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A-Priori Estimates for the Solutions of a Class of Nonlinear Convolution Equations

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We present sharp two-sided a-priori estimates for the solutions of a class of nonlinear Volterra integral equations in the cone of non-negative continuous functions. These estimates enable us to construct a complete metric space which is invariant under the nonlinear convolution operator considered here and to prove that the equation induced by this operator has a unique solution in this space as well as in the class of all non-negative continuous functions vanishing at the origin.

Key words: *Volterra equations, nonlinear equations, non-negative solutions*
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1. A-priori estimates. This paper concerns nonlinear integral equations of the form

$$u^{\alpha}(x) = a(x)\int_0^x k(x-t)u(t)\,dt, \qquad (1)$$

where $\alpha > 1$ is a given real number, the coefficient a and the kernel k are given non-negative functions, and the solution u is sought in the class Γ_0 of all non-negative continuous functions on $[0,\infty)$ for which $u(0) = 0$ and $u(x) > 0$ for $x > 0$. The results we shall establish here generalize some ones of W. Okrasinski [3,4] and make more precise part of the results contained in [1,2]. Throughout what follows the coefficient a and the kernel k are assumed to satisfy the following conditions:

(i) The function a belongs to the cone Γ of all non-negative continuous functions on $[0,\infty)$, is non-decreasing on $[0,\infty)$, and $a(x) > 0$ for $x > 0$.

(ii) The function k is non-decreasing on $[0,\infty)$ and $k(0) > 0$.

(iii) There is an $\eta > 0$ such that $k(\eta) < \alpha k(0)$.

Lemma 1: *If $u \in \Gamma_0$ is a solution of equation* (1) *and if we put*

$$F(x) = \left(\frac{\alpha-1}{\alpha}k(0)\right)^{1/(\alpha-1)} a^{1/\alpha}(x)\left(\int_0^x a^{1/\alpha}(t)\,dt\right)^{1/(\alpha-1)},$$

$$G(x) = \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)} a^{1/\alpha}(x)\left(\int_0^x a^{1/\alpha}(t)k(t)\,dt\right)^{1/(\alpha-1)},$$

then

$$F(x) \le u(x) \le G(x) \text{ for all } x \in [0,\infty). \qquad (2)$$

Proof: From (1) we infer that $u(0) = 0$ and that u^{α}, being the product of two non-decreasing non-negative functions, is also a non-decreasing function. Hence u itself is non-decreasing and thus differentiable almost everywhere on $[0,\infty)$, and we have $u'(x) \ge 0$ for almost all $x \in [0,\infty)$.

We now show that $F(x) \le u(x)$ for all $x \ge 0$. Condition (ii) implies

$$u(x) \ge k^{1/\alpha}(0)a^{1/\alpha}(x)\left(\int_0^x u(t)\,dt\right)^{1/\alpha} \qquad (3)$$

whence

$$u(x)\Big(\int_0^x u(t)\,dt\Big)^{-1/\alpha} \ge k^{1/\alpha}(0)\,a^{1/\alpha}(x), \tag{4}$$

and since

$$\int_0^x u(\xi)\Big(\int_0^\xi u(t)\,dt\Big)^{-1/\alpha} d\xi = \frac{\alpha}{\alpha-1}\Big(\int_0^x u(t)\,dt\Big)^{(\alpha-1)/\alpha},$$

integration of (4) gives

$$\Big(\int_0^x u(t)\,dt\Big)^{1/\alpha} \ge \Big(\frac{\alpha-1}{\alpha}\Big)^{1/(\alpha-1)} k(0)^{1/\alpha(\alpha-1)}\Big(\int_0^x a^{1/\alpha}(t)\,dt\Big)^{1/(\alpha-1)}. \tag{5}$$

Substituting (5) into (3) we obtain $F(x) \le u(x)$ for all $x \ge 0$.

To profe that $u(x) \le G(x)$ for all $x \ge 0$, we first show that

$$\varphi(x) := \int_0^x k(x-t)u(t)\,dt - \int_0^x k(t)u(t)\,dt \le 0 \text{ for all } x \ge 0. \tag{6}$$

It is clear that $\varphi(0) = 0$. We have already shown that $u(0) = 0$ and $u'(x) \ge 0$ for almost all $x \ge 0$. This in conjunction with (ii) yields

$$\varphi'(x) = \int_0^x k(t)u'(x-t)\,dt - k(x)u(x) \le k(x)\Big(\int_0^x u'(x-t)\,dt - u(x)\Big) \le 0$$

and thus $\varphi(x) \le \varphi(0) = 0$ for all $x \ge 0$. From (1) and (6) we deduce

$$u(x) \le a^{1/\alpha}(x)\Big(\int_0^x k(t)u(t)\,dt\Big)^{1/\alpha}. \tag{7}$$

So $k(x)u(x)\Big(\int_0^x k(t)u(t)\,dt\Big)^{-1/\alpha} \le a^{1/\alpha}(x)k(x)$, and upon integrating we find

$$\Big(\int_0^x k(t)u(t)\,dt\Big)^{1/\alpha} \le \Big(\frac{\alpha-1}{\alpha}\int_0^x a^{1/\alpha}(t)k(t)\,dt\Big)^{1/(\alpha-1)}. \tag{8}$$

Inserting (8) into (7) gives the desired inequality $u(x) \le G(x)$ ∎

Remarks : **1.** It can be verified straightforwardly that if $\alpha > 1$, then the function

$$u^*(x) = \Big(\frac{\alpha-1}{\alpha}\Big)^{1/(\alpha-1)} a^{1/\alpha}(x)\Big(\int_0^x a^{1/\alpha}(t)\,dt\Big)^{1/(\alpha-1)}$$

is a solution of the equation $u^\alpha(x) = a(x)\int_0^x u(t)\,dt$ (see [1]). Consequently, if $k = 1$, then $F(x) = u^*(x) = G(x)$ for all $x \ge 0$, which shows that, in some sense, the estimates provided by Lemma 1 are sharp. **2.** In [1] and [2], we established the lower a-priori estimate

$$u(x) \ge \Big(\frac{\alpha-1}{\alpha}k(0)\Big)^{1/(\alpha-1)}\Big(\int_0^x a(t)\,dt\Big)^{1/(\alpha-1)} \tag{9}$$

By virtue of (i) we have

$$F_0(x) := \Big(\int_0^x a(t)\,dt\Big)^{1/(\alpha-1)} \le a^{1/\alpha}(x)\Big(\int_0^x a^{1/\alpha}(t)\,dt\Big)^{1/(\alpha-1)}$$

and thus $F_0(x) \le F(x)$ for all $x \ge 0$. Moreover, if $\alpha = 2$ and $a(x) = x$, then $F_0(x) = k(0)x^2/4 < k(0)x^2/3 = F(x)$ for $x > 0$. These two observations show that the lower estimate in (2) is essentially sharper than that in (9). **3.** Our Lemmas 1 and 2 extend Theorem 2 and Lemma 1 of [4] to the case where the coefficient a is not identically one.

2. Existence and uniqueness theorem. Let F and G be as in Lemma 1. Given a number $b > 0$, we denote by P_b the collection of all functions $u \in C[a,b]$ such that $F(x) \le u(x) \le G(x)$ for all $x \in [0,b]$ (notice that P_b also depends on a and k). Define the operator T by

$$(Tu)(x) = \Big(a(x)\int_0^x k(x-t)u(t)\,dt\Big)^{1/\alpha}.$$

Lemma 2: *The operator T maps P_b into itself.*

Proof: Let $u \in P_b$. Then clearly $Tu \in C[0,b]$, and so it remains to show that $F(x) \le (Tu)(x) \le G(x)$ for all $x \in [0,b]$. Since $F(x) \le u(x)$ and k is non-decreasing, we have

$$\big((Tu)(x)\big)^{\alpha} \ge k(0)a(x)\int_0^x F(t)\,dt$$
$$= \left(\frac{\alpha-1}{\alpha}k(0)\right)^{1/(\alpha-1)} k(0)a(x)\int_0^x\left(\int_0^t a^{1/\alpha}(\tau)\,d\tau\right)^{1/(\alpha-1)} d\left(\int_0^t a^{1/\alpha}(\tau)\,d\tau\right),$$

and as the latter expression is nothing but $F^{\alpha}(x)$, it follows that $(Tu)(x) \ge F(x)$ for all $x \in [0,b]$. Taking into account that $u(x) \le G(x)$, we see that

$$\big((Tu)(x)\big)^{\alpha} \le \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)} a(x)\int_0^x k(x-t)g(t)\,dt, \tag{10}$$

where $g(t) := a^{1/\alpha}(t)\left(\int_0^t a^{1/\alpha}(\tau)k(\tau)\,d\tau\right)^{1/(\alpha-1)}$. Clearly, $g(0)=0$ and g is non-decreasing on $[0,b]$. Therefore $g'(t) \ge 0$ for almost all $t \in [0,b]$. Thus, g enjoys the same properties as the function u occuring in (6), which implies

$$\int_0^x k(x-t)g(t)\,dt \le \int_0^x k(t)g(t)\,dt. \tag{11}$$

Combining (10) and (11) we get

$$\big((Tu)(x)\big)^{\alpha} \le \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)} a(x)\int_0^x k(t)g(t)\,dt = G^{\alpha}(x),$$

i.e. $(Tu)(x) \le G(x)$ for all $x \in [0,b]$ ∎

Fix now any number η satisfying (iii) and any number $b > \eta$. We define

$$\beta = k(0)^{-1}\sup\{(k(x)-k(0))/x : \eta \le x \le b\}$$

The following result was established in [3].

Lemma 3: *We have*

$$k(x)e^{-\beta x} \le k(\eta) \text{ for all } x \in [0,b]. \tag{12}$$

For $u, v \in P_b$ we put

$$\rho(u,v) = \sup_{0<x\le b}\left\{|u(x)-v(x)| \Big/ a^{1/\alpha}(x)\left(\int_0^x a^{1/\alpha}(t)\,dt\right)^{1/(\alpha-1)} e^{\beta x}\right\} \tag{13}$$

(see [1 - 4]). Since $|u(x)-v(x)| \le G(x) - F(x)$ for all $x > 0$ and thus

$$|u(x)-v(x)| \Big/ a^{1/\alpha}(x)\left(\int_0^x a^{1/\alpha}(t)\,dt\right)^{1/(\alpha-1)} e^{\beta x}$$
$$\le \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)}\left[\left(\int_0^x a^{1/\alpha}(t)k(t)\,dt\right)^{1/(\alpha-1)} \Big/ \left(\int_0^x a^{1/\alpha}(t)\,dt\right)^{1/(\alpha-1)} - k(0)^{1/(\alpha-1)}\right]$$
$$\le \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)}\left[k(x)^{1/(\alpha-1)} - k(0)^{1/(\alpha-1)}\right],$$

the right-hand side of (13) is always finite. Using the arguments of [3] it can be easily checked that (P_b, ρ) is a complete metric space.

Theorem 1: *The operator $T\colon P_b \to P_b$ is a contraction. More precisely, we have*

$$\rho(Tu, Tv) \le \big(k(\eta)/\alpha k(0)\big)\rho(u,v) \text{ for all } u, v \in P_b. \tag{14}$$

Proof: Obviously,

$$|u(x) - v(x)| \le \rho(u,v)e^{\beta x}a^{1/\alpha}(x)\Big(\int_0^x a^{1/\alpha}(t)\,dt\Big)^{1/(\alpha-1)},$$

whence, by Lemma 3,

$$|(k*(v-u))(x)| \le \rho(u,v)e^{\beta x}k(\eta)\int_0^x a^{1/\alpha}(t)\Big(\int_0^t a^{1/\alpha}(\tau)\,d\tau\Big)^{1/(\alpha-1)}$$

$$= \frac{\alpha-1}{\alpha}\rho(u,v)e^{\beta x}k(\eta)\Big(\int_0^x a^{1/\alpha}(\tau)\,d\tau\Big)^{\alpha/(\alpha-1)}.$$

Here $k*(v-u)$ denotes the convolution occuring in (1). The latter inequality in conjunction with Lemma 2 and the mean-value theorem (see [1 - 4]) yields that $\rho(Tu, Tv)$ does not exceed

$$\frac{1}{\alpha}\sup_{0<x\le b}\frac{a(x)|(k*(v-u))(x)|}{\big(\min\{(Tu)(x),(Tv)(x)\}\big)^{\alpha-1}a^{1/\alpha}(x)\big(\int_0^x a^{1/\alpha}(t)\,dt\big)^{1/(\alpha-1)}e^{\beta x}} \le \frac{k(\eta)}{\alpha k(0)}\rho(u,v) \blacksquare$$

Theorem 2: *If the conditions* (i) - (iii) *are in force, then the equation* (1) *has a unique solution in Γ_0 (as well as in P_b for every $b > \eta$). This solution can be obtained by means of the method of successive approximation.*

Proof: The assertion follows from Banach's fixed point theorem along with the fact that the contraction constant $k(\eta)/\alpha k(0)$ in (14) is independent of $b > \eta$ $\blacksquare$

Remark 4: Results similar to those of the present paper can also be proved for classes of so-called almost increasing functions (see [2]).

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