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A Generalized Commutativity Theorem

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Let H be a complex separable Hilbert space, $\mathcal{C}$ the class of contractions with $C_{\cdot 0}$ completely non-unitary parts, $\mathcal{C}_0$ the class of $A \in \mathcal{C}$ wich satisfy the property (called *property* (P2)) that if the restriction of A to an invariant subspace M is normal, then M reduces A, and let $\mathcal{C}_1$ be the class of $A \in \mathcal{C}_0$ with defect operator D_A being of the Hilbert-Schmidt class C_2 and which are such that either the pure part of A has empty point spectrum or the eigen-values of A are all simple. It is known that if $A \in \mathcal{C}_0$ and $B^* \in \mathcal{C}_1$, then $AX = XB$ implies $A^*X = XB^*$. This implication fails to hold for the case in which $A \in \mathcal{C}$. It is shown here that if $A \in \mathcal{C}$ and $B^* \in \mathcal{C}_1$, then $AX = XB$ implies either (i) $A|\overline{\operatorname{ran}}\, X$ and $(B^*|\ker^{\perp}X)^*$ are quasi-similar C_0 contractions (with $B^*|\ker^{\perp}X$ normal), or (ii) $A^*X = XB^*$. Let $\mathcal{C}^1$ denote the class of contractions E satisfying property (P2), the inclusion $D_E \in C_2$ and which are such that the pure part of E has empty point spectrum. Choosing the intertwining operator X to be compact it is shown that $AX = XB$ implies $A^*X = XB^*$ for $A \in \mathcal{C}_0$ and $B^* \in \mathcal{C}^1$. Recall that quasi-similar operators need not to be unitarily equivalent (or, even, similar). We show that if $A \in \mathcal{C}_0$ and $B \in \mathcal{C}^1$ are quasi-similar with one of the implementing quasi-affinities compact, then A and B are unitarily equivalent normal contractions. Also it is shown that a compact operator $A \in \mathcal{C}^1$ is normal.

Key words: *commutativity property, contraction, Hilbert-Schmidt operator, quasi-similar operators*

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1. Introduction

We consider operators, i.e. elements of the algebra $B(H)$ of bounded linear transformations, on a complex infinite-dimensional separable Hilbert space H. Given Hilbert spaces H_1 and H_2, and operators $A \in B(H_1)$ and $B \in B(H_2)$, define the commutator $C(A,B)$: $B(H_2,H_1) \to B(H_2,H_1)$ by $C(A,B)X = AX - XB$. Let $\mathcal{C}$ denote the class of contractions with $C_{\cdot 0}$ completely non-unitary parts, $\mathcal{C}_0$ the class of contractions $A \in \mathcal{C}$ which satisfy the property

(P2) *if the restriction of A to an invariant subspace M is normal, then M reduces A,*

and let $\mathcal{C}_1$ be the class of contractions $A \in \mathcal{C}_0$ with defect operator $D_A = (1 - A^*A)^{1/2}$ being of the Hilbert-Schmidt class C_2 and which are such that either the pure part of A has empty point spectrum or the eigen-values of A are all simple.

The classical *Putnam-Fuglede Commutativity Theorem* says that if A and B are normal operators, then $C(A,B)X = 0$ for some operator X implies $C(A^*,B^*)X = 0$. Generalizing this result it has been shown in [5: Theorem 7] that the pair $(\mathcal{C}_0,\mathcal{C}_1)$ has the *Putnam-Fuglede Property*, i.e., given $A \in \mathcal{C}_0$ and $B^* \in \mathcal{C}_1$, if $C(A,B)X = 0$ for some operator X, then $C(A^*,B^*)X = 0$. Here the hypothesis that the elements of $\mathcal{C}_0$ satisfy property (P2) is essential in as much as that the pair $(\mathcal{C},\mathcal{C}_1)$ fails to have the Putnam-Fuglede Property: There exist con tractions $A \in \mathcal{C}$ and $B^* \in \mathcal{C}_1$, and an operator X, such that

$C(A,B)X = 0$, $B^*|\ker^{\perp}X$ is normal, and $A|\overline{\text{ran}}\,X$ and $(B^*|\ker^{\perp}X)^*$ are quasi-similar C_0 contractions but $C(A^*,B^*)X \neq 0$ (see the example in [5: Remark 1]). In this note we show that this is precisely the way in which the pair $(\mathbb{C},\mathbb{C}_1)$ may fail to satisfy the Putnam-Fuglede Property, i.e. we show that if $A \in \mathbb{C}$ and $B^* \in \mathbb{C}_1$ are such that $C(A,B)X = 0$ for some operator X, then either $A|\overline{\text{ran}}\,X$ and $(B^*|\ker^{\perp}X)^*$ are quasi-similar C_0 contractions (with $B^*|\ker^{\perp}X$ normal) or $C(A^*,B^*)X = 0$. The hypothesis that the elements of $\mathbb{C}_1$, in the pair $(\mathbb{C}_0,\mathbb{C}_1)$, have $C_{\cdot 0}$ completely non-unitary parts can not be replaced by the hypothesis that they have $C_{0\cdot}$ completely non-unitary parts. By requiring the intertwining operator X to be compact it will be shown that $C(A,B)X = 0$ implies $C(A^*,B^*)X = 0$ for contractions $A \in \mathbb{C}_0$ and $B^* \in \mathbb{C}^1$, where $\mathbb{C}^1$ is the class of contractions E such that E satisfies Property (P2), $D_E \in C_2$ and the pure part of E has empty point spectrum.

Recall that quasi-similarity of operators does not in general imply their equivalence (or, even, similarity) even in the case in which the implementing quasi-affinities are both compact. We show that if $A \in \mathbb{C}_0$ and $B \in \mathbb{C}^1$ are quasi-similar with one of the implementing quasi-affinities compact, then A and B are unitarily equivalent normal contractions. A compact contraction $A \in \mathbb{C}_0$ such that $D_A \in C_2$ and the pure part of A has empty point spectrum is normal [5]; we show here that this result extends to all $A \in \mathbb{C}^1$.

2. Notation and terminology

In addition to the notation and terminology already defined we shall in the following denote the range, the closure of the range, the kernel and the orthogonal complement of the kernel of an operator A by $\text{ran}\,A$, $\overline{\text{ran}}\,A$, $\ker A$ and $\ker^{\perp}A$, respectively. The restriction of A to a subspace M will be denoted by $A|M$. The spectrum and the point spectrum of A will be denoted by $\sigma(A)$ and $\sigma_p(A)$, respectively. The open unit disk (in the complex plane) will be denoted by D and C will denote the unit circle. The Fredholm index of A will be denoted by $\text{ind}\,A$, and $\dim M$ will denote the dimension of the subspace M of H. We say that the operator X is a quasi-affinity if both X and X^* have dense range. We shall denote the fact that $C(A,B)X = 0$ for some operator X with dense range (injective operator X) by $B \overset{d}{<} A$ (respectively, $B \overset{i}{<} A$), and we shall denote the fact that $C(A,B)X = 0$ for some quasi-affinity X by $B < A$. We say that A and B are quasi-similar, denoted $A \sim B$, if $A < B < A$. The operator A will be said to be pure if there exists no non-trivial reducing subspace M of A such that $A|M$ is normal. Recall that every operator has a direct sum decomposition of the type *normal* $\oplus$ *pure*.

We say that the contraction A is completely non-unitary if there exists no non-trivial reducing subspace M of A such that $A|M$ is unitary. The contraction A is said to belong to the class $C_{\cdot 0}$ (class $C_{\cdot 1}$) of contractions if $A^{*n} \to 0$ strongly as $n \to \infty$ ($\inf_n \|A^{*n}x\| > 0$ for all non-zero $x \in H$). The classes $C_{0\cdot}$ and $C_{1\cdot}$ are defined by considering A^* instead of A, and, for $\alpha,\beta = 0,1$, the class $C_{\alpha\beta}$ is defined by $C_{\alpha\cdot} \cap C_{\cdot\beta}$. We say that the completely non-unitary contraction A belongs to the class C_0 if there exists an inner function Φ such that $\Phi(A) = 0$. Recall that if $A \in C_0$, then amongst all inner functions Φ such that $\Phi(A) = 0$ there is a minimal one (i.e., one which is a divisor in the Hardy space H^{∞} of all others), called the minimal function of A [7]. The contraction A is said to be a weak contraction if the defect operator D_A $(= (1 - A^*A)^{1/2})$ is of the Hilbert-Schmidt class C_2 and $\sigma(A)$ does not fill the open unit disc D.

3. The results

We start by stating some lemmas. Lemma 1 follows from [5: Corollary 4] and Lemma 2 is [5: Theorem 5].

Lemma 1: *If $C(A,B)X = 0$ for some normal contraction A and contraction $B^* \in C_0$ such that $D_{B^*} \in C_2$, then $C(A^*,B^*)X = 0$.*

Lemma 2: *If $A \in C_0$ and $\sigma(A) \subseteq C$, then A does not satisfy property* (P2).

Lemma 3: *Let A be a completely non-unitary contraction of the class $C_{\cdot 0}$ and let $B^* \in C_1$ be a pure contraction. Then there exists no non-trivial operator X such that $C(A,B)X = 0$.*

Proof: Suppose that there exists a non-trivial solution X to $C(A,B)X = 0$. Letting $E = A|\overline{\text{ran}}\,X$, $F^* = B^*|\ker^{\perp}X$ and defining the quasi-affinity Y: $\ker^{\perp}X \to \overline{\text{ran}}\,X$ by setting $Yx = Xx$ for each $x \in \ker^{\perp}X$ we have that $C(E,F)Y = 0$, where $E \in C_{\cdot 0}$ (and $F^* \in C_1$ is pure). Clearly $F^* \in C_{00}$; hence, since $D_{B^*} \in C_2$ implies $D_{F^*} \in C_2$, $F^* \in C_0$ [12: Theorem 1]. Now if $\sigma_p(B^*) \cap D = \Phi$, then $\sigma_p(F^*) \cap D = \Phi$, and so $\sigma(F^*) \subseteq C$ [7: Theorem III.5.1]. This, since B^* satisfies property (P2) implies F^* satisfies property (P2), is a contradiction (by Lemma 2). Hence $X = 0$ in this case. If, on the other hand, the eigen-values of B^* are all simple, then the eigen-values of F^* are all simple. Recall that a C_0 contraction F^* with minimal function m has a triangulation $F^* = \begin{pmatrix} F_1 & * \\ 0 & F_2 \end{pmatrix}$, where the minimal function of F_1 is a "Blaschke product" m_1 and the minimal function of F_2 is a "singular inner function" m_2 (such that $m = m_1 m_2$ except for a constant factor of modulus one [7: p. 129]). Since F^* has simple eigen-values, m_1 has simple zeros. The eigen-spaces corresponding to distinct eigen-values of F_1 describe a "basic system" (in the sense of [1]) of invariant subspaces of F_1 and the restriction of F_1 to each of these subspaces is normal [7: p. 135]. Since F^* satisfies property (P2) implies F_1 satisfies property (P2), these invariant subspaces reduce F_1, i.e. F_1 is "reductive". Hence F_1 is normal and $F^* = F_1 \oplus F_2$. But then B^* has a normal direct summand - a contradiction since B^* is pure. Hence, once again, $X = 0$ ∎

Theorem 1: *Let $A \in C$ and $B^* \in C_1$* be such that *$C(A,B)X = 0$ for some non-trivial operator X. Then either*

(a) *$E = A|\overline{\text{ran}}\,X$ and $F = (B^*|\ker^{\perp}X)^*$ are quasi-similar C_0 contractions (with F normal), or*

(b) *$C(A^*,B^*)X = 0$ (and $A|\overline{\text{ran}}\,X$ and $B|\ker^{\perp}X$ are unitarily equivalent normal contractions).*

Proof: We consider the cases (i) A is pure; (ii) A is normal; and (iii) A has a normal direct summand separately, and show that whereas hypothesis (i) implies conclusion (a), hypotheses (ii) and (iii) imply conclusion (b).

Suppose that A is pure. Then, upon defining E, F^* and Y as in the proof of Lemma 3, we have that $F^* \in C_0$ and $F < E$, and hence that E, F are quasi-similar C_0 contractions (use [7: Prop. III.4.6]). The non-triviality of X implies, by the argument of the proof of Lemma 3 leading to the conclusion $\sigma(F^*) \subseteq C$, that $\sigma_p(F^*) \neq 0$, and so we must have that the eigen-values of F^* are all simple. Consequently, as in the proof of Lemma 3, F^* is normal, and so conclusion (a) holds.

Since Lemma 1 implies that conclusion (b) holds in the case in which A is normal, to complete the proof we consider the case in which A has a normal direct summand. We show to start with that B^* has a normal direct summand in such a case. Suppose that the (non-trivial) subspace M of H reduces A and $A_1 = A|M$ is normal. Set $N = \overline{X^*M}$ (= the closure of the Range of X^* acting on M); then N is invariant for B^*. Let $B_1^* = B^*|N$, and define the operator $X_1^*: M \to N$ by setting $X_1^*x = X^*x$ for each $x \in M$. Then X_1^* has dense range and $C(B_1^*, A_1^*)X_1^* = 0$. Clearly, B_1^* satisfies property (P2) and $D_{B_1^*} \in C_2$. Lemma 1 applies, and we have $C(B_1, A_1)X_1^* = 0$. Hence B_1^* is normal, and so B^* has a normal direct summand. Now define $\tilde{A}, \tilde{B}^*$ and $\tilde{X}$, on $\tilde{H} = H \oplus H$, by

$$\tilde{A} = A \oplus 0,\ \tilde{B}^* = 0 \oplus B^* \text{ and } \tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

Then $C(\tilde{A}, \tilde{B})\tilde{X} = 0$. Decompose $\tilde{A}$ and $\tilde{B}^*$ into their normal and pure parts by $\tilde{A} = E_1 \oplus E_2$, $\tilde{B}^* = F_1^* \oplus F_2^*$, and let $\tilde{X}$ have the corresponding matrix representation

$$\tilde{X} = [X_{ij}]_{i,j=1}^{2},\ X_{12}^* = X_{21},\ X_{11} \text{ and } X_{22} \text{ self-adjoint.}$$

It is then clear that $E_2 \in C_{\cdot 0}$ is completely non-unitary and $F_2^* \in C_1$ is the pure part of $\tilde{B}^*$. Applying Lemmas 1 and 3 to the equations $C(E_1, F_2)X_{12} = 0$ and $C(E_2, F_2)X_{22} = 0$, respectively, it follows that $X_{12} = 0 = X_{22}$. Since $C(E_1, F_1)X_{11} = 0$ implies $C(E_1^*, F_1^*)X_{11} = 0$ (by the Putnam-Fuglede Theorem), we have $C(\tilde{A}^*, \tilde{B}^*)\tilde{X} = 0$. Hence $C(A^*, B^*)X = 0$. Clearly, $\overline{\operatorname{ran}}\, X$ reduces A, $\ker^{\perp} X$ reduces B, and $A|\overline{\operatorname{ran}}\, X$ and $B|\ker^{\perp} X$ are unitarily equivalent normal contractions ∎

In the particular case in which the contraction B^* is chosen to be an isometry, Theorem 1 implies (in view of [5: Theorem 4]) the following generalization of [6: Theorem 1] and [14: Theorem 2.3].

Corollary 1: *If $C(A,B)X = 0$ for some contraction $A \in C$ and isometry B^*, then $A|\overline{\operatorname{ran}}\, X$ and $B|\ker^{\perp} X$ are unitarily equivalent unitary operators.*

Proof: Since B^* has no C_{00} part, Theorem 1 implies that $C(A^*, B^*)X = 0$. Hence $\overline{\operatorname{ran}}\, X$ reduces A, $\ker^{\perp} X$ reduces B, and $A|\overline{\operatorname{ran}}\, X$ and $B|\ker^{\perp} X$ are unitarily equivalent normal contractions. Since necessarily $B|\ker^{\perp} X \in C_{11}$, $B|\ker^{\perp} X$ is unitary ∎

Recall that given a C_{10} contraction B^* there exists an isometry V such that $B^* < V$ [7: Proposition II.3.5]. Hence the following corollary, which generalizes [4: Remark 4.1] and [8: Theorem 3], is immediate from Corollary 1.

Corollary 2: *Given $A \in C$ and $B^* \in C_{10}$, there exists no non-trivial operator X such that $C(A,B)X = 0$.*

An algebra $\mathcal{A}$ of operators on a Hilbert space is said to be *reflexive* if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$, where $\operatorname{Lat}\mathcal{A}$ denotes the family of subspaces invariant under all elements of $\mathcal{A}$, $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ is the algebra of all operators X for which $XM \subseteq M$ for every $M \in \operatorname{Lat}\mathcal{A}$. The reflexivity of the commutant $\{A\}'$ of the operator A is preserved under quasi-similarity [2]. Let $\{A\}''$ denote the double commutant of A. We have the following

Corollary 3: *Let $A \in \mathcal{C}$. If $C(A,B)X = 0$ for some $B^* \in \mathcal{C}_1$ and quasi-affinity X, then $\{A\}''$ is reflexive.*

Proof: By Theorem 1, either A is normal or (the pure operator) A is quasi-similar to a (C_0) normal contraction.In either case [10: Theorem] implies that $\{A\}''$ is reflexive ∎

Since quasi-similar C_0 contractions have the same spectrum [7: Proposition III.4.6 and Theorem III.5.1], the hypothesis that the eigen—values of B^* are all simple may be replaced by the hypothesis that the eigen-values of A are all simple (in Theorem 1). If, however, one replace the hypothesis that the pure part of B^* has empty point spectrum by the hypothesis that the pure part of A has empty spectrum, then conclusion (a) of Theorem 1 is not possible (for the reason that in such a case the operator Y in the proof of Theorem 1 must be trivial). Also, it is seen (in such a case) that if A has a normal direct summand $A = A_1 \oplus A_2$, then upon letting $B^* = B_1^* \oplus B_2^*$ be the normal direct summand of B^* and $X = [X_{ij}]_{i,j=1}^2$ that $X_{12} = X_{21} = X_{22} = 0$. Hence we have the following

Theorem 1′ (a Putnam-Fuglede Theorem): *Let $A \in \mathcal{C}$ be such that the pure part of A has empty point spectrum. If $B^* \in \mathcal{C}$ satisfies property* (P2) *and $D_{B^*} \in C_2$, then $C(A,B)X = 0$ implies $C(A^*,B^*)X = 0$.*

Remark 1: As seen in [5: Remark 6] the hypothesis that B^* has $C_{\cdot 0}$ completely non-unitary part in Theorem 1 (or, Theorem 1′) can not be replaced by the hypothesis that B^* is of such type. Since a C_{11} completely non-unitary contraction is quasi-similar to a unitary operator [7: p. 79], Theorems 1 and 1′ fail if A (or B^*) has a C_{11} completely non-unitary part. The hypothesis that $D_{B^*} \in C_2$ can not be replaced by the hypothesis that $\operatorname{trace}(1 - BB^*)^p < \infty$ for any $p > 1$. To see this, let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of H, and let B^* be the weighted shift $B^*e_n = \alpha_n e_{n+1}$, $\alpha_n = 1 - (n+2)^{-1}$. Then $B^* \in C_{\cdot 0}$ is a (non-normal) hyponormal contraction (so that B^* satisfies property (P2)) with empty point spectrum. Since $\sum(1 - \alpha_n) = \infty$, $\alpha_1\alpha_2\cdots\alpha_n \to 0$ as $n \to \infty$, and so $B^* \in C_{00}$. Since $\{e_n\}$ is a complete system of eigen-vectors of $1 - BB^*$ corresponding to the system of eigen-values $\{1 - \alpha_n^2\}$, $\operatorname{trace}(1 - BB^*)^p \le 2^p \sum(n+1)^{-p} < \infty$ for any $p > 1$. Choosing $A = B$ $(\in C_{\cdot 0})$ it is seen that the hypotheses of Theorem 1 are satisfied (with $X = 1$ and $\operatorname{trace}(1 - BB^*)^p < \infty$) but A is not normal, or quasi-similar to a normal contraction of the class C_0 (since $B^* \notin C_0$).

Recall that a completely non-unitary contraction B^* such that $D_{B^*} \in C_2$ has a triangulation

$$\begin{bmatrix} E_1 & & & * \\ 0 & E_2 & & \\ 0 & 0 & E_3 & \\ 0 & 0 & 0 & E_4 \end{bmatrix} \text{ of the type } \begin{bmatrix} c_{01} & & & * \\ 0 & c_0 & & \\ 0 & 0 & c_{11} & \\ 0 & 0 & 0 & c_{10} \end{bmatrix}, \tag{1}$$

where $D_{E_i} \in C_2$ for all $i = 1,2,3,4$ [13: Theorem 1.5]. It is clear from Remark 1 that given $A \in \mathcal{C}$, the hypothesis that $B^* \in \mathcal{C}^1$ is not sufficient for $C(A,B)X = 0$ to imply $C(A^*,B^*)X = 0$ for a general operator X; that this hypothesis is sufficient in the case in which X is compact is the content of our next theorem. The following lemmas will be required.

Lemma 4: *Let $A \in \mathcal{C}$ be completely non-unitary, and let $B^* \in \mathcal{C}^1$ be such that $\sigma_p(B^*) \cap D = \Phi$. Then there exists no non-trivial solution X to the equation $C(A,B)X = 0$* ∎

Proof: Suppose that there exists a non-trivial X with $C(A,B)X = 0$. Let $A_0 = A|\overline{\text{ran}}X$, $B_0^* = B^*|\ker^{\perp}X$ and define the quasi-affinity $X_0 : \ker^{\perp}X \to \overline{\text{ran}}X$ by setting $X_0 x = Xx$ for each $x \in \ker^{\perp}X$. Then $C(A_0,B_0)X_0 = 0$, and so, since $A_0 \in C_{\cdot 0}$, $B_0 \in C_{\cdot 0}$. Clearly, $D_{B_0^*} \in C_2$; hence B_0^* has a triangulation

$$\begin{bmatrix} B_1^* & * \\ 0 & B_2^* \end{bmatrix} \text{ of the type } \begin{bmatrix} C_{01} & * \\ 0 & C_0 \end{bmatrix},$$

where $D_{B^*} \in C_2$. We show that B_0^* is non-existent: This contradiction will then imply that X could not have been non-trivial. Since $B_1 \in C_{10}$,

$$\dim\ker(B_0^* - \lambda) = \dim\ker(B_1^* - \lambda) + \dim\ker(B_2^* - \lambda) = \text{ind}(B_1^* - \lambda) + \dim\ker(B_2^* - \lambda)$$

for all $\lambda \in D$. Since $\sigma_p(B^*) \cap D = \Phi$, and $\ker^{\perp}X$ is invariant for B^*, $\sigma_p(B_0^*) \cap D = \Phi$. Hence, since $B_2^* \in C_0$ implies that $\sigma_p(B_2^*) \cap D$ is countable [7: Theorem III.5.1],

$$\min\left\{\dim\ker(B_0^* - \lambda) : \lambda \in D\right\} = \text{ind}(B_1^* - \lambda) = 0.$$

This implies that B_1^* is a weak contraction (and so has a C_0-C_{11} decomposition [7: p. 327]). Consequently, B_0^* has no C_{01} part, and so $B_0^* \in C_0$. But then, since $\sigma_p(B_0^*) \cap D = \Phi$ and B^* satisfies property (P2), $\sigma_p(B_0^*) \subseteq C$ and B_0^* satisfies property (P2) - a contradiction by Lemma 2. Hence B_0^* is non-existent ∎

Lemma 5: *If $A \in C_{10}$ and B^* is a normal contraction such that $D_{B^*} \in C_2$, then there exists no non-trivial solution X to the equation $C(A,B)X = 0$.*

Proof: Suppose that there exists a non-trivial solution X of the equation $C(A,B)X = 0$. Then, upon defining A_0, B_0 and X_0 as in the proof of Lemma 4, we have $C(A_0,B_0)X_0 = 0$, where $B_0^* \in C_{0\cdot}$ is subnormal. We assert that B_0^* is normal. For if not, then B_0^* has a pure part B_1 (say) such that $B_1 \in C_0$ (this follows from the fact that $B_1 \in C_{00}$ and $D_{B_1^*} \in C_2$), $\sigma_p(B_1) \cap D = 0$ and B_1 satisfies property (P2) - a contradiction by Lemma 2. Consequently, $B_0^* \in C_0$, which implies that $A_0 \in C_0$ (and $A_0 \sim B_0$). Since $A \in C_{10}$, and $\overline{\text{ran}}\,X$ is invariant for A, this is a contradiction. Hence X must have been trivial ∎

Lemma 6: *If A is a pure C_{00} contraction satisfying property* (P2) *and B^* is a normal contraction such that $D_{B^*} \in C_2$, then there exists no non—trivial solution X to the equation $C(A,B)X = 0$.*

Proof: If there exists a non-trivial X satisfying $C(A,B)X = 0$, then upon proceeding as in the proof of Lemma 5 we have $A_0 \sim B_0$, where $B_0^* \in C_0$ is normal. This as in the proof of Lemma 3 implies that A_0 is normal, and hence that A has a normal direct summand - a contradiction. Hence, $X = 0$ ∎

Lemma 7: *If $A \in C_{11}$ and B^* is a completely non-unitary contraction, then there exists no non-trivial compact operator X such that $C(A,B)X = 0$.*

Proof: Suppose that X is a non-trivial compact solution of $C(A,B)X = 0$. Since $A \in C_{11}$, there exists a unitary U and a quasi-affinity T such that $C(U,A)T = 0$ [7: p. 79]. Set $TX = Z$. Then Z is compact and $C(U,B)Z = 0$. Letting $Z^*Z = S$, this implies that $B^*SB = S$, where S is a positive compact operator. Applying [3: Theorem 8 and Corollary 6.5], we have that $\overline{\text{ran}}\, S = \ker^{\perp} Z$ reduces B and $B|\ker^{\perp} Z$ is unitary. This, since B^* is completely non-unitary, is impossible. Hence Z, and so also X, is the trivial operator ∎

Theorem 2: *If $A \in \mathcal{C}_0$ and $B^* \in \mathcal{C}^1$, then $C(A,B)X = 0$ implies $C(A^*,B^*)X = 0$ for all compact operators X.*

Proof: As in the proof of Theorem 1, we consider the cases (i) A is pure, (ii) A is normal, and (iii) A has a normal direct summand separately.

(i) If A is pure, then $A \in C_{.0}$, and so it has a triangulation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \text{ of the type } \begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}.$$

Decompose B^* into its normal and pure parts by $B^* = B_1^* \oplus B_2^*$, and let X have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then Lemma 4 applied to $C(A_2,B_2)X_{22} = 0$ implies $X_{22} = 0$, Lemma 5 applied to $C(A_2,B_1)X_{21} = 0$ implies $X_{21} = 0$, Lemma 6 applied to $C(A_1,B_1)X_{11} = 0$ implies $X_{11} = 0$, and Lemma 4 applied to $C(A_1,B_2)X_{12} = 0$ implies $X_{12} = 0$. Hence $X = 0$, and the conclusion holds trivially.

(ii) If A is normal, then $A = A_1 \oplus A_2$, where $A_1 \in C_{11}$ is unitary and $A_{22} \in C_{00}$ is normal. Decompose B^* into its normal and pure parts as in (i) above, and let X have the representation $X = [X_{ij}]_{i,j=1}^2$. Then Lemma 4 applied to $C(A_2,B_2)X_{22} = 0$ implies $X_{22} = 0$ and Lemma 7 applied to $C(A_1,B_2)X_{12} = 0$ implies $X_{12} = 0$. (Notice that if X is compact and $X_{22} = 0$, then X_{12} is compact.) Hence, since $C(A_1,B_1)X_{11} = 0 = C(A_2,B_1)X_{21}$ implies (by the Putnam-Fuglede Theorem) that $C(A_1^*,B_1^*)X_{11} = 0 = C(A_2^*,B_1^*)X_{21}$, we get $C(A^*,B^*)X = 0$.

(iii) Assume now that $A = A_1 \oplus A_2$, where A_1 is normal and A_2 is pure. Then, upon letting B^* and X have the representations of (i) of the proof, it is seen that $X_{22} = 0 = X_{21}$ in the equations $C(A_2,B_2)X_{22} = 0 = C(A_2,B_1)X_{21}$ (proceed as in (i)). Also, see (ii), $C(A_1,B_2)X_{12} = 0$ implies $C(A_1^*,B_2^*)X_{12} = 0$, so that $\ker^{\perp} X_{12}$ reduces B_2 and $B_2|\ker^{\perp} X_{12}$ is normal. Since B_2 is pure, we must have $X_{12} = 0$. The fact that $C(A^*,B^*)X = 0$ now follows since $C(A_1,B_1)X_{11} = 0$ implies $C(A_1^*,B_1^*)X_{11} = 0$ (by the Putnam-Fuglede Theorem) ∎

Remark 2: If the operator X is non-trivial, then the operator $B^* \in \mathcal{C}^1$ in Theorem 2 can not be pure (and so must have a normal direct summand). To see this we notice that if $B^* \in \mathcal{C}^1$ is pure, then $B^* \in C_1$ (see (1) and the proof of Lemma 4). Since $D_{B^*} \in C_2$, it follows that there exists an isometry V and a quasi-affinity T such that $C(V,B^*)T = 0$ [9: Theorem 1]. The operator X being compact, this then implies that $C(A,V^*)Z = 0$, where Z

$= XT^*$ is compact. But then $\overline{\text{ran}}\, Z = \overline{\text{ran}}\, X$ reduces A and $A|\overline{\text{ran}}\, X$ is unitary [4: Theorem 2]. Let $A_1 = A|\overline{\text{ran}}\, X$, $B_1^* = B^*|\ker^{\perp}X$, and define the compact quasi-affinity $Y: \ker^{\perp}X \to \overline{\text{ran}}\, X$ in the usual way. Then $C(A_1, B_1)Y = 0$, where B_1^* is completely non-unitary. By Lemma 7 such an Y can not exist. Hence B^* could not have been pure.

The quasi-similarity of contractions does not in general imply their unitary equivalence (or, even, similarity), even in the case in which they satisfy property (P2), their pure parts have empty point spectrum and the quasi-affinities implementing the quasi-similarity are compact. Thus there exist pure quasi-normal contractions A and B satisfying $A \sim B$ with the intertwining quasi-affinities both compact such that A is not similar to B [14: Example 2.2]. If, however, $D_A \in C_2$ or $D_B \in C_2$, then one has the following

Theorem 3: *If $A \in \mathcal{C}_0$ and $B^* \in \mathcal{C}^1$ are such that $A \sim B$ with one of the implementing quasi-affinities compact, then A and B are unitarily equivalent normal contractions.*

Proof: Since the pure part of B has empty point spectrum and $D_B \in C_2$, the argument of the proof of Lemma 4 and (1) imply that B has a triangulation

$$B = \begin{bmatrix} B_n & 0 & 0 \\ 0 & B_{11} & * \\ 0 & 0 & B_{10} \end{bmatrix},$$

where B_n is normal, $B_{11} \in C_{11}$ is completely non-unitary, $B_{10} \in C_{10}$, and $D_{B_n}, D_{B_{11}}, D_{B_{10}} \in C_2$. Since $A \in \mathcal{C}_0$, A has a triangulation

$$A = \begin{bmatrix} A_u & 0 & 0 \\ 0 & A_{00} & * \\ 0 & 0 & A_{10} \end{bmatrix},$$

where A_u is unitary, $A_{00} \in C_{00}$ and $A_{10} \in C_{10}$. Assume, for definiteness, that $BX = XA$ and $AY = YB$, where X and Y are quasi-affinities with Y compact. Let X have the representation $X = [X_{ij}]_{i,j=1}^{3}$. Then $X_{31} = X_{32} = X_{22} = 0$. (Sample argument: Since $C(B_{10}, A_u)X_{31} = 0$, $\|X_{31}^*x\| = \|A_u^{n}X_{31}^*B_{10}^{*n}x\| \le \|X_{31}^*\|\,\|B_{10}^{*n}x\| \to 0$ as $n \to \infty$ for all $x \in \overline{\text{ran}}\, X_{31}$. Hence $X_{31} = 0$.) Consequently, X_{12} is injective. Applying [5: Theorem 7] to $C(A_{00}^*, B_n^*)X_{12}^* = 0$, it follows that A_{00} is normal. This then implies that $A = A_n \oplus A_{10}$, where $A_n = A_u \oplus A_{00}$ is normal. We now show that A has no C_{10} part. Suppose that A_{10} is non-trivial. Consider the equation $AT = TA$, where the compact quasi-affinity $T = YX$ has the representation $T = [T_{ij}]_{i,j=1}^{2}$. Since $A_{10} \in C_{10}$, there exists a quasi-affinity Z and an isometry V such that $C(V, A_{10})Z = 0$ [7: Proposition II.3.5]. Since $C(A_{10}, A_n)T_{21} = 0$, we have $C(V, A_n)ZT_{21} = 0$. Hence, by the Putnam-Fuglede Theorem for subnormal V and normal A_n^*, we have that $\ker^{\perp}ZT_{21}$ reduces A_n and that $A_n|\ker^{\perp}T_{21}$ is unitary. But then for all non-trivial $x \in \overline{\text{ran}}\, T_{21}$, $\|A_n^{*n}T_{21}^*x\| = \|T_{21}^*x\| = \|T_{21}^*A_{10}^{*n}x\| \to 0$ as $n \to \infty$. Hence $T_{21} = 0$. Consequently, T_{22} is compact (and has dense range). The equation $C(A_{10}, A_{10})T_{22} = 0$ implies the equation $C(V, A_{10})ZT_{22} = 0$, and so, since ZT_{22} is compact, $\ker^{\perp}ZT_{22}$ reduces A_{10} and $A_{10}|\ker^{\perp}T_{22}$ is unitary (see Remark 2). This contradiction implies that A_{10} must have been trivial. Part (ii) of the proof of Theorem 2 now implies that $C(A^*, B^*)X = 0$, and hence that A and B are unitarily equivalent normal contractions ∎

Remark 3: It can be seen, we leave the detail to the reader, that if the contraction $B \in \mathbf{C}^1$ in Theorem 3 is such that the completely non-unitary part B has empty point spectrum, then A and B are unitarily equivalent unitary operators

Remark 4: Starting with particular quasi-similar contractions A and B, it is sometimes possible to deduce their unitary equivalence even when neither of the intertwining quasi-affinities is compact. Let T_f, $f \in H^\infty$, denote the analytic Toeplitz operator $T_f h = fh$ (for each $h \in H^2$). Suppose that the quasi-similar contractions A and B have triangulations

$$A = \begin{bmatrix} A_u & 0 & 0 \\ 0 & A_{00} & * \\ 0 & 0 & T_f \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_n & 0 \\ 0 & T_g \end{bmatrix},$$

where A_u is unitary, $A_{00} \in C_{00}$, B_n is normal, $f \in H^\infty$ satisfies $\|f\|_\infty \le 1$ and g is a non-constant inner function. Suppose further that A satisfies property (P2) and $D_B \in C_2$. Then A and B are unitarily equivalent. To see this, let X and Y be quasi-affinities such that $C(B,A)X = 0 = C(A,B)Y$. Letting X have the representation $X = [X_{ij}]$, $1 \le i \le 2$ and $1 \le j \le 3$, it is then seen that $X_{21} = 0 = X_{22}$. (Recall that the analytic Toeplitz operator T_g is a unilateral shift if and only if g is a non-constant inner function; our hypotheses imply that T_g is a unilateral shift.) Consequently, X_{12} is injective. Since $C(B_n, A_{00})X_{12} = 0$, and $D_{B_n} \in C_2$, A_{00} is normal (see [5: Theorem 7]). Hence, since A satisfies property (P2), $A = A_n \oplus T_f$, where A_n is normal. Letting X and Y (now) have the representations $X = [X_{ij}]_{i,j=1}^2$ and $Y = [Y_{ij}]_{i,j=1}^2$, it is seen that $X_{21} = 0 = Y_{21}$. Thus $A_n \overset{i}{<} B_n \overset{i}{<} A_n$ and $T_f \overset{d}{<} T_g \overset{d}{<} T_f$. Clearly, A_n and B_n are unitarily equivalent. Applying [11: Corollary 1] to the relation $T_f \overset{d}{<} T_g \overset{d}{<} T_f$ it is seen that T_f and T_g are unitarily equivalent. Hence A and B are unitarily equivalent.

Recall that a compact hyponormal (or M-hyponormal) operator is normal, but there exists a compact quasi-nilpotent dominant operator [8]. (The operator A is said to be *dominant* if to each complex number λ there corresponds a real number $M_\lambda \ge 1$ such that $\|(A-\lambda)^* x\| \le M_\lambda \|(A-\lambda)x\|$ for all $x \in H$; if there exists a real number M such that $M_\lambda \le M$ for all λ, then the dominant operator A is said to be *M-hyponormal.* A 1-hyponormal operator is hyponormal.) Extending this result it was shown in [5] (see the note following Corollary 7) that a compact contraction $A \in \mathbf{C}_0$ such that $D_A \in C_2$ and the pure part of A has empty point spectrum is normal. That this result generalizes to $A \in \mathbf{C}^1$ is the content of our next theorem.

Theorem 4: *A pure contraction $A \in \mathbf{C}^1$ can not be compact.*

Proof: Let $A \in \mathbf{C}^1$ be a non-trivial pure contraction. Then, by the proof of Lemma 4 and (1), $A \in C_1$. is a pure (and so completely non-unitary) contraction such that $D_A \in C_2$. As such there exists an isometry V and a quasi-affinity X such that $C(V,A)X = 0$ [9: Theorem 1]. Now if A is compact, then $X = V^*XA$ is compact, and we conclude (as in Remark 2) that A is unitary. This contradiction implies that A can not be compact ∎

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