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Stability of Nonlinear Systems with Periodically Nonstationary Linear Part

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In memory of Solomon G. Michlin (1908 – 1990)

Multidimensional nonlinear systems with periodically nonstationary linear part are considered. It is supposed that nonlinear blocks satisfy some integral quadratic constraint. The necessary and sufficient condition of absolute stability with respect to output for such systems are established in terms of certain properties of solutions of a linear Hamiltonian system with periodic coefficients. In the case of constant coefficients this condition transforms into a well-known frequency criterion of absolute stability.

Key Words: Nonlinear system, quadratic integral constraint, absolute stability, frequency criteria, Hamiltonian system, nonoscillativity, quadratic functional, positive definiteness.

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Introduction

Frequency domain absolute stability criteria for nonlinear systems with stationary linear part have been known for a long time, – see, for example, [1, 3, 5, 8, 9] and references there. First sufficient criteria of absolute stability for systems with periodically nonstationary linear part were obtained, as it seems, in [6, 7]. These criteria have the form of positive definiteness of an infinite Hermitian matrix depending on a frequency parameter. In [10], the absolute stability criterion was obtained in another form characterized by the properties of the solutions of a linear Hamiltonian system. Unlike [10] this paper presents the absolute stability criterion for arbitrary given output. As it is known [3–9 and others], this makes possible a more detailed study of the nonlinear system. The obtained criterion has a form similar to [10], and in contrast to [6, 7] it is shown to be not only sufficient but necessary for absolute stability. If the "complete output" (state and output of nonlinear blocks) is taken as the system's output, then this criterion coincides with the criterion [10]. In the case of stationary linear part it reduces to the frequency "quadratic" criterion [3, 9, 16].

1. Problem Formulation

First let us agree upon the following notations:

$\mathbb{C}^k$ $(\mathbb{R}^k)$	– space of complex (real) k-vectors
$*$	– Hermitian conjugation (transposition for real vectors and matrices)
I_k	– the identity $k \times k$ matrix, $i = \sqrt{-1}$
$\mathbb{W}_2^1\{(0,t_0) \to \mathbb{X}\}$	– vector Sobolev space (Hilbert space of absolutely continuous functions $f(\cdot):(0,t_0) \to \mathbb{X}$ with inner product $\langle f_1, f_2\rangle = \int_0^{t_0} \left\{ f_2(t)^* f_1(t) + \dot f_2(t)^* \dot f_1(t) \right\} dt$ and the norm $\lVert f(\cdot)\rVert = \sqrt{\langle f, f\rangle}$; here $t_0 \le +\infty$, $\mathbb{X} = \mathbb{C}^k$ or $\mathbb{R}^k$.)

The symbols $L_2\{(0,t_0) \to \mathbb{X}\}$ and others have similar sense.

Consider the system whose linear part is described by the vector equation

$$dx/dt = A(t)x(t) + b(t)u(t). \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A(t+T) = A(t)$, $b(t+T) = b(t)$ are real, T-periodic, $n \times m$ matrix functions with measurable bounded elements. The nonlinear part of the system may be described by the equations

$$u(t) = \varphi_1[t, x(t)], \quad u(t) = \varphi_2\left[t, x(\cdot)\big|_0^t\right]$$

and others. Below, the equation of the nonlinear part is not used explicitely. Instead of this we assume that input $x(t)$ and output $u(t)$ of the nonlinear part satisfy the following integral quadratic constraint:

$$\exists\gamma > 0,\ \exists k_j \to \infty : \quad \int_0^{k_jT} \mathcal{G}(t, x(t), u(t))\, dt \geq -\gamma. \tag{1.2}$$

Here k_j are integers, $\mathcal{G}(t, x, u)$ is a given real quadratic form

$$\mathcal{G}(t, x, \xi) = \frac{1}{2}\left[x^*G(t)x + 2x^*g(t)\xi + \xi^*\Gamma(t)\xi\right] \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m) \tag{1.3}$$

with T-periodic (measurable, bounded) coefficients

$$G(t+T) = G(t) = G(t)^*, \quad g(t+T) = g(t), \quad \Gamma(t+T) = \Gamma(t) = \Gamma(t)^*.$$

The numbers k_j and γ in (1.2) may depend on the process $x(\cdot)$, $u(\cdot)$ and usually $\gamma = \gamma[x(0)] \to 0$ as $|x(0)| \to 0$.

Various examples of nonlinearities and corresponding quadratic constraints (1.2) can be found in [1, 3–9]. Often instead of (1.2) the stronger "local" quadratic constraint $\mathcal{G}(t, x(t), u(t)) \geq 0$ is satisfied (then obviously (1.2) holds). If, for example, $m = 1$, $u = \varphi(t, \sigma)$ is a scalar nonlinearity satisfying the usual "sector condition" $\mu_1 \leq \varphi(t, \sigma)/\sigma \leq \mu_2$ and $\sigma = c(t)^*x, c(t+T) = c(t)$, then the quadratic constraint $\mathcal{G}(t, x, u) = (\mu_2\sigma - u)(u - \mu_1\sigma) \geq 0$ is valid.

If $m = 1$, $u = \varphi(\sigma)$ and the same sector condition is valid, then (1.2) holds with the form $\mathcal{G}(t, x, u) = (\mu_2\sigma - u)(u - \mu_1\sigma) + \Theta(u - \mu_1\sigma)^*\dot{\sigma}$, where $\dot{\sigma} = \dot{c}(t)^*x + c(t)^*(A(t)x + b(t)u), \Theta \geq 0$.

If $m = 2$, $u = \mathrm{col}\ [u_1, u_2]$, $u_1 = \sigma^2$, $u_2 = \sigma^3$ and as above $\sigma = c(t)^*x$, $c(t+T) = c(t)$, $\Theta \geq 0$, then the integral constraint (1.2) with the form $\mathcal{G}(t, x, u) = (\sigma u_2 - u_1^2) + \Theta u_2\dot{\sigma}$ is fulfilled.

Special kinds of hysteresis functions, pulse modulators with various types of modulation satisfy the constraints (1.2) with some forms $\mathcal{G}$ (see [3, 9]). As a matter of fact all papers on absolute stability use the constraints (1.2), although often this is not formulated explicitely.

Consider the system (1.1), (1.2). In this system the processes $x(\cdot)$, $u(\cdot)$ are determined on $(0, \infty)$ and are locally quadratically summable (then the integral in (1.2) makes sense), $x(t)$ is absolutely continuous and (1.1) is valid almost everywhere.

Let $d(t+T) = d(t)$, $d_0(t+T) = d_0(t)$ be real (bounded, measurable) $n \times l$ and $l \times m$ matrix functions, $|d(t)| + |d_0(t)| \neq 0$ and

$$\eta(t) = d(t)^*x(t) + d_0(t)u(t) \tag{1.4}$$

be a given system's output. An output $\eta_C = \mathrm{col}\ [x, u]$ is called the *complete output* (then $[d^*, d_0] = I_{n+m}$).

The system (1.1), (1.2) is called *absolutely stable with respect to the output* η if there exists a constant $C > 0$ such that $|\eta(\cdot)| \in L_2(0, \infty)$ for any of its solutions $x(\cdot)$, $u(\cdot)$ and

$$\|\eta(\cdot)\|^2 = \int_0^\infty |\eta(t)|^2\, dt < C(|x(0)|^2 + \gamma) \tag{1.5}$$

is fullfilled. If the system (1.1), (1.2) is absolutely stable with respect to the complete output η_C (and consequently to any output (1.4)) it will be called *absolutely stable*. For an absolutely stable system it follows from (1.1) that $|x(\cdot)| \in L_2(0, \infty)$ and, therefore $|x(t)| \to 0$ as $t \to \infty$. Let $\mathfrak{N} = \left\{\varphi(t, x(\cdot))\,\big|_0^t\right\}$ be a set of nonlinear blocks. If for any solution of (1.1) with $u = \varphi(t, x(\cdot))|_0^t \in \mathfrak{N}$ (1.5) holds with a common constant $C = C_{\mathfrak{N}}$, then we say that the system (1.1) is *absolutely stable in class* $\mathfrak{N}$ *with respect to the output* η. If $\eta = \eta_C = \mathrm{col}\ [x, u]$, then we shall speak of absolute stability in class $\mathfrak{N}$.

The system (1.1), (1.2) is called *strongly minimally stable* if there exists a feedback $u(t) = c(t)^*x(t)$ $(|c(\cdot)| \in L_\infty,\ c(t+T) = c(t))$ such that $\mathcal{G}(t, x, c(t)^*x) \geq 0$ for any t, x and $|x(t)| \to 0$ as $t \to \infty$ for any solution of (1.1) with $u(t) = c(t)^*x(t)$.

The system (1.1), (1.2) is called *minimally stable* if for any $a \in \mathbb{R}^n$ a solution $x^M(\cdot), u^M(\cdot)$ of (1.1), (1.2) exists (with numbers $\gamma^M = \gamma^M(a), k_j^M$ in (1.2)) such that

$$x^M(0) = a,\ x^M\left(k_j^M T\right) \to 0 \quad \text{as} \quad k_j^M \to \infty \quad \text{and} \quad \inf_{\tau > 0}\left[\tau^{-2}\gamma^M(\tau a)\right] \le 0.$$

Obviously, a strongly minimally stable system is also minimally stable. (Indeed, in this case take x^M, u^M to be a solution of (1.1), $u^M = c(t)^* x^M$, $\gamma^M = 0$, and $k_j^M \to \infty$ are arbitrary.)

It is assumed below that (1.1) is stabilizable in the following sense: there exist $n \times m$ matrices $c_j(t) = c_j(t+T) (j = 1,2), |c_j(\cdot)| \in L_\infty$, such that the system (1.1) with $u = c_1(t)^* x$ is asymptotically stable as $t \to \infty$ and the system (1.1) with $u = c_2(t)^* x$ is asymptotically stable as $t \to -\infty$. The criteria for this condition can be found, for example, in [2].

It will be shown below that for absolute stability with respect to an output η it is necessary that

$$\exists \varepsilon > 0 : \mathcal{G}(t,0,u) + \varepsilon |d_0(t)u|^2 \le 0 \quad (\forall t,\ \forall u \in \mathbb{R}^m). \tag{1.6}$$

Hence it follows that $\Gamma(t) = \Gamma(t)^* \le 0$ in (1.3).

The essential difference from [10] is that here we consider the cases when the matrix $\Gamma(t)$ may be singular. (In [10] it was assumed that $\Gamma(t) < -\gamma_0 I_m < 0$; in view of (1.6) this is a necessary condition for the absolute stability with respect to the complete output.) There are many practically important examples with the singular matrix $\Gamma(t)$ [9].

2. Formulation of the result

Consider the *adjoint Hamiltonian system*

$$dx/dt = (\partial \mathcal{H}/\partial \psi)^*, \quad d\psi/dt = -(\partial \mathcal{H}/\partial x)^*, \quad \partial \mathcal{H}/\partial u = 0, \tag{2.1}$$

where

$$\begin{aligned} \mathcal{H} &= \psi^*(A(t)x + b(t)u) + \mathcal{G}_\delta \\ \mathcal{G}_\delta &= \mathcal{G}(t,x,u) - \delta\left(|x|^2 + |u|^2\right) + \varepsilon|\eta|^2 \end{aligned} \tag{2.2}$$

and $\delta \ge 0$, $\varepsilon \ge 0$. This system will play an important role below. The last equation (2.1) has the form $[\Gamma(t) - \delta I_m + \varepsilon d_0^* d_0]u + \ldots = 0$, where dots denote an expression independent of u. From (1.6) it follows that $\Gamma(t) - \delta I_m + \varepsilon d_0(t)^* d_0(t) < -\delta I_m$ for $\varepsilon > 0$ sufficiently small. Hence for $\delta > 0$ the last equation (2.1) implies that $u = \alpha(t)x + \beta(t)\psi$, where $|\alpha(t)|, |\beta(t)| \in L_\infty$. Denoting $\mathcal{H}_0(t,x,\psi) = \mathcal{H}(t,x,\psi,u)$ for $u = \alpha(t)x + \beta(t)\psi$, we transform (2.1) into the usual Hamiltonian system

$$dx/dt = (\partial \mathcal{H}_0/\partial \psi)^*, \quad d\psi/dt = -(\partial \mathcal{H}_0/\partial x)^*. \tag{2.3}$$

Indeed, $\partial \mathcal{H}_0/\partial \psi = \partial \mathcal{H}/\partial \psi + (\partial \mathcal{H}/\partial u)(\partial u/\partial \psi) = \partial \mathcal{H}/\partial \psi$ since $\partial \mathcal{H}/\partial u = 0$. Similarly, $\partial \mathcal{H}_0/\partial x = \partial \mathcal{H}/\partial x$. The system (2.3) may be rewritten as a vector equation

$$J\frac{dz}{dt} = H(t)z, \quad \text{where} \quad z = \begin{bmatrix} x \\ \psi \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \tag{2.4}$$

$$H(t) = \begin{bmatrix} G_0 - g_0\Gamma_0^{-1}g_0^* & \left(A - b\Gamma_0^{-1}g_0^*\right)^* \\ A - b\Gamma_0^{-1}g_0^* & -b\Gamma_0^{-1}b^* \end{bmatrix}, \tag{2.5}$$

$$g_0 = g + \varepsilon d d_0, \quad G_0 = G - \delta I_n + \varepsilon d d^*, \quad \Gamma_0 = \Gamma - \delta I_m + \varepsilon d_0 d_0^*. \tag{2.6}$$

From (1.6) we have $\Gamma_0(t) \le -\delta I_m < 0$, therefore $|\Gamma_0^{-1}| \in L_\infty$ if $\varepsilon > 0$ is sufficiently small.

Let $Z(t)$ be the evolution matrix of (2.4) (i. e. $dZ/dt = J^{-1}HZ$, $Z(0) = I_n$). $Z(T)$ is the monodromy matrix of (2.4). The system (2.4) (the system (2.1) and (2.3)) is called *completely unstable* if it has no solution that is bounded on $(-\infty, \infty)$, i. e. (see [11, Ch. II]) if the following frequency condition is satisfied:

$$\det\left[Z(T) - e^{i\omega} I_{2n}\right] \ne 0 \quad (\forall \omega : 0 \le \omega \le 2\pi). \tag{2.7}$$

Let (2.7) be satisfied. Then n linear independent solutions $z_j(t) = \mathrm{col}\ [x_j(t), \psi_j(t)]$ of (2.4) may be constructed such that $|z_j(t)| \to 0$ as $t \to \infty$. Consider $n \times n$ matrices

$$X(t) = [x_1(t), \ldots, x_n(t)], \quad \Psi(t) = [\psi_1(t), \ldots, \psi_n(t)]. \tag{2.8}$$

The completely unstable equation (2.4) is called *nonoscillatory* if

$$\det X(t) \ne 0 \quad (\forall t \in [0,T]). \tag{2.9}$$

Other equivalent nonoscillatory criteria may be found in [12].

Theorem 1. *Let the system (1.1), (1.2) be minimally stable. For absolute stability of the system (1.1), (1.2) with respect to the output η it is necessary and sufficient that one of the following equivalent conditions hold.*

(i) $\Gamma(t) + \varepsilon d_0(t)^* d_0(t) \le 0$ for some $\varepsilon > 0$ and for this ε and for all sufficiently small $\delta > 0$ (2.7) and (2.9) hold (i. e. the system (2.1) is completely unstable and nonoscillatory).

(ii) For some $\varepsilon > 0$ and for any $x(\cdot) \in W_2^1\{[0,T] \to \mathbb{C}^n\}$, $u(\cdot) \in L_2\{[0,T] \to \mathbb{C}^m\}$, $\varrho \in \mathbb{C}$, $|\varrho| = 1$, such that $dx/dt = A(t)x(t) + b(t)u(t)$, $x(T) = \varrho x(0)$ the inequality

$$\int_0^T \mathcal{G}(t, x(t), u(t))\, dt \le -\varepsilon \int_0^T |\eta(t)|^2\, dt \tag{2.10}$$

holds.

We note that in (2.10) and below if x, u are complex, then $\mathcal{G} = \frac{1}{2}(x^* G x + 2\mathrm{Re}(x^* g u) + u^* \Gamma u)$ is the Hermitian extension of the real form (1.3).

Corollaries. *Let the system (1.1), (1.2) be minimally stable.*

1. *If $\Gamma(t) + \varepsilon d_0(t)^* d_0(t) < -\gamma_0 I_m < 0$ for some $\varepsilon > 0$ and if for $\delta = 0$ (2.7) and (2.9) hold, then the system (1.1), (1.2) is absolutely stable with respect to the output η.*

2. *Absolute stability with respect to the complete output $\eta_C = \mathrm{col}\,[x, u]$ is equivalent to the conditions:*

(a) $\exists \gamma_0 > 0 : \Gamma(t) \le -\gamma_0 I_m < 0$,

(b) *(2.7), (2.9) are satisfied for $\delta = 0$, $\varepsilon = 0$.*

(This criterion supplements the result in [10].)

Proof of the corollaries. 1. Since (2.7) and (2.9) hold for $\delta = 0$, these conditions will also hold for small $\delta > 0$. (Matrices $Z(t)$ and $X(t)$ are continuous with respect to δ.)

2. The sufficiency follows from (i) and from the continuity with respect to ε, δ of the matrices $Z(t), X(t)$. By Theorem 1 the necessity is equivalent to $\Gamma(t) \le -\gamma_0 I_m < 0$ and (ii) with $|\eta|^2 = |\eta_C|^2 = |x|^2 + |u|^2$. According to Theorem 2 of [14], the last condition is equivalent to (2.7) and (2.9) with $\delta = 0, \varepsilon = 0$ ∎

The condition (ii) is close to the criterion of Shilman [6, 7]; efficient methods of its verification may be found in [7]. In many cases (i) seems to be more convenient to apply. In general cases both conditions are used only together with computer devices. Note however that conditions (2.7), (2.9) refer to a certain linear system (2.3) and they characterize behaviour of all solutions of nonlinear systems belonging to an infinite set.

Example 1. Consider the system (1.1) for $m = 1$, $u = \varphi(t, \sigma)$, $\sigma = c(t)^* x$, $c(t+T) = c(t)$, $|c(\cdot)| \in L_\infty$. Let $\mathfrak{N}_0$ be the class of functions $\varphi(t, \sigma)$ satisfying the inequality $\mu_1(t) \le \varphi(t,\sigma)/\sigma \le \mu_2(t)$, where $\mu_j(t+T) = \mu_j(t)$ are fixed functions from L_∞. Let us find the absolute stability conditions in the class $\mathfrak{N}_0$ with respect to the complete output. In our case the local quadratic constraint

$$\mathcal{G} = \frac{1}{2}(u - \mu_1\sigma)(\mu_2\sigma - u) \ge 0, \quad \sigma = c(t)^* x \tag{2.11}$$

is satisfied; hence $\Gamma = -1$. According to Corollary 2 we set $\delta = \varepsilon = 0$ in (2.2), therefore

$$\mathcal{H} = \psi^*(Ax + bu) + \frac{1}{2}(u - \mu_1\sigma)(\mu_2\sigma - u).$$

The system (2.1) in our case reduces to (2.4) with the matrix Hamiltonian

$$H(t) = \begin{bmatrix} \left(\frac{1}{2}(\mu_1 - \mu_2)\right)^2 cc^* & A^* + \frac{1}{2}(\mu_1 + \mu_2)cb^* \\ A + \frac{1}{2}(\mu_1 + \mu_2)bc^* & bb^* \end{bmatrix} \tag{2.12}$$

(see (2.5); the formulae (2.1), (2.2) may also be used). The strong minimal stability takes place if equation (1.1) is asymptotically stable with $u = (\mu_1 + \mu_2)\sigma/2$. Suppose that this condition is satisfied. By Corollary

2 the absolute stability in the class $\mathfrak{N}_0$ takes place if the Hamiltonian equation (2.4) with the Hamiltonian (2.12) is completely unstable and nonoscillatory, i. e. (2.7) and (2.9) hold.

This condition is similar to the circle criterion for stationary systems and transforms to the circle criterion if the system (1.1) and μ_1, μ_2 are stationary. This condition is necessary and sufficient for (1.1) to be absolutely stable in the class of nonlinearities such that their input $\sigma(t)$ and output $u(t)$ satisfy the integral quadratic constraint:

$$\exists \gamma \geq 0, \exists k_j \to \infty : \int_0^{k_j T} (u - \mu_1 \sigma)(\mu_2 \sigma - u)\, dt \geq -\gamma. \tag{2.13}$$

Some pulse and frequency modulators satisfy (2.13). (See [9].) Note that the system (1.1), (2.13) is strongly minimally stable if the equation (1.1) with some feedback $u = \mu_0(t)\sigma$, $\sigma = c(t)^* x$, $\mu_1(t) \leq \mu_0(t) \leq \mu_2(t)$, is asymptotically stable. Obviously this condition is also necessary for absolute stability in the class $\mathfrak{N}_0$.

Example 2. Consider the system (1.1) for $m = 1$, $u = \varphi(\sigma), \sigma = c(t)^* x$, $c(t+T) = c(t)$, $|c(\cdot)| \in L_\infty$. Suppose that we know only that $\sigma\varphi(\sigma) \geq 0$ and we have to find the absolute stability conditions for all such nonlinear systems. More precisely: Let $\mathfrak{N}_1$ be the class of functions $\varphi(\sigma)$ (they may be discontinuous or multivalued) which satisfy the existence theorem [3, Ch. 2] and $\sigma\varphi(\sigma) \geq 0$ (examples: $\varphi_1(\sigma) = \sigma^3$, $\varphi_2(\sigma) = \operatorname{sign} \sigma$, $\varphi_3(\sigma) = (1+\sigma^2)\operatorname{sign} \sigma$ if $\sigma \neq 0$, $\varphi_3(0) = [-\Delta, \Delta]$, $\Delta > 1$). We want to find the absolute stability conditions in the class $\mathfrak{N}_1$. Consider at first the complete output. It is simple to verify that the integral quadratic constraint (1.2) is fulfilled with the form $\mathcal{G} = \tau\sigma u + \Theta\dot{\sigma}u$, where $\tau \geq 0$, $\Theta \geq 0$ are the parameters. As $\varphi(\sigma) \equiv 0 \in \mathfrak{N}_1$, we have to suppose that the equation $dx/dt = A(t)x$ is asymptotically stable. Then the system (1.1), (1.2) is strongly minimally stable (the corresponding feedback is $u = 0$). As $\mathcal{G} = x^*(\tau c + \Theta\dot{c} + \Theta A^* c)u + \Theta b^* c u^2$ in our case we have $G = 0$, $\Gamma(t) = \Theta b(t)^* c(t)$. According to Theorem 1, (i) the inequality $\Gamma(t) \leq 0$ must hold.

Assume at first for simplicity that $\alpha(t) = -b(t)^* c(t) \geq \delta_0 > 0$. Without loss of generality we can put $\Theta = 1$. Consequently $g(t) = 1/2(\tau c + \dot{c} + A^* c)$. Using formulae (2.5), (2.6) from Theoerem 1, (i) we have that system (1.1) is absolutely stable in class $\mathfrak{N}_1$ if (2.7), (2.9) are fulfilled for the system (2.1) with the Hamiltonian

$$H(t) = \begin{bmatrix} gg^*/\alpha & (A + bg^*/\alpha)^* \\ A + bg^*/\alpha & bb^*/\alpha \end{bmatrix}.$$

Now suppose that $\alpha(t) = -b(t)^* c(t) \geq 0$. Consider the absolute stability problem with respect to an output $\eta = x$. We have $d = I_n$, $d_0 = 0$, $G_0 = (\epsilon - \delta)I_n$, $g_0 = g$, $\Gamma_0 = -(\alpha(t) + \delta)$, $\delta > 0$. According to Theorem 1, (i) the system (1.1) is absolutely stable with respect to the output η in the class $\mathfrak{N}_1$ if (2.7), (2.9) are fulfilled for the system (2.1) with the Hamiltonian

$$H(t) = \begin{bmatrix} (\epsilon - \delta)I_n - gg^*/(\alpha + \delta) & (A + bg^*/\alpha)^* \\ A + bg^*/(\alpha + \delta) & bb^*/(\alpha + \delta) \end{bmatrix}$$

for some $\epsilon > 0$ and for all $\delta > 0$.

Return to the general case. The proof of Theorem 1 (given later in Section 4) uses essentially the following proposition which is itself of considerable interest and is an "integral" variant of the frequency theorem for periodic systems [14, 15].

Theorem 2. *Let equation (1.1) be stabilizable (in the sense mentioned above) and $\mathcal{G}$ be the form (1.3). The following conditions are equivalent:*

(i) There exists a real $n \times n$ matrix $H = H^*$ such that for any $x(\cdot) \in W_2^1[(0,T) \to \mathbb{R}^n]$, $u(\cdot) \in L_2[(0,T) \to \mathbb{R}^m]$ satisfying (1.1) on $0 < t < T$, the inequality

$$\int_0^T \mathcal{G}[t, x(t), u(t)]\, dt \leq x(T)^* H x(t) - x(0)^* H x(0) \tag{2.14}$$

holds.

(ii) For any $x(\cdot) \in W_2^1[(0,T) \to \mathbb{C}^n]$, $u(\cdot) \in L_2[(0,T) \to \mathbb{C}^m]$, $\varrho \in \mathbb{C}$, $|\varrho| = 1$ satisfying (1.1) and $x(T) = \varrho x(0)$ the inequality

$$\int_0^T \mathcal{G}[t, x(t), u(t)]\, dt \leq 0 \tag{2.15}$$

holds.

Any of these conditions being satisfied, there exist a real $n \times n$ matrix $H = H^*$, a real $n \times m$ matrix-function $h(t)$ with entries from $L_2(0,T)$ and a bounded linear operator $\kappa = \kappa^* : L_2\{[0,T] \to \mathbb{R}^m\} \to L_2\{[0,T] \to \mathbb{R}^m\}$ such that for any functions $x(\cdot)$, $u(\cdot)$ satisfying (1.1) on $(0,T)$, the identity

$$\int_0^T \mathcal{G}[t, x(t), u(t)]\, dt = x(T)^* H x(T) - x(0)^* H x(0) - \int_0^T |\kappa u - h^* x(0)|^2\, dt \tag{2.16}$$

holds.

Remark. The theorem remains valid if the stabilizability condition of equation (1.1) is replaced by the following less strong (but less effective) condition: the pair $\{A(\cdot), b(\cdot)\}$ is exponentially stabilizable as $t \to \infty$ and for any $a \in \mathbb{R}^n$ the functional

$$\Phi[x(\cdot), u(\cdot)] = \int_0^\infty \mathcal{G}[t, x(t), u(t)]\, dt$$

is bounded from below on the set of processes $x(\cdot) \in W_2^1\{(0,\infty) \to \mathbb{R}^n\}$, $u(\cdot) \in L_2\{(0,\infty) \to \mathbb{R}^m\}$ satisfying (1.1) and $x(0) = a$.

3. Proof of Theorem 2

Obviously, (i)$\Rightarrow$(ii). (Indeed (2.14) for real $x(\cdot), u(\cdot)$ implies (2.14) for complex $x(\cdot), u(\cdot)$; if $x(T) = \varrho x(0)$, $|\varrho| = 1$, then (2.14) implies (2.15).) Thus it is necessary to prove that (i) and (2.16) follow from (ii).

Let (ii) hold. Apply Theorem 2 of [4]. For this let

$$\mathbb{U} = L_2\{[0,T] \to \mathbb{C}^m\} = \{u(\cdot)\},\ \mathbb{X} = \mathbb{C}^n,\ \mathbb{Y} = W_2^1\{[0,T] \to \mathbb{C}^n\} = \{y(\cdot)\}$$

be the spaces of controls, states and outputs [4]. Define linear bounded operators

$$\hat{A} : \mathbb{X} \to \mathbb{X},\quad \hat{b} : \mathbb{U} \to \mathbb{X},\quad \hat{C} : \mathbb{X} \to \mathbb{Y},\quad \hat{D} : \mathbb{U} \to \mathbb{Y},$$

assuming that for $x_0 \in \mathbb{X}$, $u(\cdot) \in \mathbb{U}$ the relations

$$y_0 = \hat{A}x_0 + \hat{b}u(\cdot) \in \mathbb{X},\quad x(\cdot) = \hat{C}x_0 + \hat{D}u(\cdot) \in \mathbb{Y} \tag{3.1}$$

are equivalent to

$$dx(t)/dt = A(t)x(t) + b(t)u(t)\ (0 \le t \le T),\quad x(0) = x_0,\quad x(T) = y_0. \tag{3.2}$$

Define the Hermitian form $F[x_0, u(\cdot)]$ on $\mathbb{X} \times \mathbb{U}$ by

$$F[x_0, u(\cdot)] \equiv -\int_0^T \mathcal{G}[t, x(t), u(t)]\, dt, \tag{3.3}$$

where $\mathcal{G}$ is a form (1.3) and $x(t)$ is defined by (3.2). Now let us show that the conditions of Theorem 2 of [4] are satisfied. As noted in [4, p. 70], l_2-controllability of the pair $(\hat{A}, \hat{b})$ follows from its exponential stabilizability. The pair $(\hat{A}, \hat{b})$ is exponentially stabilizable according to the assumption made at the end of Section 1. Indeed, the first of the equations (3.2) is asymptotically stable for $u(t) = c_1(t)^* x(t)$, $c_1(t+T) = c_1(t)$ as $t \to +\infty$. Therefore, all $|\lambda_j(K_1)| < 1$, where K_1 is the operator defined by $dx(t)/dt = [A(t) + b(t)c_1(t)^*]x(t)$, $x(0) = x_0$, $x(T) = K_1 x_0$. The feedback $u(t) = c_1(t)^* x(t)$ defines a bounded operator $\hat{C}_1 : \mathbb{X} \to \mathbb{U}$, $u(\cdot) = \hat{C}_1 x_0$ and from (3.1) we have $K_1 = \hat{A} + \hat{b}\hat{C}$. Thus, the pair $(\hat{A}, \hat{b})$ is exponentially stabilizable and therefore l_2-controllable.

Similarly, existence of the feedback $u(t) = c_2(t)^* x(t)$, $c_2(t+T) = c_2(t)$ such that the equation $dx(t)/dt = [A(t) + b(t)c_2(t)^*]x$ is asymptotically stable as $t \to -\infty$ means that a bounded operator

$\hat{C}_2 : \mathbb{X} \to \mathbb{U}$ exists such that the spectrum of the operator (matrix) $K_2 = \hat{A} + \hat{b}\hat{C}_2$ lies strictly outside the unit circle. According to the remark to Theorem 2 of [4] the condition b) of this theorem is satisfied. The last condition also follows from the assumption made in the remark just mentioned. Condition a) of Theorem 2 of [4]

$$F[x_0, u(\cdot)] \geq 0 \quad (\forall x_0, u(\cdot), \varrho : |\varrho| = 1, \varrho x_0 = \hat{A}x_0 + \hat{b}u(\cdot))$$

also holds since it coincides with (ii). According to this theorem, there exist such bounded linear operators $H = H^* : \mathbb{X} \to \mathbb{X}$, $h^* : \mathbb{X} \to \mathbb{U}$, $\kappa : \mathbb{U} \to \mathbb{U}$ that the identity (1.11) from [4] holds [1]:

$$\begin{aligned} F(x_0, u(\cdot)) + \left(\hat{A}x_0 + \hat{b}u(\cdot)\right)^* H \left(\hat{A}x_0 + \hat{b}u(\cdot)\right) - x_0^* H x_0 \\ = |\kappa u(\cdot) - h^* x_0|^2 \qquad (\forall x_0 \in \mathbb{X}, \forall u(\cdot) \in \mathbb{U}). \end{aligned} \tag{3.4}$$

Obviously, $h^* = h^*(t)$ is an $m \times n$ matrix with entries from $L_2(0, T)$. By (3.1)–(3.3) the identity (3.4) coincides with (2.16).

The inequality (2.14) follows from (2.16). Let us show that $H = H^*$, h, κ are real. According to Remark 2 to Theorem 1 in [4], the operators H, h, κ in that theorem are real in the case of real Hilbert spaces $\mathbb{X}, \mathbb{U}$. From the proof of Theorem 2 of [4] it follows that in this case the operators $H = H^*$, h, κ in this theorem are also real. Thus (ii)$\Rightarrow$(2.16), (i) ∎

4. Proof of Theorem 1

Consider the Hilbert space $W = W_2^1\{(0,\infty) \to \mathbb{R}^n\} \times L_2\{(0,\infty) \to \mathbb{R}^m\}$ of processes $w = [x(\cdot), u(\cdot)]$ and the affine manifold $\mathfrak{M}(x_0) \subset W$ of processes satisfying (1.1) and $x(0) = x_0$. Clearly $\mathfrak{M}(0)$ is a linear space. Let $\mathcal{G}(t, x, u)$ be a form of the type (1.3) (or its Hermitian extension if x, u are complex vectors).

Lemma 1. *Let $(A(\cdot), b(\cdot))$ be a T-periodic exponentially stabilizable pair as $t \to \infty$ (there exists a feedback $u(t) = c_1(t)^* x(t)$, $c_1(t+T) = c_1(t)$, such that the equation $dx/dt = [A(t) + b(t)c_1(t)^*]$ is asymptotically stable as $t \to \infty$). The following conditions are equivalent:*

(A) $\Phi = \int\limits_0^\infty \mathcal{G}[t, x(t), u(t)]\, dt \geq 0$ *for all* $[x(\cdot), u(\cdot)] \in \mathfrak{M}(0)$.

(B) $\Phi_T = \int\limits_0^T \mathcal{G}[t, \tilde{x}(t), \tilde{u}(t)]\, dt \geq 0$

for all

$$\tilde{x}(\cdot) \in W_2^1\{[0,T] \to \mathbb{C}^n\}, \ \tilde{u}(\cdot) \in L_2\{[0,T] \to \mathbb{C}^m\}, \ \varrho \in \mathbb{C}, \ |\varrho| = 1$$

satisfying the equation

$$d\tilde{x}/dt = A(t)\tilde{x} + b(t)\tilde{u}, \ \tilde{x}(T) = \varrho\tilde{x}(0). \tag{4.1}$$

Proof. A similar statement with strong inequalities is contained in [14]. By [14], the following conditions (A_+) and (B_+) are equivalent:

(A_+) $\exists\delta > 0 : \Phi \geq \delta\left(\|x(\cdot)\|^2 + \|u(\cdot)\|^2\right)$ for all $[x(\cdot), u(\cdot)] \in \mathfrak{M}(0)$.

(B_+) $\exists\delta > 0 : \Phi_T \geq \delta\left(\|\tilde{x}(\cdot)\|_T^2 + \|\tilde{u}(\cdot)\|_T^2\right)$ for all $\tilde{x}(\cdot)$, $\tilde{u}(\cdot)$, ϱ, $|\varrho| = 1$ satisfying (4.1).

Here $\|\cdot\|$ and $\|\cdot\|_T$ are $L_2(0,\infty)$- and $L_2(0,T)$-norms.

Let (A) hold. Then obviously (A_+) is satisfied for $\mathcal{G}_\epsilon = \mathcal{G} + \epsilon(|x|^2 + |u|^2)$, $\epsilon > 0$. Therefore, (B_+) also holds: $\exists\delta = \delta(\epsilon) > 0$, $\Phi_T + \epsilon\left(\|\tilde{x}(\cdot)\|_T^2 + \|\tilde{u}(\cdot)\|_T^2\right) \geq \delta\left(\|\tilde{x}(\cdot)\|_T^2 + \|\tilde{u}(\cdot)\|_T^2\right) \geq 0$ for all $\tilde{x}(\cdot)$, $\tilde{u}(\cdot)$, ϱ denoted. Since $\epsilon > 0$ is an arbitrary number (B) holds. Thus, (A) implies (B). Similarly, (B) implies (A) ∎

1 Notice that in formula (1.11) of [4] there is a misprint: in the right part instead of $|\kappa_0(u - h_0^* x)|^2$ must stand $|\kappa_0 u - h_0^* x|^2$. (Indeed, according to the proof on p. 74 of [4] the right part in (1.11) of [4] is the limit of the right part of (3.5) in [4], i. e. of $|\kappa_\delta u - h_\delta^* x|^2$ as $\delta = \delta_k \to 0$.)

Lemma 2. *Let $(A(\cdot), b(\cdot))$ be a T-periodic exponentially stabilizable pair, $\varrho \in \mathbb{C}$ be fixed, $|\varrho| = 1$, and let $\Phi_T \geq 0$ for all $\tilde{x}(\cdot)$, $\tilde{u}(\cdot)$ satisfying the equations (4.1). Then $\mathcal{G}(t, 0, \tilde{u}) \geq 0$ for all $t \in [0, T]$, $\tilde{u} \in \mathbb{C}^m$. In particular, both (A) and (B) imply $\mathcal{G}(t, 0, \tilde{u}) \geq 0$.*

Proof. Suppose the contrary: there exist $u_0 \in \mathbb{C}^m$ and $\Delta \subset [0, T]$ of measure $\delta_0 = \operatorname{mes} \Delta > 0$ such that $u_0^* \Gamma(t) u_0 \leq -\gamma_0 < 0$ for $t \in \Delta$. For arbitrary δ, $\delta_0 > \delta > 0$ define a subset Δ_δ of Δ, $\operatorname{mes} \Delta_\delta = \delta$. Let $X(t, s)$ be the evolution matrix of the equation $dx/dt = A(t)x$ and $X(T, 0)$ be its monodromy matrix. Without loss of generality assume that all $|\lambda_j[X(T, 0)]| < 1$; otherwise a substitution $u = u_1 + c_1(t)^* x$ can be made. Then for any $\tilde{u}(\cdot) \in L_2\{[0, T] \to \mathbb{C}^m\}$ the boundary problem $d\tilde{x}/dt = A(t)\tilde{x} + b(t)\tilde{u}$, $\tilde{x}(T) = \varrho \tilde{x}(0)$ is solvable and has a solution of the form $\tilde{x}(t) = \int_0^T \Omega(t, s) \tilde{u}(s)\, ds$, where $|\Omega(t, s)| \leq \text{const}$. Take $\tilde{u}(t) = u_0$ for $t \in \Delta_\delta$, $\tilde{u}(t) = 0$ for $t \in [0, T] \setminus \Delta_\delta$. Then $|\tilde{x}(t)| \leq C_1 \operatorname{mes} \Delta_\delta = C_1 \delta$,

$$\int_0^T \tilde{x}(t)^* G(t) \tilde{x}(t)\, dt = O(\delta^2), \tag{4.2}$$

$$\int_0^T \tilde{x}(t)^* g(t) \tilde{u}(t)\, dt = \int_{\Delta_\delta} \tilde{x}(t)^* g(t) \tilde{u}(t)\, dt = O(\delta^2). \tag{4.3}$$

On the other side,

$$\int_0^T \tilde{u}(t)^* \Gamma(t) \tilde{u}(t)\, dt = \int_{\Delta_\delta} u_0^* \Gamma(t) u_0\, dt \leq -\gamma_0 \delta. \tag{4.4}$$

From (4.2)–(4.4) we have $\Phi_T < 0$ for small δ. Thus we obtained the contradiction $\Gamma(t) \geq 0$ ∎

Lemma 3. *Suppose that the pair $(A(\cdot), b(\cdot))$ is stabilizable (this assumption was made at the end of section 1). Then there exist $t_0 > 0$ and a function u on $[0, t_0]$ such that $\int_0^{t_0} |\eta|^2\, dt > 0$ for a solution of (1.1) with $x(0) = 0$ and for an output (1.4).*

Proof. If $d(t) \equiv 0$ (i. e. $d(t) = 0$ almost everywhere), then $d_0(t) \not\equiv 0$ and the statement is obvious. Let $d(t) \not\equiv 0$. Suppose the contrary: for all $u(\cdot)$ almost everywhere

$$\eta(t) = d(t) \int_0^t X(t, s) b(s) u(s)\, ds + d_0(t) u(t) = 0.$$

Here $X(t, s)$ is an evolution matrix of equation $dx/dt = A(t)x$. Then for any $x(0) = a \neq 0$ the output $\eta(t)$ does not depend on $u(\cdot)$. Putting $u(t) = c_j(t)^* x(t)$ $(j = 1, 2)$ we obtain $\eta(t) = d(t)^* X_1(t) a = d(t)^* X_2(t) a$, where $X_j(t) = X_j(t, 0)$ and $X_j(t, s)$ is the evolution matrix of the equation $dx/dt = (A + bc_j^*)x$. Here a is an arbitrary vector, therefore $\Theta(t) = d(t)^* X_1(t) = d(t)^* X_2(t)$ almost everywhere. Moreover $|d(t)| \neq 0$ for $t \in E$, $\operatorname{mes} E \neq 0$. There exists a t_0, $d_1 = d(t_0) \neq 0$. Then $\Theta(t_0 + kT) = d_1^* X_1(t_0 + kT) = d_1^* X_1(t_0) \cdot X_1(T)^k$ and $\Theta(t_0 + kT) = d_1^* X_2(t_0) \cdot X_2(T)^k$. By supposition all $|\lambda_j[X_1(T)]| < 1$ and all $|\lambda_j[X_2(T)]| > 1$. Thus simultaneously $|\Theta(t_0 + kT)| \to 0$ and $|\Theta(t_0 + kT)| \to +\infty$ as $k \to \infty$. The contradiction proves the lemma ∎

Lemma 4. *Let $W_0 = \{w\}$ be a real linear space and let $\mathfrak{F}$, $\mathfrak{G}$ be quadratic functionals on W_0 and $\mathfrak{G}(w_0) > 0$ for some $w_0 \in W_0$. Then*

$$\sup_{w : \mathfrak{G}(w) \geq 0} \mathfrak{F}(w) = \inf_{\tau \geq 0} \sup_{W_0} [\mathfrak{F}(w) + \tau \mathfrak{G}(w)] \tag{4.5}$$

(here we assume that $\inf \psi(\tau) = +\infty$ if $\psi(\tau) \equiv \infty$).

The proof is given in [16] (it is only necessary to change $\mathfrak{F}$ to $(-\mathfrak{F})$) ∎

Let us prove the necessity of (ii) in Theorem 1. Let $\mathfrak{M}(a, \gamma_0)$ be the set of process $[x(\cdot), u(\cdot)] \in \mathfrak{M}(a)$ such that

$$\mathfrak{G}(w) = \int_0^\infty \mathcal{G}(t, x(t), u(t))\, dt + \gamma_0 \geq 0.$$

By the property of exponential stabilizability of the pair $(A(\cdot), b(\cdot))$ the set $\mathfrak{M}(a)$ is not en any $a \in \mathbb{R}^n$. Hence there exist $\gamma_0 = \gamma_0(a) \geq 0$ such that $\mathfrak{M}(a, \gamma_0)$ is not empty and $\mathfrak{G}(w_0)$ some $w_0 \in \mathfrak{M}(a, \gamma_0)$. Suppose (ii) does not hold. It is sufficient to show that

$$\sup_{\mathfrak{M}(a,\gamma_0)} \|\eta\|^2 = +\infty.$$

Indeed, this contradicts the definition (1.5) of absolute stability. (Indeed, (4.6) implies (1. $\gamma = \gamma_0 + \varepsilon_0$, $\varepsilon_0 > 0$, and from (1.5) we have $\|\eta\|^2 \leq C[\gamma_0 + \varepsilon_0 + |x(0)|^2]$ instead of (4.7).)

For $w \in \mathfrak{M}(a)$ we have $\Delta w = w - w_0 \in \mathfrak{M}(0)$. Clearly, $\mathfrak{F}(w) = \|\eta\|^2$ and $\mathfrak{G}(w)$ may be cor as quadratic functionals of $\Delta w = w - w_0$ on the linear space $W_0 = \mathfrak{M}(0) = \{\Delta w\}$. By Lemr have

$$\sup_{w \in \mathfrak{M}(a,\gamma_0)} \mathfrak{F}(w) = \inf_{\tau \geq 0} \sup_{w \in \mathfrak{M}(a)} [\mathfrak{F}(w) + \tau \mathfrak{G}(w)].$$

Remind that the right part is equal to $+\infty$ if $\sup[\ldots] = +\infty$ for all $\tau \geq 0$.

Obviously, $\sup_{\mathfrak{M}(a)}[\mathfrak{F}(w) + \tau\mathfrak{G}(w)] = +\infty$ if

$$\mathfrak{F}(w_1) + \tau\mathfrak{G}(w_1) > 0 \text{ for some } w_1 \in \mathfrak{M}(0).$$

(It suffices to put $w = w_0 + \zeta w_1$, $\zeta \to +\infty$.) By Lemma 1, (ii) is equivalent to conditi $\mathfrak{G}(w) + \varepsilon\mathfrak{F}(w) \leq 0$ for some $\varepsilon > 0$ and for all $w = [x(\cdot), u(\cdot)] \in \mathfrak{M}(0)$. According to the assu (ii) does not hold. Therefore, (A') is also not satisfied, i. e. for any $\varepsilon > 0$ there exists w_ε such that $\mathfrak{G}(w_\varepsilon) + \varepsilon\mathfrak{F}(w_\varepsilon) > 0$. We have obtained (4.9) for any $\tau = \varepsilon^{-1} > 0$. Let us sh (4.9) holds also for $\tau = 0$. Let $x(\cdot), u(\cdot)$ be the process on $0 \leq t \leq t_0$ defined by Lemma $u(t) = c_1(t)^* x(t)$ for $t \geq t_0$. Then $w_1 = [x(\cdot), u(\cdot)] \in \mathfrak{M}(0)$ and $\|\eta\|^2 = \mathfrak{F}(w_1) > 0$. Thus, satisfied for all $\tau \geq 0$. Hence, (4.7) holds ∎

Let us prove the equivalence of (i) and (ii) in Theorem 1.

(i)⇒(ii): By Theorem 2 of [14] (viz., by equivalence of the conditions (C) and (G) of this th it follows that fulfilment of conditions (2.9) and (2.7) (with fixed $\delta > 0$) implies that

$$\Phi_\delta = \int_0^T [-\mathcal{G}_\delta(t, x, u)]\, dt \geq \delta_0 \int_0^T (|x|^2 + |u|^2)\, dt$$

for some $\delta_0 > 0$ and for all $[x(\cdot), u(\cdot), \varrho]$ mentioned in (ii). Therefore,

$$\Phi_\delta = -\int_0^T (\mathcal{G} + \varepsilon|\eta|^2)\, dt + \delta \int_0^T (|x|^2 + |u|^2)\, dt \geq 0.$$

Here $\delta > 0$ is an arbitrarily small number. Hence $\Phi_0 \geq 0$ for all $[x(\cdot), u(\cdot), \varrho]$, (ii) holds.

(ii)⇒(i): Let (ii) hold. By Lemma 2, $\Gamma(t) + \varepsilon d_0(t)^* d_0(t) \leq 0$. Therefore $\mathcal{G}_\delta(t, 0, u) = u^*\Gamma$ $-\delta|u|^2$ for $\delta > 0$, where $\mathcal{G}_\delta$ is the form from (2.2). By Theorem 2 of [14] (viz., by the equival (C) and (G)) it follows that the system (2.1) is completely unstable and nonoscillatory. (W to substitute in [14] $\mathcal{G}$ by $(-\mathcal{G}_\delta)$.) Thus, (i) holds ∎

Let us prove the sufficiency of (ii). Let (ii) of Theorem 1 hold. By Theorem 2 there exists a matrix $H = H^*$ such that (2.14) holds for $\mathcal{G}_\varepsilon = \mathcal{G} + \varepsilon|\eta|^2$:

$$\int_0^T (\mathcal{G} + \varepsilon|\eta|^2)\,dt \leq x(T)^* H x(T) - x(0)^* H x(0). \tag{4.10}$$

Let $x(\cdot), u(\cdot)$ be an arbitrary solution of (1.1), (1.2). Since A, b and the coefficients of $\mathcal{G}$ are T-periodic we have from (4.10)

$$\int_0^{kT} (\mathcal{G} + \varepsilon|\eta|^2)\,dt \leq x(kT)^* H x(kT) - x(0)^* H x(0) \tag{4.11}$$

for any integer k. Let us show that $H \leq 0$. Substituting $x^M(\cdot)$, $u^M(\cdot)$ from the definition of minimal stability in (4.11) and using (1.2), we obtain

$$-\gamma^M(a) \leq x^M(k_j^M T)^* H x^M(k_j^M T) - a^* H a.$$

If $k_j^M \to \infty$, then $|x^M(k_j^M)| \to 0$ and therefore $a^* H a < \gamma^M(a)$. Here a is an arbitrary vector. Substituting a by τa, we obtain

$$a^* H a \leq \tau^{-2}\gamma^M(\tau a), \quad a^* H a \leq \inf_{\tau \geq 0} \tau^{-2}\gamma^M(\tau a) \leq 0.$$

Thus, $H \leq 0$. For any solution $x(\cdot)$, $u(\cdot)$ of (1.1), (1.2) from (4.11) we have

$$-\gamma + \varepsilon \int_0^{k_j T} |\eta|^2\,dt \leq \int_0^{k_j T} (\mathcal{G} + \varepsilon|\eta|^2)\,dt \leq x(0)^* H x(0).$$

Here $k_j \to +\infty$. Hence $\|\eta\| < \infty$, (1.5) holds, and the system (1.1), (1.2) is absolutely stable with respect to the output η ∎

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Book reviews

JOHN B. CONWAY: **A Course in Functional Analysis** (Graduate Texts in Mathematics: Vol. 96). Second Edition. XVI + 399 pp., 1 fig. Berlin - Heidelberg - New York: Springer-Verlag 1990.

Functional analysis has developed to a vast field of mathematics such that connections between its different parts are rather loose now. (One should think perhaps of locally convex spaces, partially ordered vector spaces, operators in Hilbert spaces, or C^*- and W^*-algebras.) Therefore it strongly depends on the author, where and how emphasis is shifted concerning the selection of topics for a book about functional analysis.

The approach of John Conway in his book meets completely the taste and the point of view of the reviewer: The basic methods of functional analysis as well as operators in Hilbert space are treated extensively.

The book begins with Hilbert spaces and operators in it in the first two chapters. The following four chapters represent the fundamental techniques and notations of functional analysis with increasing abstraction (closed graph theorem, Hahn-Banach theorem, weak topologies, dual space etc.). The author gives many applications and cross connections to other fields of analysis, contained in star-marked sections. Such topics are e.g. the Banach limit, Runge's theorem, and extension of positiv linear forms as applications of the Hahn-Banach theorem, and the Stone-Weierstraß theorem as application of the Krein-Milman theorem.