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Parametric Quadratic Splines with Minimal Curvature

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Dedicated to Prof. L. Berg on the occasion of his 60th birthday

The concept of curvature-minimizing is extended to parametric polynomial splines of degree two. In contrast to the non-parametric case the resulting smooth curve is invariant under rotation of the co-ordinate system. Moreover, for a certain choice of the parameters (defining the functional to be minimized) it may be interpreted as a minimizer of the strain energy. For the case that the given data are points on a sufficiently smooth curve there is given an $O(h^2)$ error estimation (h - steplength).

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1. Introduction. The approximation of plane curves by interpolating parametric polynomial splines of degree three is a method well-known and often used (see, e.g., HANNA, EVANS and SCHWEITZER [6] and the references cited there). The usual, i.e. non-parametric, polynomial splines of degree two have been studied in papers by STEČKIN and SUBBOTIN [12], METTKE, PFEIFER and NEUMAN [11], MAESS [8,10] and others. Very often the main interest is in interpolating polynomials or polynomial splines preserving some characteristic properties of the given data, such as convexity or monotonicity (cf. BERG [1-3]) or possessing certain extremal properties, for instance a minimal curvature (cf. DIETZE and SCHMIDT [4]) or minimality in the sense of BERG [2].

In the present paper we extend the concept of minimizing the curvature (applied in [8, 10] to the common spline interpolation of degree two) to the parametric case. An interpretation of the functional used is given in Section 2. Our main results are the followings: the new concept is a generalization of the minimization of the total curvature considered in [8], but in contrast to the non-parametric case the resulting curve is invariant under rotation and thus preserving symmetry (Theorems 1 and 2 of Section 3). Finally we extend the error estimations from [8] to the parametric case.

In order to keep the paper self-contained we recall the basic results from [8-10] about non-parametric quadratic splines minimizing the total curvature. We define

$$p(x) = p_n(x) \quad \text{for} \quad x \in [x_{n-1}, x_n], \quad n = 1(1)N, \tag{1}$$

by

$$p_n(x_{n-1} + \tau h_n) = y_{n-1} + h_n d_{n-1}\tau + (g_n - d_{n-1}) h_n \tau^2, \tag{2}$$
$$0 \le \tau \le 1, \quad h_n = x_n - x_{n-1}, \quad g_n = (y_n - y_{n-1})/h_n,$$

with certain unknown parameters d_n. For given data points $(x_n, y_n) \in \mathbb{R}^2$, $n = 0(1)N$, we assume that (x_n) is a strictly monotonous sequence, i.e.,

$$\Pi_x : x_0 < x_1 < \ldots < x_N \tag{3}$$

is a partition of $[x_0, x_N]$. The parameters d_n are determined by the following conditions:

(i) The first derivative of p is continuous, $p \in C^1[x_0, x_N]$.

(ii) The function p minimizes the functional

$$Q(p) = \sum_{n=1}^{N} \omega_n \int_{x_{n-1}}^{x_n} p''(x)^2 dx \quad \text{(with certain weights } \omega_n > 0\text{)}. \tag{4}$$

From (i) we obtain the relation

$$d_n = 2g_n - d_{n-1}, \quad n = 1(1)N, \tag{5}$$

the starting value d_0 being uniquely defined by (ii) as

$$d_0 = g_1 - a_1^{-1} \sum_{j=2}^{N} (-1)^j a_j (g_j - g_{j-1}), \quad a_j = \sum_{n=j}^{N} \omega_n / h_n . \tag{6}$$

The special choice of the values $\omega_n = 1$ in (4) reduces Q to the functional from Holladays' well-known theorem on cubic splines. However, instead of minimizing the L^2-norm of the second derivative we may minimize the L^2-norm of the approximated total curvature or the approximated strain energy by choosing $\omega_n = (1+g_n^2)^{-3}$ or $\omega_n = (1+g_n^2)^{-5/2}$, respectively.

Especially the physical interpretation of the last choice of the functional together with the computational ease makes the method attractive for applications. Indeed, our numerical experience affirms that the quadratic splines defined above are considerably cheaper than cubic ones and, moreover, they are well-suited for convex data yielding even "visually smoother" results than interpolating splines of higher degree (cf. Figure 1 and [10]).

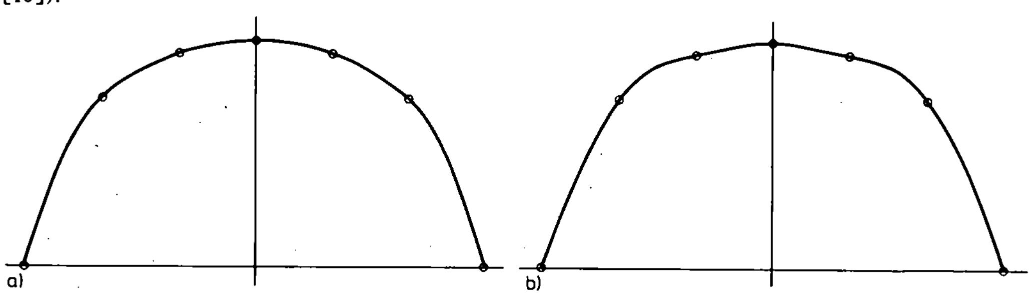

Fig. 1. Interpolation a) by quadratic splines with minimal total curvature, and b) by natural cubic splines

However, there are drawbacks of the method, too. So the interpolation is not invariant under rotation, although in the case of $\omega_n = (1+g_n^2)^{-5/2}$ the rotation invariant strain energy is approximated. For larger rotations, and generally for non-monotonous data ($\{x_n\}$ as well as $\{y_n\}$), the method breaks down at all. For this reason a generalization is needed, especially if

- closed curves and/or
- curves exhibiting some symmetry to be preserved

are considered as, e.g., for the interpolation of isolines or other graphical tools for diverse applications. In the present paper we use the concept of componentwise parametric interpolation by polynomial splines of degree two. For this case we choose parameter sets and weights in such a way that

- computational ease and physical interpretation are preserved,
- invariance under rotation and robustness are gained, and
- error estimates in terms of the steplength h of the same order as in the non-parametrized case (cf. [8]) may be given.

We conclude the paper with some refinements, so an iterative process for reparametrization and an interactive approach are proposed.

As an illustration of our results - closely related to the dedication of the paper - Figure 2 shows some curves computed by parametric quadratic spline interpolation. In all cases (with exception of the "3") 5 points $P_0, \ldots, P_4$ were given, with $P_4 = P_0$ for the "0" and $P_4 = P_1$ for the "6" and the "9", respectively. The "3" consists of two 4-point interpolations computed independently because of the singularity. The digits "6" and "9" demonstrate the rotation invariance. Note that the smooth closed curve for the digit "0" is obtained without a C^1-condition at the point $P_0 = P_4$.

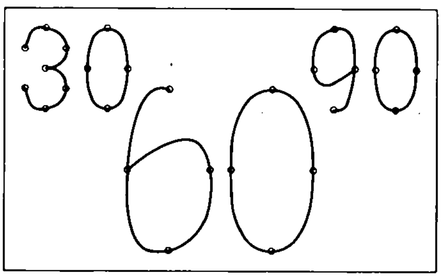

Fig. 2. Some digits interpolated by parametric quadratic splines

2. Componentwise quadratic interpolation. The idea is very simple and used frequently in the literature (see, e.g.,[6]). We define a partition of a parameter intervall $[0,T]$ by

$$\Pi_t:\ 0 = t_0 < \ldots < t_N = T$$

and calculate separately the two polynomial splines x and y satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad x \in C^{x\deg-1}[0,T],\ y \in C^{y\deg-1}[0,T],\ x\deg, y\deg \in \{1,2\}, \\ &\text{(ii)} \quad Q^x(x) \Rightarrow \min,\quad Q^y(y) \Rightarrow \min, \end{aligned} \tag{7}$$

with Q^x and Q^y defined analogously to the functional Q (see (4)) by

$$Q^x(x) = \sum_{n=1}^{N} \omega_n^x \int_{t_{n-1}}^{t_n} \dot{x}(t)^2\,dt \quad \text{and} \quad Q^y(y) = \sum_{n=1}^{N} \omega_n^y \int_{t_{n-1}}^{t_n} \dot{y}(t)^2\,dt. \tag{8}$$

Degres of freedom of this method are the choice of the partition Π_t and of the weights ω_n^x and ω_n^y. The nontrivial choices of the weights ω_n in Section 1 were motivated physically. In the present case, for arbitrary ω_n^x and ω_n^y, the functionals Q^x and Q^y are lakking in any physical interpretation. However, for weights equal to 1 and a proper choice of Π_t, the sum of both functionals

$$Q^{xy}(x,y) = Q^x(x) + Q^y(y) \tag{9}$$

allows the interpretation of an approximative strain energy, thus being a generalization of $Q(p)$ with $\omega_n = (1+g_n^2)^{-5/2}$. To make this clear, we recollect that the square of the curvature can be written in the form

$$c^2 = (\dot{x}\ddot{y} - \ddot{x}\dot{y})^2/(\dot{x}^2+\dot{y}^2)^3 = (\ddot{x}^2+\ddot{y}^2)(1-\cos^2\varphi)/(\dot{x}^2+\dot{y}^2)^2,$$

where $\cos\varphi$ is defined by

$$\cos\varphi = (\dot{x}\ddot{x}+\dot{y}\ddot{y})/[(\dot{x}^2+\dot{y}^2)(\ddot{x}^2+\ddot{y}^2)]^{1/2}.$$

Hence φ is the angle between the tangent direction and the direction of the acceleration - which is just equal to $\pi/2$ if t is proportional to the arc length. If we parametrize by the arc length, then we obtain

$$c(t)^2 = \ddot{x}(t)^2 + \ddot{y}(t)^2 \tag{10}$$

which motivates the interpretation above. However, for polynomial splines $x(t)$, $y(t)$ in general it is impossible to demand $(\dot{x}^2+\dot{y}^2)^{1/2} = \text{const} = 1$. But for practical purposes we can choose the cumulative chordal distance (as, e.g., in the case of parametric cubic splines, cf. [6])

$$t_0 = 0, \quad t_n = t_{n-1} + \left((x_n - x_{n-1})^2 + (y_n - y_{n-1})^2\right)^{1/2}, \quad n = 1(1)N, \tag{11}$$

so that (10) holds approximately. A refinement of this choice will be proposed in the algorithm in Section 5. If one of the conditions

$$|\dot{x}^2+\dot{y}^2-1| \ll 1, \quad |\dot{x}\ddot{x}+\dot{y}\ddot{y}| \ll \left((\dot{x}^2+\dot{y}^2)(\ddot{x}^2+\ddot{y}^2)\right)^{1/2}$$

is violated in some interval $[t_{n-1}, t_n]$, then rareness of data is indicated and additional points should be inserted.

3. Properties of the parametric splines. First let us show that for monotonous $\{x_n\}$ the parametric splines defined above, with (7), (11) and weights equal to 1, belong to the class of curves defined by (1) - (4) if a proper specification is used.

Theorem 1: *Let the partition Π_t be defined by* (11) *and let the x_n satisfy the monotonicity condition* (3). *Let $x\deg = 1$, $y\deg = 2$ and $\omega_n^y = 1$. Then the minimizer of the functional* (9) *with Q^x, Q^y from* (8) *yields the same curve as the quadratic spline p minimizing* (4) *with the weights $\omega_n = (1+g_n)^{-3/2}$.*

Proof: Since $\ddot{x} = 0$ and $x = x_{n-1} + (t - t_{n-1})(1+g_n^2)^{-1/2}$ it follows $y' = \dot{y}(1+g_n^2)^{1/2}$ and $y'' = y(1+g_n^2)$. Thus

$$\int_{x_{n-1}}^{x_n} y''^2 dx = \int_{t_{n-1}}^{t_n} \ddot{y}^2(1+g_n^2)^2 \dot{x}\, dt = (1+g_n^2)^{3/2} \int_{t_{n-1}}^{t_n} \ddot{y}^2 dt.$$

Consequently,

$$\sum_{n=1}^{N} (1+g_n^2)^{-3/2} \int_{x_{n-1}}^{x_n} y''^2 dx = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} [\ddot{x}^2 + \ddot{y}^2] dt$$

and the assertion follows ∎

Now, we are going to study the behaviour of the interpolating curve under rotation. We assume here and later on that Π_t is an arbitrary, but fixed partition of the parameter interval and that the weights are constant and equal to 1. The quadratic splines x, y defined by (7), (8) (with $x$deg = $y$deg = 2) and interpolating the data x_n, y_n, $n = 0(1)N$, then take the form

$$\begin{aligned} x(t_{n-1} + \tau h_n) &= x_{n-1} + d^x_{n-1} h_n \tau + (g^x_n - d^x_{n-1}) h_n \tau^2 \\ y(t_{n-1} + \tau h_n) &= y_{n-1} + d^y_{n-1} h_n \tau + (g^y_n - d^y_{n-1}) h_n \tau^2 \end{aligned} \qquad (0 \le \tau \le 1;\ n = 1(1)N).$$

Here

$$h_n = t_n - t_{n-1},\ g^x_n = (x_n - x_{n-1})/h_n,\ g^y_n = (y_n - y_{n-1})/h_n,$$

and

$$\begin{aligned} d^x_0 &= \textstyle\sum_{n=1}^N (-1)^n c_n g^x_n,\quad d^x_n = 2 g^x_n - d^x_{n-1}, \\ d^y_0 &= \textstyle\sum_{n=1}^N (-1)^n c_n g^y_n,\quad d^y_n = 2 g^y_n - d^y_{n-1}, \end{aligned} \tag{12}$$

where the c_n are obtained from (6) by rearranging the sum and using $\omega^x_n = \omega^y_n = 1$:

$$c_n = \left[1/h_n + 2 \sum_{j=n+1}^{N} 1/h_j\right] \Big/ \sum_{j=n+1}^{N} 1/h_j .$$

Using these formulae it is easy to prove the following

Theorem 2: *Let $x(t)$, $y(t)$ denote the above defined quadratic splines, interpolating the data x_n, y_n, $n = 0(1)N$, and let $\check{x}(t)$, $\check{y}(t)$ denote the quadratic splines, defined in the same manner, but interpolating the rotated data*

$$\check{x}_n = x_n \cos\alpha - y_n \sin\alpha,\quad \check{y}_n = x_n \sin\alpha + y_n \cos\alpha.$$

Then for each $t \in [0,T]$ it follows

$$\check{x}(t) = x(t)\cos\alpha - y(t)\sin\alpha \quad \text{and} \quad \check{y}(t) = x(t)\sin\alpha + y(t)\cos\alpha.$$

Proof: The step sizes are invariant under rotation, since Π_t is assumed to be fixed (note that for Π_t defined by (11) this assumption is fulfilled). Due to the linear dependence the slopes g^x_n, g^y_n "rotate" in the same manner as x_n, y_n:

$$\begin{aligned} \check{g}^x_n &= (\check{x}_n - \check{x}_{n-1})/h_n = g^x_n \cos\alpha - g^y_n \sin\alpha, \\ \check{g}^y_n &= (\check{y}_n - \check{y}_{n-1})/h_n = g^x_n \sin\alpha + g^y_n \cos\alpha. \end{aligned} \tag{13}$$

The same is true for the parameters d^x_n, d^y_n. For instance we obtain eassily from (12) and $\check{c}_n = c_n$ the equalities

$$\check{d}^x_0 = \sum_{n=1}^{N} (-1)^n c_n \check{g}^x_n = \sum_{n=1}^{N} (-1)^n c_n (g^x_n \cos\alpha - g^y_n \sin\alpha) = d^x_0 \cos\alpha - d^y_0 \sin\alpha,$$

and

$$\check{d}^x_n = 2\check{g}^x_n - \check{d}^x_{n-1} = d^x_n \cos\alpha - d^y_n \sin\alpha. \tag{14}$$

Now the assertion of the theorem follows immediately from the representation

$$x(t_{n-1} + \tau h_n) = x_{n-1} + d_{n-1} h_n \tau + (g_n - d_{n-1}) h_n \tau$$

and the corresponding one for $y(t_{n-1} + \tau h_n)$ by inserting the rotation formulae (13),(14) ∎

We conclude this section with the remark that Theorem 2 remains valid for generalizations to Euclidean spaces of any dimension.

4. Error estimation. Now we assume that the interpolation nodes (x_n, y_n), $n = 0(1)N$, are points from a smooth plane curve

$$\bar{C} = \{(u(s), v(s)) : 0 \le s \le S ;\ u, v \in C^3[0,S]\},$$

where s denotes a certain parameter, for instance the arc length, with the partition Π_s: $0 = s_0 < s_1 < \dots < s_N$. The interpolation nodes then read as $u(s_n) = x_n$, $v(s_n) = y_n$. By $C = \{(x(t), y(t)) : 0 \le t \le T\}$ we denote the interpolating curve, where x and y are polynomial splines determined separately by (7), (8) and (11), with $x\deg = y\deg = 2$ and weights ω_n^x $\omega_n^y = 1$, $n = 1(1)N$. Each point of the curve $\bar{C}$ may be written in the form

$$\left(u(s_{n-1} + \sigma h_n), v(s_{n-1} + \sigma h_n)\right), \quad 0 \le \sigma \le 1,\ h_n = s_n - s_{n-1}.$$

In the same manner we write the points of C in the form $\left(x(t_{n-1} + \sigma k_n),\ y(t_{n-1} + \sigma k_n)\right)$,

$$\left(x(t_{n-1} + \sigma k_n),\ y(t_{n-1} + \sigma k_n)\right), \quad 0 \le \sigma \le 1,\ k_n = t_n - t_{n-1},$$

where $\Pi_t : 0 = t_0 < \dots < t_N$ is a second given partition (cf. (11)). Since C interpolates $\bar{C}$, the points $(x(t_n), y(t_n))$ and $(u(s_n), v(s_n))$, $n = 0(1)N$, coincide. In each subinterval we associate the point $\left(u(s_{n-1} + \sigma h_n), v(s_{n-1} + \sigma h_n)\right)$ of the curve $\bar{C}$ with the point $\left(x(t_{n-1} + \sigma k_n),\right.$ $\left.y(t_{n-1} + \sigma k_n)\right)$ of the curve C. So we get a mapping between the parameters s and t which is linear on each subinterval:

$$(t - t_{n-1})/k_n = (s - s_{n-1})/h_n \qquad \left(t \in [t_{n-1}, t_n],\ s \in [s_{n-1}, s_n]\right),$$

and the functions u, v become functions of the parameter t:

$$u(s) = u(s_{n-1} + (t - t_{n-1})h_n/k_n) =: \tilde{u}(t),$$
$$v(s) = v(s_{n-1} + (t - t_{n-1})h_n/k_n) =: \tilde{v}(t).$$

Using the error estimation [7] separately for $x(t) - \tilde{u}(t)$ and $y(t) - \tilde{v}(t)$ we get [1)]

$$|x(t) - \tilde{u}(t)| \le C^x k^2, \quad |y(t) - \tilde{v}(t)| \le C^y k^2,$$

with

$$C^x = \tfrac{3}{8} M_2^x + T M_3^x, \quad C^y = \tfrac{3}{8} M_2^y + T M_3^y, \quad k = \max_{n=1(1)N} (t_n - t_{n-1}),$$

where M_i^x, M_i^y are bounds for the i-th derivatives of $\tilde{u}$ and $\tilde{v}$, respectively:

$$M_i^x = \max_{n=1(1)N} \sup_{t \in [t_n - t_{n-1}]} |\tilde{u}^{(i)}| = H^i \max_{n=1(1)N} \sup_{s \in [s_n - s_{n-1}]} |u^{(i)}| ,$$

$$H = \max_{n=1(1)N} (s_n - s_{n-1})/(t_n - t_{n-1}).$$

Note that even in the case $C = \bar{C}$ we may not expect generally the equalities $|x(t) - \tilde{u}(t)| = 0$ and $|y(t) - \tilde{v}(t)| = 0$.

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1) By combining the results of [7] with some estimates given in [9] one can obtain bounds for the error in other norms, especially norms containing the first derivative (R. STRAUSS, private communication).

5. Concluding remarks. For the interpretation of the functional used as a strain energy it is essential to have satisfied the equality

$$\dot{x}^2 + \dot{y}^2 = 1 , \tag{15}$$

at least approximately. Especially, for larger deviations from (15) there may occur singular points with both $\dot{x}$, $\dot{y}$ vanishing. It is easy to see that for quadratic x and y such singular points are always connected with a corner of angle O. The occurence of such situations seems to be very unlikely, since both of the coordinates should exhibit a local extremum at the same point of a certain subinterval of the given partition, but nevertheless if

- there are rare data points and
- the data points possess some special symmetry,

then the relevance of corners was practically observed. For these cases we apply the following heuristic procedure:

Step 1: Choose as an initial partition that one defined by the chordal lengths (sf. (11)).

Step 2: Calculate the parametric quadratic spline interpolant with the actual partition and weights equal to 1.

Step 3: Define a new partition of a new parameter interval basing on the arc lengths of the spline interpolant.

Step 4: If there are major changes in the partition, then go back to Step 1, else END.

We are lacking in analytic results about the convergence of this procedure, but the behaviour observed indicated a rather fast convergence (3...5 cycles) in all cases considered. For the limit curve the relation (15) holds in the sense of an integral mean value for each subinterval of the terminal partition. However, for practical use an interactive approach seems to be best. For unacceptable deviations from (15) a completion of the data as well as a splitting into several interpolating curves (without the C^1-condition) should be at the disposal of the user. We close the paper with an example to which such an interactive implementation of our parametric quadratic spline interpolation was applied. The data points (x_n, y_n), $n = 0(1)55$, are points on the boundary of a plane domain Ω, for which a boundary value problem from marine hydrodynamics is considered in [11]. The splitting of the interpolating curve is motivated by change of type of the boundary conditions or by singularities of the estuary.

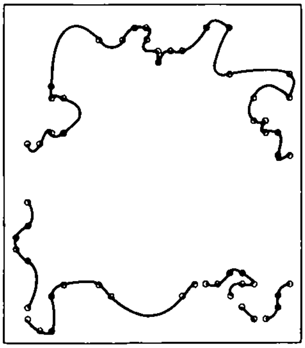

Fig. 3. Interpolation of the boundary of a bay of the Baltic Sea.

In this context it is interesting to note (as one of the referees remarked) that parametric quadratic splines yield algebraic curves of degree two. This fact makes our concept compatible with second order isoparametric finite elements (c.f. [12]).

REFERENCES

[1] BERG, L.: *Über vollmonotone Polynome durch vier gegebene Punkte.* Optimization **16** (1985), 265 - 272.

[2] BERG, L.: *Existence, characterization and application of interpolating minimal polynomials.* Optimization **18** (1987), 55 - 63.

[3] BERG, L.: *Interpolation and approximation by completely monotonous polynomials.* In: Coll. Math. Soc. Janos Bolyai 50: Numerical Methods. Miskolc 1986 (Ed.: P. Rozsa). Amsterdam: North-Holland 1988, 589 - 604.

[4] DIETZEL, S., and J.W. SCHMIDT: *Determination of shape preserving splines interpolants with minimal curvature via dual programs.* J. Appr. Theory **52** (1988), 43 - 57.

[5] FRISCHMUTH, K., ROSSMANN, J., and R. STRAUSS: *Solvability and regularity results for the hydrodynamic Ekman model.* Comp. Mech. (to appear).

[6] HANNA, M.S., EVANS, D.G., and P. N. SCHWEITZER: *On the approximation of plane curves by parametric cubic splines.* BIT **26** (1986), 217 - 232.

[7] KIKUCHI, N.: *Finite Element Methods in Mechanics.* Cambridge: University Press 1986.

[8] MAESS, G.: *An error estimation for quadratic splines with minimal curvature.* Rostock. Math. Koll. **35** (1988), 57 - 60.

[9] MAESS, G.: *Vorlesungen über numerische Mathematik II.* Berlin: Akademie-Verlag und Basel: Birkhäuser Verlag 1988.

[10] MAESS, B., and G. MAESS: *Interpolating quadratic splines with norm-minimal curvature.* Rostock. Math. Koll. **26** (1984), 83 - 88.

[11] METTKE, H., PFEIFFER, E., and E. NEUMANN: *Quadratic spline interpolation with coinciding interpolation and spline grids.* J. Comp. Appl. **8** (1982), 57 - 62.

[12] STEČKIN, S. B., and J. N. SUBBOTIN: *Supplements to the Russian translation of* AHLBERG, J.H., NILSON, E.N., and J.L. WALSH: *The Theory of splines and their Applications.* Moscow: MIR 1972.

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