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On a Certain Differential Equation of Non-Integer Order

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Some differential equation of non-integer order is considered. It plays an important role in polarography.

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1. Introduction. The equation

$$y^{(1/2)}(x) - \upsilon x^{\beta} y(x) = x^{-1/2} \qquad (x > 0;\ -1/2 < \beta \le 0),$$

where $y^{(1/2)}$ is the derivative of order 1/2, plays an important role in the polarography (for chemical background cf. [2,5-7]). K. WIENER in his papers [8-10] examines the above-said equation assuming that the derivative of non-integer order appearing in the equation is defined in the Hadamard sense. In further reasoning we denote

$$y_0(x) = y(x) \text{ and } y_\alpha(x) = \int_0^x (x-t)^{\alpha-1} y(t)dt/\Gamma(\alpha) \text{ for } \alpha, x > 0,$$

where the integral is understood in the Lebesgue sense.

The symbols $y_{p-\alpha}$ and s_α in the sequel will be defined analogously. Let $\alpha > 0$ and set $p = -[-\alpha]$, $[\alpha]$ being the greatest integer not exceeding α. We will consider the equation

$$y^{(\alpha)}(x) - \upsilon x^{\beta} y(x) = h(x), \quad x > 0, \tag{1}$$

with the initial conditions $y_{p-\alpha}^{(p-k)}(0) = c_k$ $(k=1,2,\dots,p)$, where $y^{(\alpha)} = y_{p-\alpha}^{(p)}$ is the Riemann-Liouville derivative of order α (cf. DZHRBASHYAN [1, Ch. IX, 3 and 4]), provided that y is Lebesgue locally integrable and $y_{p-\alpha}^{(p-1)}$ is absolutely continuous. The above problem for equation (1) is called the *Cauchy problem*.

One can observe in this class of functions that the Hadamard and the Riemann-Liouville derivatives coincide. It is known for $\beta \ge 0$ and $0 < x < A$, with arbitrary positive A and $h \in L(0,A)$, that the Cauchy problem is well-posed (cf. [3]). It means that there exists exactly one solution and the solution depends continuously on initial data in the following sense: if y^1 and y^2 are solutions of (1) with the initial data c_k^1 and c_k^2 ($k=1,2,\ldots,p$), respectively, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|y^1 - y^2\| < \varepsilon$, provided that $|c_k^1 - c_k^2| < \delta$, where $\|\cdot\|$ denotes the norm in $L(0,A)$.

2. The Cauchy problem for equation (1). In further part of the paper we will examine equation (1) with negative β. Introduce the Banach space

$$L_\tau(I) = \left\{ f: I \longrightarrow \mathbb{R}: \text{ f measurable and } \|f\|_\tau < \infty \right\},$$

where $\tau \ge 0$, $I \subset \mathbb{R}_+$ and $\|f\|_\tau = \int_I |f(t)|\exp(-\tau t)dt$. Since $|f(t)|\exp(-\tau A) \le |f(t)|\exp(-\tau t) \le |f(t)|$ the norms $\|\cdot\|_0$ and $\|\cdot\|_\tau$ are equivalent for the bounded interval $I = (0,A)$. Generally, for arbitrary I the relations $L_0(I) \subset L_\tau(I) \subset L_{loc}(I)$ hold, where $L_0(I)$ and $L_{loc}(I)$ are the spaces of the Lebesgue integrable and the Lebesgue locally integrable functions, respectively.

Denote $s = y^{(\alpha)}$ and $w(x) = \sum_{k=1}^{p} c_k x^{\alpha-k}/\Gamma(1+\alpha-k)$. The function s is a solution of the equation

$$s(x) = \upsilon x^\beta w(x) + h(x) + \upsilon x^\beta s_\alpha(x). \tag{2}$$

Hence, the function $y = w + s_\alpha$ is a solution of the Cauchy problem for equation (1). Define the transformation T by the right-hand side of equation (2). Assume that $h \in L_\tau(\mathbb{R}_+)$ and β, $-\alpha < \beta \le 0$ is such that $x^\beta w(x)$ belongs to $L_\tau(\mathbb{R}_+)$ with τ sufficiently large to fulfil the relation

$$\upsilon\Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}/\Gamma(\alpha) \le q < 1. \tag{3}$$

One can observe that the inequalities

$$\|Ts^1 - Ts^2\|_\tau \le \upsilon\int_0^\infty x^\beta \exp(-\tau x)\left(\int_0^x (x-t)^{\alpha-1}|s^1(t) - s^2(t)|dt\right)dx/\Gamma(\alpha)$$

$$\le \upsilon\int_0^\infty |s^1(t)-s^2(t)|\exp(-\tau t)\left(\int_t^\infty (x-t)^{\alpha-1}x^\beta \exp(-\tau(x-t))dx\right)dt/\Gamma(\alpha)$$

hold. Examine the integral with respect to x. Changing the variable of integration and bearing in mind the definition of Euler's gamma function we have

$$\int_t^\infty (x-t)^{\alpha-1}x^\beta \exp(-\tau(x-t))dx \le t^{\alpha+\beta}\int_0^\infty \xi^{\alpha-1}(1+\xi)^\beta \exp(-\tau t\xi)d\xi \le \Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}$$

what implies

$$\|Ts^1 - Ts^2\|_\tau \le \upsilon\Gamma(\alpha+\beta)\tau^{-(\alpha+\beta)}\|s^1 - s^2\|_\tau/\Gamma(\alpha).$$

Thus, for τ satisfying relation (3), the transformation T is contractive, with the contraction constant equal q. Moreover, one can notice that T maps

$L_\tau(\mathbb{R}_+)$ into itself. Using the Banach fixed point theorem to the transformation T and the set $L_\tau(\mathbb{R}_+)$ we can assert that there exists in $L_\tau(\mathbb{R}_+)$ exactly one solution of equation (2). Hence, the function $y = w + s_\alpha$ is the only solution of the Cauchy problem such that $y^{(\alpha)} \in L_\tau(\mathbb{R}_+)$.

Following the method presented above the function

$$y(x) = \sum_{k=1}^{p} c_k x^{\alpha-k}\left(1 + \sum_{l=1}^{\infty}\left(\upsilon x^{\alpha+\beta}/\Gamma(\alpha)\right)^{l} \prod_{j=1}^{l} B\left(\alpha, 1-k+j(\alpha+\beta)\right)\right)\Big/\Gamma(1+\alpha-k)$$

$$+ \sum_{n=0}^{\infty} (\upsilon/\Gamma(\alpha))^n \int_0^x K_{n+1}(x,t)t^{-\beta}h(t)dt/\Gamma(\alpha), \qquad (4)$$

where B is Euler's beta function and

$$K_{n+1}(x,t) = \begin{cases} (x-t)^{\alpha-1}t^{\beta} & \text{for } n = 0 \\ \int_t^x K_n(x,\xi)K_1(\xi,t)d\xi & \text{for } n \in \mathbb{N} \end{cases}$$

is a solution of the Cauchy problem for equation (1).

It is seen that the solution of the Cauchy problem for equation (1) is a sum of two functions, say y_H and y_N, which are the solutions of homogeneous equation (1) with given initial conditions and non-homogeneous equation (1) with homogeneuos initial conditions, respectively. Moreover, every solution depends on constants being arbitrarily up to p (the value of β determines the number of independent constants - cf. Remark 1 in the sequel).

The aforegoing considerations imply

Theorem 1: *If the above assumptions are fulfilled, then there exists exactly one solution* y *of the Cauchy problem for the equation* (1) *such that* $y^{(\alpha)} \in L_\tau(\mathbb{R}_+)$ *and* y *is given by formula* (4).

Remark 1: The improper integral $\int_0^\infty t^{\kappa-1}\exp(-\tau t)dt$ $(\tau > 0)$ is convergent if and only if $\kappa > 0$. Hence, if $c_k = 0$ for $k \in \{1,2, \ldots, p\}$ and such that $1+\alpha+\beta-k \le 0$, then the function $x^\beta w(x)$ belongs to $L_\tau(\mathbb{R}_+)$.

Remark 2: We want to point at two special cases. For $\beta = 0$ the solution of equation (1) denoted by y^0 has the form

$$y^0(x) = \sum_{k=1}^{p} c_k x^{\alpha-k} E_{\alpha,1+\alpha-k}(\upsilon x^\alpha) + \int_0^x t^{\alpha-1}E_{\alpha,\alpha}(\upsilon t^\alpha)h(x-t)dt,$$

where $E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} z^n/\Gamma(n\rho+\mu)$ with $z \in \mathbb{C}$, $\rho,\mu > 0$ is the Mittag-Leffler function. In the case $h(x) = \gamma x^{\kappa-1}$ $(\kappa > 0)$ the function

$$y(x) = \sum_{k=1}^{p} c_k x^{\alpha-k}\left(1 + \sum_{l=1}^{\infty}\left(\upsilon x^{\alpha+\beta}/\Gamma(\alpha)\right)^{l} \prod_{j=1}^{l} B\left(\alpha, 1-k+j(\alpha+\beta)\right)\right)\Big/\Gamma(1+\alpha-k)$$

$$+ \ (\gamma/\Gamma(\alpha))x^{\alpha+\kappa-1} \sum_{n=0}^{\infty} \left(\upsilon x^{\alpha+\beta}/\Gamma(\alpha)\right)^n \prod_{j=0}^{n} B\left(\alpha, \kappa+j(\alpha+\beta)\right)$$

is a solution of the Cauchy problem for equation (1). Moreover, the function $y(x) = \gamma x^{\alpha+\kappa-1}/(\Gamma(\alpha+\kappa)/\Gamma(\kappa) - \upsilon)$ is a solution of equation (1) with $\beta = -\alpha$ and $\upsilon \neq \Gamma(\alpha+\kappa)/\Gamma(\kappa)$ (cf. also WIENER [10, pp. 165-166]).

Remark 3: The presented method allows to examine the more general equation $y^{(\alpha)}(x) - g(x)y(x) = h(x)$, $x > 0$, with the function g being measurable and such that the estimate $|g(x)| \le \upsilon x^{\beta}$ holds a.e. on $\mathbb{R}_+$.

The above-obtained theorem yields the following corollaries.

Corollary 1: *Assume that* $c_k \ge 0$ $(k=1,2,\ldots,p)$, $\upsilon \ge 0$ *and* $h \ge 0$ *a.e. on* $(0,\delta) \subset (0,1)$. *If* y *and* y^0 *are two solutions of the Cauchy problem for equation* (1) *with* $\beta < 0$ *and* $\beta = 0$, *respectively, then* $y > y^0 > 0$ *a.e. on* $(0,\delta)$. *Moreover, if* $\alpha \notin \mathbb{N}$ *and* $c_p > 0$, *then* $y(0+) = \infty$.

Proof. By the method of mathematical induction one can assert that for every $n \in \mathbb{N}$ the relation $K_n(x,t)t^{-\beta} > (\Gamma(\alpha))^n(x-t)^{n\alpha-1}/\Gamma(n\alpha)$ holds. And if $x < 1$ then

$$x^{\alpha+\beta}B\left(\alpha, 1-k+j(\alpha+\beta)\right) = x^{\alpha+\beta}\int_0^1 t^{\alpha-1}(1-t)^{j(\alpha+\beta)-k}dt$$

$$> x^{\alpha}\int_0^1 t^{\alpha-1}(1-t)^{j\alpha-k}dt = x^{\alpha}B(\alpha, 1-k+j\alpha).$$

Hence, bearing in mind formula (4), we get our assertion ∎

Corollary 2: *If* $c_k \ge 0$ $(k=1,2,\ldots,p)$, $\sum_{k=1}^{p}|c_k| > 0$, $\upsilon \ge 0$ *and* $h \ge 0$ *a. e. on* $\mathbb{R}_+$, *then* $y > 0$ *a.e. on* $\mathbb{R}_+$ *and* $y(\infty) = \infty$, *provided that* y *is a solution of the Cauchy problem for equation* (1).

3. Continuous dependence of solution. If $s = y^{(\alpha)} \in L_\tau(\mathbb{R}_+)$, then $y = w + s_\alpha$ is measurable and, moreover, $\|y\|_\tau \le \tau^{-\alpha}\left(\sum_{k=1}^{p} c_k \tau^{k-1} + \|s\|_\tau\right)$, what imply that $y \in L_\tau(\mathbb{R}_+)$. Now, we can establish

Theorem 2: *Let the assumptions of Section 2 be fulfilled, let* y^1 *and* y^2 *be the solutions of the Cauchy problem for equation* (1) *with the initial constants* c_k^1 *and* c_k^2 $(k=1,2,\ldots,p)$, *and the right-hand sides* h^1 *and* h^2, *respectively. If for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *with* $\max_k |c_k^1 - c_k^2| < \delta$ *and* $\|h^1 - h^2\|_\tau < \delta$, *then* $\|y^1 - y^2\|_\tau < \varepsilon$.

Proof. Denote $\delta_k = |c_k^1 - c_k^2|$ and $h(t) = |h^1(t) - h^2(t)|$. Using mathematical induction it be can proved that the estimate $K_n(x,t)t^{-\beta} \le (x-t)^{n\alpha+\beta-1} \times\prod_{j=1}^{n-1}B(\alpha, j\alpha+\beta)$ holds for integers $n \ge 2$. Hence, bearing in mind formula (4) and the definition of, the norm $\|\cdot\|_\tau$, we have

$$\|y^1 - y^2\|_\tau \le \|h\|_\tau\left(\tau^{-\alpha} + (\tau^{-\beta}/\upsilon)\sum_{l=2}^{\infty}\left(\upsilon\tau^{-\alpha}/\Gamma(\alpha)\right)^l\Gamma\left((l+1)\alpha+\beta\right)\prod_{j=1}^{l-1}B(\alpha, j\alpha+\beta)\right)$$

$$+ \sum_{k=1}^{p}\delta_k\tau^{-(1+\alpha-k)}\left(1 + \sum_{l=1}^{\infty}\left(\upsilon\tau^{-(\alpha+\beta)}/\Gamma(\alpha)\right)^l\Gamma\left(1+\alpha-k+l(\alpha+\beta)\right)\Big/\Gamma(1+\alpha-k)\right.$$

$$\left.\times\prod_{j=1}^{l}B\left(\alpha, 1-k+j(\alpha+\beta)\right)\right).$$

One can prove that for every finite value of τ the infinite series appearing on the right-hand side of the above inequality are finite. Thus, the relations $\|y^1 - y^2\|_\tau \le \text{const}\cdot\delta = \varepsilon$, with $\max_k \delta_k < \delta$ and $\|h\|_\tau < \delta$, hold ∎

4. The multipoint problem for equation (1). Let r be the largest integer such that $1+\alpha+\beta-r > 0$. We will consider equation (1) with the multipoint conditions $y_{p-\alpha}^{(p-m)}(x_m) = \eta_m$ $(m=1,2,\dots,r)$, where $x_m \in \mathbb{R}_+$ and $\eta_m \in \mathbb{R}$ are given (for more general multipoint problems cf. [4]).

Imposing the multipoint conditions on the general solution (4) of equation (1) we get the following algebraic system with respect to c_k $(k=1,2,\dots,r$; obviously $c_k = 0$ for $r < k \le p$, cf. Remark 1):

$$\sum_{k=1}^{r}c_k x_m^{m-k}\left(1/\Gamma(1+m-k) + \sum_{l=1}^{\infty}\left(\upsilon x_m^{\alpha+\beta}/\Gamma(\alpha)\right)^l\prod_{j=1}^{l}B\left(\alpha, 1-k+j(\alpha+\beta)\right)\right. \tag{5}$$

$$\left.\times\,\Gamma\left(1+\alpha-k+l(\alpha+\beta)\right)\Big/\left(\Gamma\left(1+m-k+l(\alpha+\beta)\right)\Gamma(1+\alpha-k)\right)\right) = \eta_m - y_N^{(\alpha-m)}(x_m),$$

where y_N is a solution of the non-homogeneous equation (1) with homogeneous initial conditions. Bearing in mind Section 2, the above considerations and the theory of the algebraic systems, one can formulate

Theorem 3: *If the assumptions of Section 2 are fulfilled and the algebraic system* (5) *has a solution, then the multipoint problem for equation* (1) *has a solution in the class of locally integrable functions such that their derivatives of order α belongs to $L_\tau(\mathbb{R}_+)$. Moreover, if the determinant of the system* (5) *is not equal* 0, *then the solution is unique.*

Remark 4: Consider a solution y of the above-set multipoint problem. We say that the solution y is *stable at infinity* if there exists a > 0 such that for every $\varepsilon > 0$ there is a $\delta > 0$ with $\max_k |\eta_k - \tilde{\eta}_k| < \delta$ for every $(\tilde{\eta}_k)_1^r \in \mathbb{R}^r$, then $|y(x) - \tilde{y}(x)| < \varepsilon$ a. e. for $x > a$, where $\tilde{y}$ is an arbitrary solution of the multipoint problem for equation (1), with the multipoint constants $\tilde{\eta}_m$ (m=1,2,...,r).

In virtue of this definition and formula (4), one can immediately notice that the solution y of the Cauchy problem for equation (1) is stable at infinity if and only if y = 0 is stable at infinity solution of the homogeneous Cauchy problem for the homogeneous equation (1) (i. e. $h \equiv 0$ and $c_k = 0$ for k=1,2,...,p). Let us observe that if $\upsilon > 0$ the solution is not stable at infinity.

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