval is proved. In our paper [7] this method is extended to the general equations of Satsuma-Mimura. Besides in [8] we use the complex analogue of the Hopf-Cole transformation for linearizing the equations of Satsuma-Mimura and studying the general Cauchy problem on the real axis. In the present paper this method is employed to the equations of Satsuma-Mimura in the spatially periodic case. We derive representation formulas in integral and series form for the solution of the general periodic Cauchy problem, study the asymptotic behaviour of the solution, and give some simple criteria for existence and non-existence (i.e., blow up) of the solution for all times. The somewhat involved general picture of the existence of such a solution is demonstrated by examples.

1. Statement of problem

We deal with the following initial-value problem (cp. [1]).

Problem P. Find a real-valued sufficiently smooth function $u(s, t)$, $0 \le t \le T$, periodic in s with period 2π, which satisfies the equation

$$u_t - du_{ss} + [\,Hu \cdot u\,]_s = 0 \quad \text{in} \quad -\pi \le s \le \pi\,,\ 0 < t < T \tag{1}$$

and the initial condition

$$u(s,0) = f(s) \quad \text{in} \quad -\pi \le s \le \pi\,, \tag{2}$$

where $d > 0$ is a given real constant, f is a given real Hölder continuous 2π-periodic function, and H denotes the Hilbert transform

$$(Hu)(s) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} u(\sigma)\cot\frac{\sigma - s}{2}\,d\sigma\,.$$

For dealing with Problem P it is advantageous to use complex variables. We interpret $u(s,t)$ as boundary values $u(e^{is},t)$ of a regular harmonic function $u(z,t)$, $z = re^{is}$, $0 \le r < 1$, in the unit disk G on Γ: $z = e^{is}$, $-\pi \le s \le \pi$. If $v(z,t)$ denotes the conjugate harmonic function to $u(z,t)$ normalized by the condition $v = 0$ at $z = 0$, we have $v(s,t) = -(Hu)(s,t)$ for its boundary values on Γ and the holomorphic function $w(z,t) = u(z,t) + i\,v(z,t)$ in G fulfills the boundary condition

$$u_t - d\,u_{ss} = [\,u\,v\,]_s \ \text{on}\ \Gamma. \tag{3}$$

Writing (3) in the form

$$\mathrm{Re}\left[\,w_t - d\,w_{ss} + (1/2)\,i\,(w^2)_s\right] = 0 \ \text{on}\ \Gamma\,,$$

we see that $w(z,t)$ satisfies the *complex differential equation*

$$w_t + d\,z\,(z\,w')' = (1/2)\,z\,(w^2)' \quad \text{in}\ G \tag{4}$$

together with the initial condition $w(z,0) = S[f](z)$ in G, where the prime denotes differentiation with respect to z and S is the Schwarz integral.

From [1] (cp. also [7]) it follows that a continuous (generalized) solution of Problem P is unique and exists in some sufficiently small time intervall [0,T]. In particular, if

$$M < 0{,}13509\,\pi\,d \quad \text{for}\ M = \max\{|\varphi(s)| : -\pi \le s \le \pi\}\,, \tag{5}$$

where $\varphi := f - iHf$, the solution exists for all times $t > 0$.

2. Integral representation of solution

The transformation

$$w = -2\,d\,z\,W'/W \tag{6}$$

with an analytic function $W = W(z,t)$ in G reduces the equation (4) for w to the equation

$$W_t + d\,z\,(z\,W')' = 0 \tag{7}$$

for W with the initial condition

$$W(z,0) = \Phi(z)\,,\ \Phi(z) = z^{-\alpha}\,\Phi_0(z)\,, \tag{8}$$

where $\alpha = f_0/2d$, $f_0 = (1/2\pi)\int_{-\pi}^{+\pi} f\,ds$ and

$$\Phi_0(z) = \exp\left\{-\frac{1}{2d}\int_0^z S[\bar{f}](\zeta)\frac{d\zeta}{\zeta}\right\},\ \bar{f} = f - f_0 . \tag{9}$$

Since $S[\bar{f}](0) = 0$ the function Φ_0 is holomorphic in G and continuous on $\bar{G}$. It satisfies the condition $\Phi_0(0) = 1$. Further, for definiteness, $z^{-\alpha}$ is understood as principal value.

According to (8) we make the ansatz

$$W(z,t) = z^{-\alpha} W_0(z,t) \tag{10}$$

with a holomorphic function W_0 in G continuous with its first derivative on $\bar{G}$ and satisfying the initial condition $W_0(z,0) = \Phi_0(z)$. In view of equation (7) the boundary function $W(s,t) = W(e^{is},t)$ fulfills the heat equation

$$W_t - d W_{ss} = 0,\quad -\pi < s < \pi,\quad 0 < t < T, \tag{11}$$

by (8) the initial condition

$$W(s,0) = \Phi(e^{is}) = e^{-i\alpha s}\Phi_0(e^{is}), \tag{12}$$

and by (10) the modified periodicity conditions

$$W(-\pi,t) = e^{2\pi i\alpha} W(\pi,t),\quad W_s(-\pi,t) = e^{2\pi i\alpha} W_s(\pi,t). \tag{13}$$

Using the Fourier expansion of $\Phi_0(e^{is})$, the problem (11)-(13) has the solution

$$W(s,t) = e^{-i\alpha s}\frac{1}{2\pi}\int_{-\pi}^{+\pi}\Phi_0(e^{i\sigma})\sum_{k=-\infty}^{\infty} e^{ik(s-\sigma)}e^{-d(k-\alpha)^2 t}d\sigma .$$

Putting $\zeta = e^{i\eta}$, $\eta = \sigma - s$, for W we obtain the formula (10) with the function

$$W_0(z,t) = \frac{1}{2\pi i}\int_\Gamma \Phi_0(z\zeta)\, G_\alpha(\zeta,t)\frac{d\zeta}{\zeta}, \tag{14}$$

where

$$G_\alpha(\zeta,t) = \sum_{k=-\infty}^{\infty} \zeta^{-k} e^{-d(k-\alpha)^2 t}. \tag{15}$$

In particular,

$$W_0(0,t) = \frac{1}{2\pi i}\int_\Gamma G_\alpha(\zeta,t)\frac{d\zeta}{\zeta} = e^{-d\alpha^2 t} > 0 . \tag{16}$$

Theorem 1 : *If the function $W_0(z,t)$ given by* (14) *does not vanish in $\bar{G}$: $|z| \le 1$ for $0 \le t \le T$ (or $0 \le t < T$, respectively), then the function $u(s,t) = \operatorname{Re} w(e^{is},t)$, where w is given by* (6) *with W defined by* (10), (14), *represents the unique solution of Problem P in $[0,T]$ (or $[0,T)$, respectively).*

Remark: If $\alpha = m$ is an integer, we have $G_\alpha(\zeta,t) = \zeta^{-m} G_0(\zeta,t)$ and

$$W(z,t) = \frac{1}{2\pi i}\int_\Gamma \Phi(z\zeta)\, G_0(\zeta,t)\frac{d\zeta}{\zeta}, \tag{17}$$

where

$$G_0(\zeta,t) = \sum_{k=-\infty}^{\infty} \zeta^{-k} e^{-dk^2 t} \tag{18}$$

with

$$G_0(e^{is},t) = \vartheta(s/2\pi, e^{-dt}) > 0 \tag{19}$$

expressed through the third Jacobi's theta function ϑ.

Example 1: For the non-constant steady-state solutions of equation (1) we have (cp. [6])

$$f(s) = nd\frac{1-|C|^2}{|1-Ce^{ins}|^2}, \quad \varphi(s) = nd\frac{1+Ce^{ins}}{1-Ce^{ins}},$$

$$w(z) = nd\frac{1+Cz^n}{1-Cz^n}, \quad n = 1,2,\ldots, \text{ where } |C| < 1.$$

Here $f_0 = \operatorname{Re} w(0) = nd$, hence $\alpha = n/2$. Further

$$\Phi_0(z) = 1 - Cz^n, \quad \Phi(z) = z^{-n/2}[1-Cz^n],$$

$$W(z,t) = e^{-d(n^2/4)t}\,\Phi(z), \quad w(z,t) = w(z).$$

3. Series representation and asymptotic behaviour of solution

Let the function Φ_0 given by (9) have the Taylor expansion

$$\Phi_0(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \quad \text{in } |z| \le 1. \tag{20}$$

Then by (10), (14)

$$W(z,t) = z^{-\alpha}\left[e^{-d\alpha^2 t} + \sum_{j=1}^{\infty} a_j z^j e^{-d(j-\alpha)^2 t}\right] \tag{21}$$

with

$$\begin{aligned} W'(z,t) = {} & -\alpha z^{-\alpha-1}\left[e^{-d\alpha^2 t} + \sum_{j=1}^{\infty} a_j z^j e^{-d(j-\alpha)^2 t}\right] \\ & + z^{-\alpha}\sum_{j=1}^{\infty} j a_j z^{j-1} e^{-d(j-\alpha)^2 t}. \end{aligned} \tag{22}$$

In view of (6) from (21), (22) we obtain

Theorem 2: *If the function Φ_0 in (9) has the expansion (20), the function w corresponding to the solution u of Problem P is given by the expression*

$$w(z,t) = f_0 - 2d\,\frac{\sum_{j=1}^{\infty} j a_j z^j e^{-d(j-\alpha)^2 t}}{e^{-d\alpha^2 t} + \sum_{j=1}^{\infty} a_j z^j e^{-d(j-\alpha)^2 t}}. \tag{23}$$

From (23) the following *asymptotic behaviour* of the function w as $t \to +\infty$ is obtained:

In the case $-\infty < \alpha < 1/2$:

$$w(z,t) \sim f_0. \tag{24}$$

In the case $N - 1/2 < \alpha < N + 1/2$, $N = 1,2,\ldots$:

$$w(z,t) \sim f_0 - 2d\,P(z,t)/Q(z,t), \tag{25}$$

with

$$P = Na_N z^N e^{-d(n-\alpha)^2 t} + \ldots + a_1 z\, e^{-d(1-\alpha)^2 t}$$
$$+ (2N-1) a_{2N-1} z^{2N-1} e^{-d(2N-1-\alpha)^2 t} + 2Na_{2N} z^{2N} e^{-d(2N\alpha)^2 t}$$

and

$$Q = a_N z^N e^{-d(N-\alpha)^2 t} + \ldots + a_1 z\, e^{-d(1-\alpha)^2 t}$$
$$+ a_{2N-1} z^{2N-1} e^{-d(2N-1-\alpha)^2 t} + a_{2N} z^{2N} e^{-d(2N-\alpha)^2 t} + e^{-d\alpha^2 t} .$$

In the case $\alpha = N + (1/2)$:

$$w(z,t) \sim f_0 - 2d\, P_0(z,t)/Q_0(z,t) , \tag{26}$$

with

$$P_0 = \left(N a_N z^N + (N+1) a_{N+1} z^{N+1}\right) e^{-(d/4)t} + \ldots$$
$$+ \left(a_1 z + 2N a_{2N} z^{2N}\right) e^{-d(N-1/2)^2 t}$$
$$+ (2N+1) a_{2N+1} z^{2N+1} e^{-d(N+1/2)^2 t}$$

and

$$Q_0 = \left(a_N z^N + a_{N+1} z^{N+1}\right) e^{-(d/4)t} + \ldots$$
$$+ \left(a_1 z + a_{2N} z^{2N}\right) e^{-d(N-1/2)^2 t}$$
$$+ \left(a_{2N+1} z^{2N+1} + 1\right) e^{-d(N+1/2)^2 t} .$$

4. Existence assertions

In view of (21) the condition for the existence of a solution to Problem P can be formulated as follows.

Existence criterion: The holomorphic function

$$\Psi_0(z,t) = 1 + \sum_{j=1}^{\infty} a_j \gamma_j z^j , \; \gamma_j = e^{-dj(j-2\alpha)t} , \tag{27}$$

does not vanish in $|z| \le 1$.

The function $\Phi_0(z)$ does not vanish in $|z| \le 1$. Further, for $t > 0$ we have $0 < \gamma_j < 1$ ($j = 1,2,\ldots$) in the case $0 \le \alpha < 1/2$ and $\gamma_1 = 1$, $0 < \gamma_j < 1$ ($j = 2,3,\ldots$) in the case $\alpha = 1/2$. Hence there holds

Theorem 3: *If* $0 \le \alpha \le 1/2$ *and*

$$\sum_{j=1}^{\infty} |a_j| \le 1 , \tag{28}$$

then the solution of Problem P exists for all $t > 0$.

In the case $\alpha > 1/2$ from the asymptotic relations (25) and (26) we obtain

$$u(0,t) = \text{Re}\, w(0,t) \sim f_\infty \mp f_0$$

if one of the coefficients $a_1, a_2, \ldots a_k$ with $k = 2N-1$ as $N - 1/2 < \alpha \le N$ and

$k = N$ as $N < \alpha \le N + 1/2$ is different from zero. Following an argumentation by Satsuma and Mimura in their papers cited, we conclude that then the solution of Problem P cannot exist for all $t > 0$ (i.e., must blow up in finite time) since

$$I = \int_{-\pi}^{+\pi} u(e^{is}, t)\,ds = 2\pi\, u(0,t)$$

is independent of t for regular solutions u of problem P.

Theorem 4: *If $\alpha > 1/2$ and one of the coefficients $a_1, a_2, \ldots a_k$ in the expansion (20) of $\Phi_0(z)$ is different from zero, where k is the greatest integer smaller than 2α, then the solution of Problem P does not exist for all $t > 0$.*

Example 2: We choose

$$\Phi_0(z) = 1 - Cz^n \quad \text{with } |C| < 1, \; C \neq 0 \tag{29}$$

and $\alpha > 0$, where $\alpha \neq n/2$. By (6) or (9) to (29) there corresponds the function

$$f(s) = 2d\,\frac{\alpha - |C|^2(n-\alpha) + (n-2\alpha)\mathrm{Re}[Ce^{ins}]}{|1 - Ce^{ins}|^2}$$

with

$$\varphi(s) = 2d\,\frac{\alpha + C(n-\alpha)e^{ins}}{1 - Ce^{ins}} \;; \quad w(z) = 2d\,\frac{\alpha + C(n-\alpha)z}{1 - Cz^n}.$$

By (23) we have the solution of (4)

$$w(z,t) = f_0 + 2d\,\frac{nCz^n e^{-d(n-\alpha)^2 t}}{e^{-d\alpha^2 t} - Cz^n e^{-d(n-\alpha)^2 t}} \tag{30}$$

with $f_0 = 2d\alpha$. If $0 < \alpha < n/2$ the solution (30) exists for all $t > 0$. If $\alpha > n/2$ it blows up at time t_0, where $\exp[dn(n-2\alpha)t_0] = |C|$, i.e., at $t_0 = [dn(2\alpha - n)]^{-1}\ln(1/|C|)$.

Example 3: We further choose

$$\Phi_0(z) = (1 - c_1 z^{n_1})(1 - c_2 z^{n_2}) = 1 - c_1 z^{n_1} - c_2 z^{n_2} + c_1 c_2 z^{n_1 + n_2} \tag{31}$$

with $0 \le c_1,\ c_2 < 1$, $c_1^2 + c_2^2 > 0$ and $\alpha = (n_1 + n_2)/2$ (sums of two pulses, cf [4]). Then

$$f(s) = n_1 d\,\frac{1 - c_1^2}{1 + c_1^2 - 2c_1\cos n_1 s} + n_2 d\,\frac{1 - c_2^2}{1 + c_2^2 - 2c_2 \cos n_2 s}$$

with

$$w(z) = n_1 d\,\frac{1 + c_1 z^{n_1}}{1 - c_1 z^{n_1}} + n_2 d\,\frac{1 + c_2 z^{n_2}}{1 - c_2 z^{n_2}}.$$

By (23) we have the solution of (4)

$$w(z,t) = f_0 - 2d\,P_1(z,t)/Q_1(z,t), \tag{32}$$

where

$$P_1(z,t) = (n_1 + n_2)c_1 c_2 z^{n_1 + n_2} e^{-d(n_1+n_2)^2 t/4} - (n_1 c_1 z^{n_1} + n_2 c_2 z^{n_2})\, e^{-d(n_1 - n_2)^2 t/4},$$

$$Q_1(z,t) = (1 + c_1 c_2 z^{n_1+n_2})\, e^{-d(n_1+n_2)^2 t/4} - (c_1 z^{n_1} + c_2 z^{n_2})\, e^{-d(n_1-n_2)^2 t/4}.$$

and $f_0 = 2d\alpha = d(n_1+n_2)$. The solution blows up at time t_0, where $\exp(-dn_1n_2t_0) = (c_1+c_2)/(1+c_1c_2)$, i.e., at $t_0 = (dn_1n_2)^{-1}\ln\left[(1+c_1c_2)/(c_1+c_2)\right]$.

5. The case $\alpha = 0$

If $\alpha = 0$, by (17) we have

$$W(z,t) = \frac{1}{2\pi i}\int_\Gamma \Phi(z\zeta)\,G_0(\zeta,t)\frac{d\zeta}{\zeta}$$

with G_0 defined by (18) and

$$\Phi(z) = \Phi_0(z) = \exp\left\{-\frac{1}{2d}\int_0^z S[f](\zeta)\frac{d\zeta}{\zeta}\right\}.$$

Therefore W is a holomorphic function in $|z| < 1$. Further, $W(z,t) \neq 0$ in $|z| \leq 1$ if

$$\hat{W}(z,t) := \frac{1}{2\pi i}\int_\Gamma \hat{\Phi}(z\zeta)\,G_0(\zeta,t)\frac{d\zeta}{\zeta} \neq 0 \quad \text{in } |z| \leq 1 \tag{33}$$

with the function

$$\hat{\Phi}(z) = \exp\left\{-\frac{1}{2d}\int_1^z S[f](\zeta)\frac{d\zeta}{\zeta}\right\}.$$

On Γ we have

$$\hat{\Phi}(e^{is}) = \exp\left\{-\frac{1}{2d}\int_0^s Hf\,d\sigma\right\}\cdot\exp\left\{\frac{-i}{2d}\int_0^s f\,d\sigma\right\}$$

so that $\operatorname{Re}\hat{\Phi}(e^{is}) > 0$ if the condition

$$-\pi/2 \leq \frac{1}{2d}\int_0^s f\,d\sigma \leq \pi/2 \quad \text{as } -\pi \leq s \leq \pi \tag{34}$$

is fulfilled. Then also

$$\operatorname{Re}\hat{W}(e^{is},t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\operatorname{Re}\hat{\Phi}\left(e^{i(s+\sigma)}\right)G_0(e^{i\sigma},t)\,d\sigma > 0$$

on account of (19), and in view of the maximum principle for harmonic functions $\operatorname{Re}\hat{W}(z,t) > 0$ in $|z| \leq 1$ implying the condition (33).

Theorem 5: *If $\alpha = 0$ and the condition* (34) *is fulfilled, then the solution of Problem* P *exists for all* $t > 0$.

Example 4: We choose

$$\Phi(z) = (1-cz)^2 = 1 - 2c + c^2z^2 \quad \text{with } 0 < c < 1, \tag{35}$$

which corresponds to the function

$$f(s) = 4d\sum_{n=1}^{\infty} c^n \cos ns = 4d\,c(\cos s - c)/|1-ce^{is}|^2.$$

By (23) we have the solution of (4)

$$w(z,t) = 4d\,\frac{ze^{-dt} - c^2z^2e^{-4dt}}{1-2cze^{-dt}+c^2z^2e^{-4dt}}. \tag{36}$$

If $0 < c < 4/(3\sqrt{3}) = 0{,}76980$ the solution (36) exists for all $t > 0$, whereas for $c \geq 4/(3\sqrt{3})$ it blows up at $t_0 = (1/d)\ln\rho_0$, where $\rho_0 > 1$ is the smallest root of the equation $\rho^4 - 2c\rho^3 + c^2 = 0$. In particular, for $c = 4/(3\sqrt{3})$ we have $\rho_0 = 2/\sqrt{3}$.

The criterion (34) is fulfilled if $\arctan(c \sin s/(1 - c\cos s)) \leq \pi/4$ in $0 \leq s \leq \pi$, i.e., if $c \leq 1/\sqrt{2} = 0{,}70710$. The condition (28) is satisfied if $c \leq \sqrt{2} - 1 = 0{,}41421$. Finally, we remark that the criterion (5) yields the very restricted condition $c < 0{,}095924$.

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