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Accelerating Convergence of Univariate and Bivariate Fourier Approximations

M. TASCHE

Dedicated to Professor Lothar Berg on the occasion of his 60th birthday

Some new results concerning the accelerating convergence of univariate and bivariate Fourier expansions are presented. The rate of convergence is estimated with respect to the uniform norm. This acceleration method is a combination of algebraic and trigonometric approximation. Note that the abstract Taylor formula is an essential tool of this approach.

Key words: *Trigonometric approximation and interpolation, Krylov-Lanczos method, accelerating convergence of Fourier expansions*
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1. Introduction

If a complex-valued function f defined on $P = [0, 2\pi]$ is sufficiently smooth, then the asymptotic behaviour of its Fourier coefficients

$$a_k(f) = \frac{1}{2\pi}\int_P f(u)\, e_{-k}(u)\, du \qquad (k = 0, \pm 1, \dots),$$

where $e_k(x) = \exp(\mathrm{i}kx)$, and the rate of convergence of its Fourier series $\sum a_k(f) e_k$ depend only on the boundary values of f, more explicitly on the largest positive integer r with

$$f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j = 0, \dots, r). \tag{1.1}$$

It is known that a smooth function f with fulfilled property (1.1) possesses a rapidly convergent Fourier expansion. But there are also smooth functions, whose Fourier series converge extreme slowly. Therefore, if f is smooth but (1.1) fails to be satisfied, it has been proposed by A. N. KRYLOV and later by C. LANCZOS [11] to determine an algebraic polynomial h such that $g = f - h$ satisfies (1.1). This Krylov-Lanczos method of accelerating convergence of Fourier expansions and its natural generalizations were studied in several papers [1-3, 13, 14]. Using the abstract Taylor formula [12] and parametric extensions [4], we will present a simple approach to the Krylov-Lanczos method for univariate and bivariate Fourier expansions. Further we will prove some new C-estimates for the rate of convergence of the Krylov-Lanczos method.

In the following we use standard notations. Let $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{R}$ be the set of all integers, non-negative integers, natural and real numbers, respectively. For $r \in \mathbb{N}_0$ we define

$C^r(P)$ as the set of all complex-valued functions f defined on P with continuous derivatives $f^{(j)}$ ($j = 0, \ldots, r$; $f^{(0)} = f$). By $C^r_{2\pi}$ we denote the set of all $f \in C^r(P)$ such that every $f^{(j)}$ ($j = 0, \ldots, r$) has a continuous and 2π-periodic extension to $\mathbb{R}$, i.e. every f satisfies the boundary conditions (1.1). We write $C(P)$ and $C_{2\pi}$ instead of $C^0(P)$ and $C^0_{2\pi}$, respectively. The uniform norm with respect to P is denoted by $\|\cdot\|$. Further, let $L^1(P)$ be the usual Lebesgue space normed by $\|f\|_1 = 1/2\pi \int_P |f(u)|\, du$.

Introducing the *r-th Bernoulli polynomial* B_r ($r \in \mathbb{N}_0$) on P by

$$B_r' = B_{r-1}\,,\ B_0 = -1 \ \text{ with } \ B_1(0) = \pi\,,\ B_{2r+1}(0) = B_{2r+1}(2\pi) = 0 \quad (r \in \mathbb{N})$$

and its 2π-periodic extension b_r by

$$b_r(x) = B_r(x) \ \ (r \geq 2, x \in P), \quad b_1(x) = B_1(x) \qquad (x \in (0, 2\pi)),$$
$$b_1(0) = b_1(2\pi) = 0, \quad b_r(x + 2\pi) = b_r(x) \qquad (r \in \mathbb{N}, x \in \mathbb{R}),$$

we get

$$b_r = \sum_{|k|>0} (ik)^{-r} e_k \qquad (r \in \mathbb{N}). \tag{1.2}$$

The function b_r is called *r-th Bernoulli function*. Note that for $r \geq 2$ the Fourier series (1.2) is uniformly convergent on $\mathbb{R}$. If $r = 1$, then (1.2) converges with uniformly bounded partial sums and (1.2) is uniformly convergent on every compact subset of $\mathbb{R} \setminus 2\pi\mathbb{Z}$.

For any two functions $f \in C(P)$ and $g \in L^1(P)$ the *convolution* $f * g$ is explained by

$$(f * g)(x) = \frac{1}{2\pi}\left(\int_0^x f(x-u)g(u)\,du + \int_x^{2\pi} f(2\pi + x - u)g(u)\,du \right).$$

Then we have $f * g \in C(P)$ and $\|f * g\| \leq \|f\|\,\|g\|_1$. Since $C(P) \subset L^1(P)$, the convolution is defined on $C(P)$ and it is a commutative, associative and distributive operation. If $a_k(f)$ ($k \in \mathbb{Z}$) denotes the k-th Fourier coefficient of $f \in C(P)$, then it follows that $f * e_k = a_k(f)e_k$ and $a_k(f * g) = a_k(f)\,a_k(g)$ for all $k \in \mathbb{Z}$ and $f, g \in C(P)$. Especially, for any $k, m \in \mathbb{Z}$ we obtain $e_k * e_m = \delta_{km} e_k$, where δ_{km} denotes the Kronecker symbol.

Introducing the linear operators

$$D\colon\ C^1(P) \to C(P) \quad \text{and} \quad T\colon\ C(P) \to C^1(P) \cap C_{2\pi}$$

(see [2, 4]) by

$$Df = f' + a_0(f - f') \qquad (f \in C^1(P)),$$
$$Tf = (f * B_1) + a_0(f) = f * (B_1 + 1) \qquad (f \in C(P)),$$

we see that T is a bounded right inverse of D, since $DTf = f$ and $\|Tf\| \leq \left(\|B_1\|_1 + 1\right)\|f\|$ for all $f \in C(P)$. Further, it follows that

$$\begin{matrix} DB_1 = 0,\ DB_{r+1} = B_r \\ T1 = 1,\ TB_r = B_{r+1} \end{matrix} \qquad (r \in \mathbb{N}) \quad \text{and} \quad a_0(Tf) = a_0(f) \quad (f \in C(P)).$$

The kernel of D coincides with span$\{B_1\}$. The so-called *initial operator* F_1 of D corresponding to T (see [12]) is explained by

$$F_1 f = f - TDf \quad (f \in C^1(P)).$$

Then we find that $F_1 f = \frac{1}{2\pi}(f(0) - f(2\pi))B_1$, so that F_1 can be extended to $C(P)$. Hence, F_1 is a projector of $C(P)$ onto $\ker D = \text{span}\{B_1\}$, i.e. $F_1^2 = F_1$, $F_1(C(P)) = \ker D$ and $F_1 T = 0$. Further, $I - F_1$ is a projector of $C(P)$ onto $C_{2\pi}$, where I denotes the identity operator.

By induction it follows for $r \in \mathbb{N}$ that

$$D^r f = f^{(r)} + a_0(f - f^{(r)}) \qquad (f \in C^r(P)), \tag{1.3}$$

$$T^r f = (f * B_r) + a_0(f) = f * (B_r + 1) \qquad (f \in C(P)). \tag{1.4}$$

Then T^r is a bounded right inverse of D^r, since $D^r T^r f = f$ and $\|T^r f\| \leq (\|B_r\|_1 + 1)\|f\|$ hold for all $f \in C(P)$. Note that $\|B_1\|_1 = \pi/2$ and $\|B_r\|_1 \leq \|B_r\| \leq 2/(1 - 2^{1-r})$ $(r \geq 2)$. The kernel of D^r coincides with the set $\mathcal{P}_{r,0} = \text{span}\{B_1, \ldots, B_r\}$ of all algebraic polynomials p with degree $p \leq r$ and $a_0(p) = 0$. The initial operator F_r of D^r corresponding to T^r is defined for $f \in C^r(P)$ by

$$F_r f = f - T^r D^r f = \sum_{j=0}^{r-1} T^j F_1 D^j f = \frac{1}{2\pi} \sum_{j=0}^{r-1} \left(f^{(j)}(0) - f^{(j)}(2\pi)\right) B_{j+1} \tag{1.5}$$

(see [12]), so that F_r can be extended to $C^{r-1}(P)$. Consequently, F_r is a projector of $C^{r-1}(P)$ onto $\ker D^r = \mathcal{P}_{r,0}$. Further, $I - F_r$ is a projector of $C^{r-1}(P)$ onto $C_{2\pi}^{r-1}$. For $f \in C^r(P)$ we get the estimate

$$\|f - a_0(f) - F_r f\| = \|T^r D^r f - a_0(f)\| = \|f^{(r)} * B_r\| \leq \|B_r\|_1 \|f^{(r)}\|.$$

Then the abstract Taylor formula [12]

$$f = F_r f + T^r D^r f = \sum_{j=0}^{r-1} T^j F_1 D^j f + T^r D^r f \quad \text{for } f \in C^r(P) \tag{1.6}$$

yields the following

Theorem 1.1 (see[2]): *For every $f \in C^r(P)$ $(r \in \mathbb{N})$ we have*

$$f = F_r f + a_0(f) + f^{(r)} * B_r. \tag{1.7}$$

Further, F_r is a projector of $C^{(r-1)}(P)$ onto $\ker D^r = \mathcal{P}_{r,0}$ and $I - F_r$ is a projector of $C^{r-1}(P)$ onto $C_{2\pi}^{r-1}$.

Proof: Formula (1.7) follows immediately from the abstract Taylor formula (1.6) and from the representations (1.3) - (1.4). By (1.5) we see that $F_r f \in \ker D^r$ for $f \in C^{r-1}(P)$. Since $F_r g = (I - T^r D^r)g$ for $g \in C^r(P)$, we obtain, for arbitrary $f \in C^r(P)$, $F_r f \in C^r(P)$ and hence $F_r^2 f = (I - T^r D^r)F_r f = F_r f - T^r D^r F_r f = F_r f$. Further, if p is an arbitrary algebraic polynomial of degree $p \leq r$ with $a_0(p) = 0$, then $T^r D^r p = 0$ and $p = F_r p$. Thus

F_r is a projector of $C^{r-1}(P)$ onto $\ker D^r$. Hence $I - F_r$ is a projector, too. Setting $h = (I - F_r)f$ for given $f \in C^{r-1}(P)$, we obtain

$$h^{(j)}(0) = h^{(j)}(2\pi) = \tfrac{1}{2}\big(f^{(j)}(0) - f^{(j)}(2\pi)\big) + \delta_{0j}\, a_0(f) \quad (j = 0,\dots, r-1),$$

i.e. $h \in C^{r-1}_{2\pi}$. Obviously, if $f \in C^{r-1}_{2\pi}$, then $F_r f = 0$ and $f = (I - F_r)f$. Consequently, $I - F_r$ is a projector of $C^{r-1}(P)$ onto $C^{r-1}_{2\pi}$ ∎

Note that the representation (1.7) of f is valid on the whole of P unlike a corresponding result of [2].

Corollary 1.2 : *Any function $f \in C^r(P) \cap C^{r-1}_{2\pi}$ can be represented in the form $f = T^r D^r f = a_0(f) + f^{(r)} * B_r$.*

This follows directly from (1.7). By Corollary 1.2 we obtain the interesting estimate

$$\|f - a_0(f)\| \le \|B_r\|_1 \, \|f^{(r)}\| \le 2(1 - 2^{1-r})^{-1} \|f^{(r)}\| \text{ for } f \in C^r(P) \cap C^{r-1}_{2\pi} \quad (r \ge 2).$$

2. The Krylov - Lanczos decomposition

It is known that for a function $f \in C^r_{2\pi}$ ($r \in \mathbb{N}$) the corresponding Fourier series converges rapidly (see Corollary 2.3). Unfortunately, there are smooth functions $f \in C^r(P)$ - for example algebraic polynomials - whose Fourier series converge extreme slowly. Since very often a given function $f \in C^r(P)$ does not fulfil the boundary conditions (1.1), it has been proposed by A.N.KRYLOV and by C.LANCZOS [11] to use an algebraic pre-approximation h to f (constructed by two-point interpolation), such that the remainder $g = f - h$ belongs to $C^r(P) \cap C^{r-1}_{2\pi}$ and has a rapidly convergent Fourier expansion. Choosing $h = F_r f$, the Krylov - Lanczos method is an immediate consequence of the Taylor formula (1.7).

Theorem 2.1 [2]: *If $f \in C^r(P)$ ($r \ge 1$), then we have the Krylov - Lanczos decomposition*

$$f = F_r f + a_0(f) + \sum_{|k|>0} (ik)^{-r} a_k(f^{(r)}) e_k ,$$

where $F_r f$ is the algebraic polynomial (1.5).

Proof : From (1.2) and from Lebesgue's bounded convergence theorem it follows that

$$f^{(r)} * B_r = f^{(r)} * b_r = \sum_{|k|>0} (ik)^{-r} a_k(f^{(r)}) e_k .$$

Then, by (1.7), we obtain immediately the assertion. In the case $r \ge 2$ we get the estimate

$$\|f - a_0(f) - F_r f\| \le \|B_r\|_1 \, \|f^{(r)}\| \le 2(1 - 2^{1-r})^{-1} \|f^{(r)}\| \quad \text{for } f \in C^r(P).$$

This completes the proof ∎

Remark : A similar method can be applied for accelerating the convergence of eigenfunction expansions associated with self-adjoint boundary-value problems [13, 14].

For $n \in \mathbb{N}$ and $f \in C(P)$ let us denote by $S_n f$ the n-*th Fourier partial sum*

$$S_n f = \sum_{k=-n}^{n} a_k(f) e_k .$$

It is known [7] that the norm of $S_n : C(P) \to C(P)$ can be estimated by

$$\|S_n\| \le \frac{3}{2} + \frac{4}{\pi^2} \ln n . \tag{2.1}$$

Then we obtain the following error estimate with respect to the uniform norm.

Theorem 2.2 : *If* $f \in C^r(P)$ $(r \ge 1)$ *and* $g := f - F_{r+1} f$, *then*

$$\|g - S_n g\| \le \frac{3\pi}{2}\left(\frac{5}{2} + \frac{4}{\pi^2} \ln n\right)(n+1)^{-r} \|f^{(r)}\| \quad \textit{for all } n \in \mathbb{N} .$$

Proof: Let $\mathcal{T}_n$ be the set of all trigonometric polynomials p of degree $p \le n$. By Theorem 1.1 we have $g \in C_{2\pi}^r$. Let $p^* \in \mathcal{T}_n$ be the best approximation to g with respect to the uniform norm. Hence for $E_n(g)$, the n-th degree of approximation of g by trigonometric polynomials, it follows that $E_n(g) = \inf\{\|g - p\| : p \in \mathcal{T}_n\} = \|g - p^*\|$. Note that $p^* = S_n p^*$. Then we obtain

$$\|g - S_n g\| \le \|g - p^*\| + \|S_n(p^* - g)\| \le (1 + \|S_n\|)\,\|p^* - g\| = (1 + \|S_n\|) E_n(g).$$

Using Jackson's theorem for $g \in C_{2\pi}^r$ (see [10]) we get the estimate

$$E_n(g) \le \frac{\pi}{2}(n+1)^{-r} \|g^{(r)}\| \qquad (n \in \mathbb{N})$$

with

$$g^{(r)} = f^{(r)} - a_0(f^{(r)}) - (2\pi)^{-1}\left(f^{(r)}(0) - f^{(r)}(2\pi)\right) B_1 .$$

By $\|g^{(r)}\| \le 3\,\|f^{(r)}\|$ we obtain the stated error estimate ∎

Remark : Analogously, we can prove error estimates with respect to an L^p-norm $(1 \le p < \infty)$. A corresponding asymptotic error estimate in the uniform norm can be found in [2].

In the case $f \in C_{2\pi}^r$ $(r \in \mathbb{N})$, Theorem 2.2 yields the following known result.

Corollary 2.3 : *If* $f \in C_{2\pi}^r$ $(r \in \mathbb{N})$, *then we have the estimate*

$$\|f - S_n f\| \le \frac{\pi}{2}\left(\frac{5}{2} + \frac{4}{\pi^2} \ln n\right)(n+1)^{-r} \|f^{(r)}\| \quad \textit{for all } n \in \mathbb{N} .$$

Now we formulate an analogous result for accelerating convergence of trigonometric interpolation. For this end we consider the *trigonometric interpolation operator* $L_n : C(P)$

$\to \mathcal{T}_n$ defined by

$$L_n f = \sum_{k=-n}^{n} a_k^{(n)}(f) e_k, \quad a_k^{(n)}(f) = \frac{1}{2n+1} \sum_{m=0}^{2n} f(mh)\, e^{-ikmh}$$

($a_k^{(n)}$ - *discrete Fourier coefficients*) with $h = 2\pi/(2n+1)$. By [5], it follows that for $n \geq 4$ the norm of $L_n : C(P) \to C(P)$ can be estimated by $\|L_n\| \leq 3/2 + (2/\pi)\ln n$.

Theorem 2.4: *If $f \in C^r(P)$ ($r \in \mathbb{N}$) and $g := f - F_{r+1} f \in C_{2\pi}^r$, then we have the estimate*

$$\|g - L_n g\| \leq \frac{3\pi}{2}\left(\frac{5}{2} + \frac{2}{\pi}\ln n\right)(n+1)^{-r}\|f^{(r)}\| \quad \textit{for all } n \geq 4.$$

The **proof** is similar to that of Theorem 2.2 and is omitted here ∎

The same procedure can be also applied to an efficient computation of the Fourier coefficients and of the discrete Fourier coefficients for a given smooth function.

Theorem 2.5: *If $f \in C^r(P)$ ($r \in \mathbb{N}$) and $g = f - F_{r+1} f$, then we have*

$$a_k(g) = a_k(f) - \frac{1 - \delta_{0k}}{2\pi} \sum_{j=0}^{r} \left(f^{(j)}(0) - f^{(j)}(2\pi)\right)(ik)^{-j-1} \qquad (k \in \mathbb{Z})$$

$$a_k^{(n)}(g) = a_k^{(n)}(f) - \frac{1}{2\pi} \sum_{j=0}^{r} \left(f^{(j)}(0) - f^{(j)}(2\pi)\right) a_k^{(n)}(B_{j+1}) \quad (|k| \leq n).$$

Further, if $b_k^{(n)}(f) = a_k^{(n)}(g) + a_k(F_{r+1} f)$, then

$$b_k^{(n)}(f) = a_k^{(n)}(f) + \frac{1}{2\pi} \sum_{j=0}^{r} \left(f^{(j)}(0) - f^{(j)}(2\pi)\right)\left((ik)^{-j-1} - a_k^{(n)}(B_{j+1})\right)$$

is a better approximation of the Fourier coefficient $a_k(f)$ than $a_k^{(n)}(f)$:

$$\left|a_k(f) - b_k^{(n)}(f)\right| \leq 3\pi(n+1)^{-r}\|f^{(r)}\|.$$

Proof: The representations $a_k(g)$, $a_k^{(n)}(g)$ and $b_k^{(n)}(f)$ follow directly from (1.2) and (1.5). By Theorem 1.1 we have $g \in C_{2\pi}^r$. Let $p^* \in \mathcal{T}_n$ be the best approximation to g with respect to the uniform norm. By Jackson's theorem [10] we get for $n \in \mathbb{N}$ the estimate

$$\|g - p^*\| \leq \frac{\pi}{2}(n+1)^{-r}\|g^{(r)}\| \leq \frac{3\pi}{2}(n+1)^{-r}\|f^{(r)}\|.$$

Note that $a_k(p^*) = a_k^{(n)}(p^*)$ ($|k| \leq n$). Hence we obtain

$$\left|a_k(f) - b_k^{(n)}(f)\right| = \left|a_k(g) - a_k^{(n)}(g)\right| \leq \left|a_k(g - p^*)\right| + \left|a_k^{(n)}(p^* - g)\right|$$

$$\leq 2\,\|g - p^*\| \leq 3\pi(n+1)^{-r}\|f^{(r)}\|.$$

This completes the proof ∎

3. Parametric extensions

The above one - dimensional approach to the Krylov - Lanczos method can be extented to the bivariate case by Gordon's blending function method [2, 4, 6]. Let $Q = P \times P$. For $r \in \mathbb{N}_0$, let $C^{r,r}(Q)$ denote the set of all functions $f: Q \to \mathbb{C}$ with continuous partial derivatives $f^{(j,k)}$ $(j,k = 0,\dots,r)$. Further, $C^{r,r}_{2\pi,2\pi}$ is the set of all $f \in C^{r,r}(Q)$ with

$$f^{(j,k)}(0,y) = f^{(j,k)}(2\pi,y)\ ,\quad f^{(j,k)}(x,0) = f^{(j,k)}(x,2\pi)$$

for all $x, y \in P$ and $j,k = 0,\dots,r$. We write $C(Q)$ and $C_{2\pi,2\pi}$ instead of $C^{0,0}(Q)$ and $C^{0,0}_{2\pi,2\pi}$, respectively. The uniform norm on Q is denoted by $\|\cdot\|$.

The parametric extensions of D are defined on Q by

$$D'f = f^{(1,0)} + a_0'\left(f - f^{(1,0)}\right) \qquad \left(f \in C^{1,0}(Q)\right),$$

$$D''f = f^{(0,1)} + a_0''\left(f - f^{(0,1)}\right) \qquad \left(f \in C^{0,1}(Q)\right),$$

where

$$a_0'(f)(y) = \tfrac{1}{2\pi}\int_P f(u,y)\,du\ ,\qquad a_0''(f)(x) = \tfrac{1}{2\pi}\int_P f(x,v)\,dv.$$

Further, the parametric extensions of T are explained on Q by

$$\left.\begin{aligned}(T'f)(x,y) &= \tfrac{1}{2\pi}\int_P (b_1(x-u)+1)\,f(u,y)\,du\\ (T''f)(x,y) &= \tfrac{1}{2\pi}\int_P (b_1(y-v)+1)\,f(x,v)\,dv\end{aligned}\right\}\qquad (f \in C(Q)).$$

Finally, the parametric extensions of F_1 are introduced on Q by

$$\left.\begin{aligned}(F_1'f)(x,y) &= \tfrac{1}{2\pi}\left(f(0,y) - f(2\pi,y)\right)B_1(x)\\ (F_1''f)(x,y) &= \tfrac{1}{2\pi}\left(f(x,0) - f(x,2\pi)\right)B_1(y)\end{aligned}\right\}\qquad \left(f \in C(Q)\right).$$

Note that $D'T' = D''T'' = I$, F_1' is the initial operator of D' corresponding to T' and F_1'' is the initial operator of D'' corresponding to T''. Further, we see that parametric extensions commute, i.e. $D'D'' = D''D'$, $T'T'' = T''T'$, $F_1'F_1'' = F_1''F_1'$, $D'T'' = T''D'$, $D'F_1'' = F_1''D'$, ... For example, for $f \in C(Q)$ we have

$$4\pi^2(F_1'F_1''f)(x,y) = \left(f(0,0) - f(0,2\pi) - f(2\pi,0) + f(2\pi,2\pi)\right)B_1(x)B_1(y).$$

Now we consider the operators

$$D = D'D'':\ C^{1,1}(Q) \to C(Q)\ ,\qquad T = T'T'':\ C(Q) \to C^{1,1}(Q) \cap C_{2\pi,2\pi}$$

and the Boolean sum operator

$$F_1 = F_1' \oplus F_1'' = F_1' + F_1'' - F_1'F_1''.$$

Then these operators are given for related functions f by

$$Df = f^{(1,1)} + a_0'\left(f^{(0,1)} - f^{(1,1)}\right) + a_0''\left(f^{(1,0)} - f^{(1,1)}\right) + a_{00}\left(f - f^{(0,1)} - f^{(1,0)} + f^{(1,1)}\right),$$

where

$$4\pi^2 a_{00}(f) = \iint_Q f(u,v)\,du\,dv,$$

and

$$4\pi^2 (Tf)(x,y) = \iint_Q \left(b_1(x-u)+1\right)\left(b_1(y-v)+1\right) f(u,v)\,du\,dv,$$

$$2\pi(F_1 f)(x,y) = \left(f(0,y) - f(2\pi,y)\right)B_1(x) + \left(f(x,0) - f(x,2\pi)\right)B_1(y)$$
$$- \tfrac{1}{2\pi}\left(f(0,0) - f(0,2\pi) - f(2\pi,0) + f(2\pi,2\pi\right)B_1(x)B_1(y).$$

Since $DTf = D'(D''T'')T'f = D'T'f = f$ holds for all $f \in C(Q)$, the operator T is a linear bounded right inverse of D. For $f \in C^{1,1}(Q)$ we get the Taylor formula $TDf = f - F_1 f$ by

$$TDf = T'(T''D'')D'f = T'(I - F_1'')D'f = T'D'f - (T'D')F_1''f$$
$$= f - F_1'f - F_1''f + F_1'F_1''f = f - F_1 f.$$

Thus, F_1 is the initial operator of D corresponding to T. Further, $I - F_1 = (I - F_1')(I - F_1'')$ is a projector of $C(Q)$ onto $C_{2\pi,2\pi}$. For $f \in C^{1,1}(Q)$ we obtain the Taylor formula $f = F_1 f + TDf$ with

$$(TDf)(x,y) = a_{00}(f) + \frac{1}{4\pi^2}\iint_Q b_1(x-u) f^{(1,0)}(u,v)\,du\,dv$$
$$+ \frac{1}{4\pi^2}\iint_Q b_1(y-v) f^{(0,1)}(u,v)\,du\,dv$$
$$+ \frac{1}{4\pi^2}\iint_Q b_1(x-u)\,b_1(y-v) f^{(1,1)}(u,v)\,du\,dv.$$

In the case $f \in C^{1,1}(Q) \cap C_{2\pi,2\pi}$ we have $F_1 f = 0$ and hence [8]

$$f(x,y) = (TDf)(x,y) = -a_{00}(f) + a_0''(f)(x) + a_0'(f)(y)$$
$$+ \frac{1}{4\pi^2}\iint_Q b_1(x-u)\,b_1(y-v) f^{(1,1)}(u,v)\,du\,dv \text{ for } (x,y) \in Q.$$

Considering the operators $D^r: C^{r,r}(Q) \to C(Q)$, $T^r: C(Q) \to C^{r,r}(Q) \cap C^{r-1,r-1}_{2\pi,2\pi}$ for $r \in \mathbb{N}$, it follows that T^r is a linear bounded right inverse of D^r. The initial operator F_r of D^r corresponding to T^r is given by

$$F_r = I - T^r D^r = \sum_{j=0}^{r-1} T^j F_1 D^j \quad \text{on} \quad C^{r,r}(Q).$$

From $D = D'D''$, $T = T'T''$ and $F_1 = F_1' \dotplus F_1''$ it follows that

$$F_r = F_r' \dotplus F_r'' \quad \text{and} \quad I - F_r = (I - F_r')(I - F_r'')$$

with

$$F_r' = I - (T')^r (D')^r = \sum_{j=0}^{r-1} (T')^j F_1' (D')^j \quad \text{on} \quad C^{r,0}(Q)$$

and

$$F_r'' = I - (T'')^r (D'')^r = \sum_{j=0}^{r-1} (T'')^j F_1''(D'')^j \quad \text{on } C^{0,r}(Q).$$

This yields, by (1.5),

$$\begin{aligned} 2\pi(F_r f)(x,y) = & \sum_{j=0}^{r-1}\Big(f^{(j,0)}(0,y) - f^{(j,0)}(2\pi,y)\Big)B_{j+1}(x) \\ & + \sum_{k=0}^{r-1}\Big(f^{(0,k)}(x,0) - f^{(0,k)}(x,2\pi)\Big)B_{k+1}(y) \\ & - \tfrac{1}{2\pi}\sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\Big(f^{(j,k)}(0,0) - f^{(j,k)}(0,2\pi) \\ & - f^{(j,k)}(2\pi,0) + f^{(j,k)}(2\pi,2\pi)\Big)B_{j+1}(x)B_{k+1}(y) \end{aligned} \tag{3.1}$$

for $f \in C^{r,r}(Q)$, so that F_r and $I - F_r$ can be extended to $C^{r-1,r-1}(Q)$. Thus, F_r is a projector of $C^{r-1,r-1}(Q)$ onto $\ker D^r$. Further, $I - F_r$ is a projector of $C^{r-1,r-1}(Q)$ onto $C^{r-1,r-1}_{2\pi,2\pi}$. An application of the abstract Taylor formula (1.6) yields the following

Theorem 3.1 (see [2]): *For every $f \in C^{r,r}(Q)$ $(r \in \mathbb{N})$ we have*

$$f = F_r f + T^r D^r f \tag{3.2}$$

with

$$\begin{aligned} (T^r D^r f)(x,y) = & \, a_{00}(f) + \tfrac{1}{4\pi^2}\iint_Q b_r(x-u) f^{(r,0)}(u,v)\,du\,dv \\ & + \tfrac{1}{4\pi^2}\iint_Q b_r(y-v) f^{(0,r)}(u,v)\,du\,dv \\ & + \tfrac{1}{4\pi^2}\iint_Q b_r(x-u)\, b_r(y-v) f^{(r,r)}(u,v)\,du\,dv. \end{aligned} \tag{3.3}$$

Note that this representation is valid on the whole of Q unlike a corresponding result of [2]. In the case $f \in C^{r,r}(Q) \cap C^{r-1,r-1}_{2\pi,2\pi}$ $(r \in \mathbb{N})$ we have $F_r f = 0$ and

$$\begin{aligned} f(x,y) = (T^r D^r f)(x,y) = & - a_{00}(f) + a_0''(f)(x) + a_0'(f)(y) \\ & + \tfrac{1}{4\pi^2}\iint b_r(x-u) b_r(y-v) f^{(r,r)}(u,v)\,du\,dv \quad \text{for } (x,y) \in Q \end{aligned}$$

(see [8]) and hence for $r \geq 2$ we get the estimate

$$\|f + a_{00}(f) - a_0'(f) - a_0''(f)\| \leq \|B_r\|_1^2\, \|f^{(r,r)}\| \leq 4\,(1 - 2^{1-r})^{-2}\, \|f^{(r,r)}\|.$$

4. Bivariate Fourier approximation

It is known [9] that for a given function $f \in C^{r,r}_{2\pi,2\pi}$ $(r \in \mathbb{N})$ the corresponding *Fourier coefficients*

$$a_{kl}(f) = \tfrac{1}{4\pi^2}\iint_Q f(u,v)\, e_{-k,-l}(u,v)\,du\,dv \qquad (k,l \in \mathbb{Z})$$

converge rapidly to 0 for $|k| + |l| \to \infty$. Here we use the notation $e_{kl}(x,y) = e_k(x)e_l(y)$. The asymptotic behaviour of the Fourier coefficients depends on the order of partial derivatives of f having a continuous periodic extension to $\mathbb{R}^2$. To obtain an acceleration method of the Fourier expansion for a given function $f \in C^{r,r}(Q)$, we have to specify a function $h \in C^{r,r}(Q)$ such that $f - h$ has this extension property. By Section 3, we can choose $h = F_r f$ or $h = F_{r+1}f$. This yields $f - F_r f \in C^{r-1,r-1}_{2\pi,2\pi}$ and $f - F_{r+1}f \in C^{r,r}_{2\pi,2\pi}$, respectively. Then the Krylov - Lanczos method for bivariate Fourier expansions follows immediately from the Taylor formula (3.2).

Theorem 4.1 : *If $f \in C^{r,r}(Q)$ $(r \geq 1)$, then we have the Krylov - Lanczos decomposition*

$$\begin{aligned} f(x,y) = {} & (F_r f)(x,y) + a_{00}(f) \\ & + \sum_{|k|>0} (ik)^{-r} a_{k0}(f^{(r,0)}) e_k(x) + \sum_{|l|>0} (il)^{-r} a_{0l}(f^{(0,r)}) e_l(y) \\ & + (-1)^r \sum_{|k|>0} \sum_{|l|>0} (kl)^{-r} a_{kl}(f^{(r,r)}) e_{kl}(x,y), \end{aligned}$$

where $F_r f$ is the blending interpolation function (3.1).

Proof : From (1.2), it follows for $r \in \mathbb{N}$ that

$$b_r(x)\, b_r(y) = \sum_{|k|>0} \sum_{|l|>0} (-1)^r (kl)^{-r} e_{kl}(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Hence we obtain

$$\iint_Q b_r(x-u) f^{(r,0)}(u,v)\, du\, dv = 4\pi^2 \sum_{|k|>0} (ik)^{-r} a_{k0}(f^{(r,0)}) e_k(x),$$

$$\iint_Q b_r(y-v) f^{(0,r)}(u,v)\, du\, dv = 4\pi^2 \sum_{|l|>0} (il)^{-r} a_{0l}(f^{(0,r)}) e_l(y),$$

$$\begin{aligned} & \iint_Q b_r(x-u)\, b_r(y-v) f^{(r,r)}(u,v)\, du\, dv \\ & \quad = (-1)^r 4\pi^2 \sum_{|k|>0} \sum_{|l|>0} (kl)^{-r} a_{kl}(f^{(r,r)}) e_{kl}(x,y) \end{aligned}$$

for $f \in C^{r,r}(Q)$. Using (3.2) and (3.3), we get the assertion ■

Remark : The same method was applied in [1, 3] for accelerating the convergence of a bivariate sine series expansion.

For $n \in \mathbb{N}_0$ and $f \in C(Q)$ we introduce the *Fourier partial sum*

$$S_{nn}f = \sum_{k=-n}^{n} \sum_{l=-n}^{n} a_{kl}(f)\, e_{kl}.$$

If we denote by $S_n' : C(Q) \to C(Q)$ and $S_n'' : C(Q) \to C(Q)$ the parametric extensions of the univariate Fourier partial sum operator $S_n : C(P) \to C(P)$, then we conclude by (2.1) that $S_{nn} = S_n' S_n''$, $\|S_n'\| = \|S_n''\| = \|S_n\|$ (see [4]) and

$$\|S_{nn}\| \le \|S_n'\| \, \|S_n''\| \le \left(\tfrac{3}{2} + \tfrac{4}{\pi^2} \ln n\right)^2.$$

Finally we obtain the following error estimate with respect to the uniform norm.

Theorem 4.2: *If* $f \in C^{r,r}(Q)$ $(r \in \mathbb{N})$ *and* $g = f - F_{r+1} f$, *then*

$$\|g - S_{nn} g\| \le c_n \, (n+1)^{-r} M_r(g) \quad \text{for all } n \in \mathbb{N},$$

with

$$c_n = \left(\pi + \frac{\pi^2}{8}\right)\left(\frac{13}{4} + \frac{12}{\pi^2} \ln n + \frac{16}{\pi^2} (\ln n)^2\right)$$

and

$$M_r(g) = \max\left(\|g^{(0,r)}\|, \|g^{(r,0)}\|, \|g^{(r,r)}\|\right).$$

Proof: By Section 3 we know that $g \in C^{r,r}_{2\pi,2\pi}$. Let $\mathcal{T}_{nn}$ denote the set of all bivariate trigonometric polynomials of the form

$$p = \sum_{k=-n}^{n} \sum_{l=-n}^{n} a_{kl} e_{kl} \qquad (a_{kl} \in \mathbb{C}).$$

The infimum of $\|g - p\|$ for all $p \in \mathcal{T}_{nn}$ is the degree of approximation $E_{nn}(g)$. Bivariate Jackson estimates can be obtained by treating the variables successively. This tensor product approach leads to rather sharp constants:

$$\begin{aligned} E_{nn}(g) &\le \tfrac{\pi}{2}(n+1)^{-r}\|g^{(r,0)}\| + \tfrac{\pi}{2}(n+1)^{-r}\|g^{(0,r)}\| + \tfrac{\pi^2}{4}(n+1)^{-2r}\|g^{(r,r)}\| \\ &\le (\pi + \pi^2/8)(n+1)^{-r} M_r(g) \quad (n \in \mathbb{N}). \end{aligned}$$

Let $p^* \in \mathcal{T}_{nn}$ be the best approximation to g with respect to the uniform norm. Then $E_{nn}(g) = \|g - p^*\|$. Since $p^* = S_{nn} p^*$, we obtain

$$\|g - S_{nn} g\| \le \|g - p^*\| + \|S_{nn}(p^* - g)\| \le (1 + \|S_{nn}\|) E_{nn}(g) \le c_n (n+1)^{-r} M_r(g)$$

for all $n \in \mathbb{N}$. By (3.1) and the proof of Theorem 2.1 we conclude that

$$M_1(g) \le 4\|f^{(1,1)}\| + \max\left(\|f^{(0,1)}\|, \|f^{(1,0)}\|\right),$$

$$M_r(g) \le 4(1 - 2^{1-r})^{-1}\|f^{(r,r)}\| + \max\left(\|f^{(0,r)}\|, \|f^{(r,0)}\|\right) \quad (r \ge 2).$$

This completes the proof ∎

Note that an analogous procedure as in Section 2 can be applied to an efficient computation of the bivariate Fourier coefficients and of the discrete bivariate Fourier coefficients for a given smooth function.

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Author's address:

Prof. Dr. Manfred Tasche
Fachbereich Mathematik der Universität Rostock
Universitätsplatz 1
D(Ost) · 2500 Rostock