Sufficient Conditions for Local Optimality
in Multidimensional Control Problems with State Restrictions

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New sufficient conditions for strong local minimality in multidimensional control problems with state restrictions are presented. The results are obtained by applying duality theory and second order sufficient optimality conditions for optimization problems with functions having a locally Lipschitzian gradient mapping.

Key words: *Sufficient optimality conditions, multidimensional control problems, parametric optimization*
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1. Introduction

We consider the following optimal control problem:

(P) Minimize $J(x,u) = \int_\Omega r(t,x(t),u(t))\,dt$ $(\Omega \subset \mathbf{R}^m, \ m \geq 1)$,

subject to the *state equation*

$$x_{t_\alpha}(t) = g_\alpha(t,x(t),u(t)) \text{ a.e. on } \Omega \ (\alpha = 1,...,m), \tag{1.1}_1$$

the *state restrictions*

$$x(t) \in G(t) := \left\{\xi \in \mathbf{R}^n \mid f_i(t,\xi) \geq 0 \ (i = 1,...,l)\right\} \text{ on } \overline{\Omega}, \tag{1.1}_2$$

the *control restrictions*

$$u(t) \in U \text{ a.e. on } \Omega \ \left(U \subset \mathbf{R}^r, r \geq 1\right), \tag{1.1}_3$$

and the *boundary conditions*

$$x(s) = b(s) \text{ on } \partial\Omega, \tag{1.1}_4$$

where $\overline{\Omega}$ is the closure of Ω, $\partial\Omega$ is the boundary of Ω, x is an n-dimensional vector function with components in $D^1(\overline{\Omega})$, $x \in D^{1,n}(\overline{\Omega})$, and u is an r-dimensional vector function with components in $D^0(\overline{\Omega})$, $u \in D^{0,r}(\overline{\Omega})$. Here $D^0(\overline{\Omega})$ is the space of all continuous functions on $\overline{\Omega}^j$ for $j = 1, ... ,\nu$, where $\left[\Omega^1,...,\Omega^\nu\right]$ is a finite decomposition of Ω into domains Ω^j with piecewise smooth boundary, and $D^1(\overline{\Omega})$ is the space of all continuous functions on $\overline{\Omega}$ having continuous first partial derivatives in $\overline{\Omega}^j$ for $j = 1, ... , \nu$. We assume that the boundary $\partial\Omega$ is piecewise smooth and all given functions, r, g_α, f_i and b are continuous. A pair $(x,u) \in D^{1,n}(\overline{\Omega}) \times D^{0,r}(\overline{\Omega})$ satisfying $(1.1)_1 - (1.1)_4$ is called *admissible to* (P) and the set of all admissible pairs is denoted by Z.

The aim of our paper is to develope sufficient conditions for a strong local minimum of the problem (P). The result is obtained by applying the duality theory of R. Klötzler [7]

as well as by using the strong second order sufficient optimality condition for optimizati-
on problems described by C^1-functions having a locally Lipschitzian gradient mapping
[5,6]. Our main theorem contains the result of V. Zeidan [9] for the special case of
one-dimensional problems ($m = 1$) without state restrictions. The very restrictive as-
sumption in her paper, effecting that the optimal x has to be smooth, is omitted. Our
proofs differ essentially from the rather complicated approaches used in [9]. A special
result for multidimensional problems comparable with our main theorem was obtained by
B. V. Krotov and V. I. Gurman [8]. Some incorrectness in their proofs is omitted here
and, moreover, we avoid the very restrictive asssumption that the Hamiltonian to (P) is
twice differentiable.

2. A dual problem to (P) and the generalized maximum principle

In a general sense we call a problem

(D) maximize $L(s)$ subject to $S \in \mathbf{S}$

a *dual problem* to (P) if the weak duality relation

$$L(S) \le J(x,u) \tag{2}$$

holds for all $S \in \mathbf{S}$ and all admissible pairs $(x,u) \in Z$. This relation implies that the exi-
stence of an element $S \in \mathbf{S}$ satisfying the strong duality relation $L(S) = J(x,u)$ is a suffi-
cient optimality condition for a given admissible pair (x,u) of (P).

Using the Hamiltonian H of (P) given by

$$H(t,\xi,y) = \sup \left\{ h(t,\xi,v,y) \mid v \in U \right\} \tag{3}$$

with

$$h(t,\xi,v,y) = -r(t,\xi,v) + \sum_{\alpha=1}^{m} y^{\alpha\mathsf{T}} g_\alpha(t,\xi,v) \tag{4}$$

a dual problem to (P) can be defined in the following way (see [1]):

$$\text{maximize } L(S) = \inf_{\xi \in Q} \sum_{j=1}^{\nu} \int_{\partial\Omega^j} S(s,\zeta(s)) n^j(s) \, do(s) \text{ subject to } S \in \mathbf{S},$$

where

$$Q = \left\{ \zeta \in C^{0,m}(\overline{\Omega}) \mid \zeta(t) \in G(t), \, t \in \overline{\Omega} \text{ and } \zeta = b \text{ on } \partial\Omega \right\},$$

$n^j = (n_1^j, \ldots, n_m^j)^\mathsf{T}$ denotes the exterior normal unit vector to $\partial\Omega^j$, and $\mathbf{S}$ the set of all vec-
tor functions $S = (S^1, \ldots, S^m)$ possessing the following properties:

1. There exists a decomposition of Ω into a finite number of domains Ω^j (depending
on S) with piecewise smooth boundary such that

$$S \in C^{1,m}(X^j), \, X^j = \left\{ (t,\xi) \mid \xi \in G(t), \, t \in \overline{\Omega}^j \right\} \quad (j = 1, \ldots, \nu),$$

where $C^{1,m}(X^j)$ is the space of all m-dimensional continuously differentiable vector
functions on X^j.

2. S fulfils the Hamilton - Jacobi inequality

$$\text{div}_t \, S(t,\xi) + H(t,\xi, \text{grad}_\xi S(t,\xi)) \le 0 \text{ on } X^j \quad (j = 1, \ldots, \nu).$$

Assertion 1 (Generalized maximum principle): *An admissible pair* (x^*, u^*) *is a global minimum of* (P) *if there exists an* $S^* \epsilon S$ *satisfying the maximum condition*

(M) $\qquad H\big(t, x^*(t), \mathrm{grad}_\xi S^*(t, x^*(t))\big) = h\big(t, x^*(t), u^*(t), \mathrm{grad}_\xi S^*(t, x^*(t))\big)$

on Ω^j $(j = 1, \ldots, \nu)$, *the Hamilton - Jacobi equation*

(HJ) $\qquad \mathrm{div}_t\, S^*(t, x^*(t)) + H\big(t, x^*(t), \mathrm{grad}_\xi S^*(t, x^*(t))\big) = 0$

on Ω^j $(j = 1, \ldots, \nu)$ *and the boundary condition*

(B) $\qquad L(S^*) = \sum\limits_{j=1}^{\nu} \int\limits_{\partial\Omega^j} S^*(s, x^*(s)) n^j(s)\, do(s).$

Proof: For arbitrary elements $(x, u) \epsilon Z$ and $S \epsilon S$ we can deduce using (3) and (4) with $y(t) = \mathrm{grad}_\xi S(t, x(t))$ and Gauss' Theorem that

$$
\begin{aligned}
J(x, u) = &\sum_{j=1}^{\nu} \int_{\Omega^j} \Big\{ - h\big(t, x(t), u(t), \mathrm{grad}_\xi S(t, x(t))\big) \\
&+ \sum_{\alpha=1}^{m} \mathrm{grad}_\xi S^\alpha(t, x(t)) g_\alpha(t, x(t), u(t))\Big\}\, dt \\
\geq &- \sum_{j=1}^{\nu} \int_{\Omega^j} \Big\{ H\big(t, x(t), \mathrm{grad}_\xi S(t, x(t))\big) + \mathrm{div}_t\, S(t, x(t))\Big\}\, dt \\
&+ \sum_{j=1}^{\nu} \int_{\partial\Omega^j} S(s, x(s)) n^j(s)\, do(s) \geq L(S).
\end{aligned}
$$

The conditions (M), (HJ) and (B) effect that especially the equality $J(x^*, u^*) = L(S^*)$ holds for $(x^*, u^*) \epsilon Z$ and $S^* \epsilon S$. Thus (x^*, u^*) is a global minimizer of (P) ∎

Generally, it is a very hard problem to find an element $S \epsilon S$ satisfying the generalized maximum principle for an $(x, u) \epsilon Z$. Nevertheless it was done for some interesting geometrical problems, see [1,2]. For this reason it is also helpful to give sufficient criterions for a strong local minimum of (P).

Definition 1: An admissible pair (x^*, u^*) is a *strong local minimum for* (P) if there exists an $\epsilon > 0$ such that (x^*, u^*) minimizes $J(x, u)$ over all admissible pairs $(x, u) \epsilon Z$ with $\|x - x^*\|_{C^{0,n}} < \epsilon$.

In a similar way as in Assertion 1 we can develop conditions for local optimality of a pair $(x, u) \epsilon Z$.

Assertion 2: A pair $(x, u) \epsilon Z$ is a strong local minimizer of (P) if there exists an $\epsilon > 0$ and an $S \epsilon S_\epsilon$ satisfying the conditions (M), (HJ) and (B), where S_ϵ is the set of all functions satisfying the following conditions:

1_ϵ. There exists a decomposition of Ω (depending on S) into a finite number of domains Ω^j with piecewise smooth boundary such that

$$S \in C^{1, m}(X_\varepsilon^j), \quad X_\varepsilon^j = \{(t, \xi) \in X^j \mid \xi \in K_\varepsilon(x(t))\},$$

where $K_\varepsilon(x(t)) = \{\xi \in \mathbf{R}^n \mid \|\xi - x(t)\| < \varepsilon\}$ and $\|\cdot\|$ is the Euclidean norm.

$2_\bullet$. S fulfils the Hamilton-Jacobi inequality

$$div_t\, S(t, \xi) + H(t, \xi, grad_\xi\, S(t, \xi)) \leq 0 \text{ on } X_\varepsilon^j \quad (j = 1, \ldots, \nu).$$

Proof: The proof follows immediately from the fact that $(x, u) \in Z$ is a strong local minimizer of (P) if and only if there is an $\varepsilon > 0$ such that (x, u) is a global minimizer of (P_ε). Here (P_ε) and the dual problem (D_ε) are defined in the same way as (P) and (D), respectively, where only $G(t)$ in $(1)_2$ is replaced by $G(t) \cap K_\varepsilon(x(t))$ ∎

3. An auxiliary result on strongly stable local maximizers of parametric optimization problems

In this section we study a general parametric optimization problem of the type

$$\mathbf{P(t)} \quad maximize\; f_0(t, \xi) \text{ subject to } \xi \in G(t),\; t \in \hat{\Omega}, \tag{5}$$

where $\hat{\Omega}$ is compact. Throughout this section let x be a given continuous vector function with $x(t) \in G(t)$, $t \in \hat{\Omega}$. In what follows we develop sufficient conditions for the existence of a positive ε (independent on $t \in \hat{\Omega}$) such that

$$f_0(t, x(t)) \geq f_0(t, \xi) \text{ for all } t \in \hat{\Omega} \text{ and } \xi \in G(t) \cap K_\varepsilon(x(t)) \tag{6}$$

holds. This relation means not only that $x(t)$ is a local maximizer of P(t) for all $t \in \hat{\Omega}$, but also the existence of a uniform (with respect to the compact set $\hat{\Omega}$) positive radius ε such that $x(t)$ is even a global maximizer with respect to the restricted feasible set $G(t) \cap K_\varepsilon(x(t))$. Our considerations are motivated by the fact that for the special choice of the objective function in (5),

$$f_0(t, \xi) = div_t\, S(t, \xi) + H(t, \xi, grad_\xi\, S(t, \xi)), \tag{7}$$

relation (6) is obviously a consequence of the assumption (HJ) for an $S \in \mathbf{S}_\varepsilon$ in Assertion 2. Moreover, the aspired result will be used in the next section to form sufficient conditions for the assumption that an $S \in \mathbf{S}_\varepsilon$ satifies (HJ) in Assertion 2 and hence for the strong local minimality of an (x, u) to (P). This will be exactly our main result.

For the case of C^2-functions in (5) the announced sufficient conditions for (6) are just the well-known strong second order sufficient conditions for local optimality. However, since the Hamiltonian H is defined in (3) as an optimal value function of a parametric optimization problem it is generally not realistic to suppose that H in (7) belongs to C^2 even if all functions appearing in (P) are in C^2 or even analytic. Under certain conditions it is pertinent to assume that H belongs locally to the subclass $C^{1,+}$ of those C^1-functions for which the gradient mapping is locally Lipschitzian. More exactly, we assume that for a given $\varepsilon > 0$ and $i = 0, \ldots, l$ the following assumptions are satisfied:

$$f_i(t, \cdot) \text{ belongs to } C^1(K_\varepsilon(x(t))) \text{ for each } t \in \hat{\Omega}. \tag{8}_1$$

$f_i(\cdot,\cdot)$ and $\mathrm{grad}_\xi f_i(\cdot,\cdot)$ are continuous on $Y_\varepsilon = \left\{(t,\xi)\mid t\in\hat{\Omega},\ \xi\in K_\varepsilon(x(t))\right\}.$ $\qquad$ $(8)_2$

$\mathrm{grad}_\xi f_i(t,\cdot)$ is locally Lipschitzian on $K_\varepsilon(x(t))$ for each $t\in\hat{\Omega}.$ $\qquad$ $(8)_3$

$(t,\xi)\to\partial_\xi(\mathrm{grad}_\xi f(t,\xi))$ is closed and locally bounded on $Y_\varepsilon.$ $\qquad$ $(8)_4$

Recall that each function f satisfying $(8)_1$ - $(8)_4$ is almost everywhere twice differentiable with respect to ξ in a neighbourhood of $x(t)$. In the following we will use the generalized Hessian in the sense of J. B. Hiriart - Urruty el al [4]:

$$\partial^2 h(t,\xi) := \mathrm{conv}\left\{M(t)\mid\exists\{\xi^k\}\subset E_h(t)\ \text{with}\ \xi^k\to\xi,\ d_{\xi\xi}^2 h(t,\xi^k)\to M(t)\right\},$$

where $E_h(t)$ is the set of all ξ for which $h(t,\cdot)$ is twice continuously differentiable with the Hessian $d_{\xi\xi}^2 h(t,\xi)$ and conv denotes the convex hull.

Further on we assume that for each $t\in\hat{\Omega}$ the point $x(t)$ satisfies the *Linear Independent Constraint Qualification*

(LICQ) For each $t\in\hat{\Omega}$ the vectors $\mathrm{grad}_\xi f_i(t,x(t))$, $i\in I_0(t):=\left\{i\in\{1,...,l\}\mid f_i(t,x(t))=0\right\}$
$\qquad$ are linearly independent.

If $x(t)$ satisfies (6), then $x(t)$ is a local maximizer of $(P(t))$. Hence, (LICQ) has the consequence that for each $t\in\hat{\Omega}$ there is a unique multiplier $\lambda(t)\in\mathbf{R}^l$ such that $(x(t),\lambda(t))$ is a stationary point of $(P(t))$, i.e.

$$\mathrm{grad}_\xi f_0(t,x(t))+\sum_{i=1}^{l}\lambda_i(t)\mathrm{grad}_\xi f_i(t,x(t))=0,$$

$$\lambda_i(t)f_i(t,x(t))=0,\quad\lambda_i(t)\geq 0\ \text{for}\ t\in\hat{\Omega}\ (i=1,...,l). \qquad (9)$$

With

$$I^+(t)=\left[i\in\{1,...,l\}\mid\lambda_i(t)>0\right]$$

and

$$W^+(t)=\left\{h\in\mathbf{R}^l\mid h^\intercal\mathrm{grad}_\xi f_i(t,x(t))=0,\ i\in I^+(t)\right\}$$

we can formulate the following sufficient optimality condition (S) for (6) which is just a natural generalization of the well - known strong second order sufficient optimality condition for the C^2 - case to the $C^{1,+}$ one:

(S) Each $M(t)\in\partial_{\xi\xi}^2 f_0(t,x(t))+\sum_{i\in I^+(t)}\lambda_i(t)\partial_{\xi\xi}^2 f_i(t,x(t))$ is negative definite on $W^+(t)$, i.e. for each vector $h(t)\in W^+(t)\backslash\{0\}$ the inequality $h^\intercal(t)M(t)h(t)<0$ holds.

Now we can show the following

Assertion 3: *Assume that the function f_0 in (5) and the functions $f_1,...,f_l$ in the state restriction $(1)_2$ belong to the class described in $(8)_0$ - $(8)_4$. For each $t\in\hat{\Omega}$, let x be a stationary solution of* P(t) *such that* (LICQ) *and* (S) *are satisfied. Then there is a positive ε such that* (6) *holds.*

Proof: According to (5) for each $t\in\hat{\Omega}$ there exists a maximal value $\varepsilon(t)>0$ (possible $\varepsilon(t)=+\infty$) with

$$f_0(t, x(t)) > f_0(t, \xi) \text{ for each } \xi \in G(t) \cap K_{\varepsilon(t)}(x(t)), \xi \neq x(t).$$

Let be $\bar{\varepsilon} := \inf \{\varepsilon(t) | t \in \hat{\Omega}\}$. Then there is a sequence $\{t_k\}, t_k \to t$, with $\varepsilon(t_k) \to \bar{\varepsilon}$. We denote $\bar{\xi} = x(\bar{t})$. According to [5, Theorem 1] for some real number $r > 0$ and each $\rho \in [0, r]$, there exists a real $\delta(\rho) > 0$ such that for $t \in V_\delta(\bar{t})$ the set $U_\rho(\bar{\xi})$ contains a local maximizer $\bar{x}(t)$ of P(t) which is the only stationary point of (P(t)) in $\bar{U} := \bar{U}_r(\bar{\xi})$ and $\bar{x}$ is continuous in $\bar{t}$. Because of the uniqueness of the stationary point $\bar{x}(t)$ in $\bar{U}$ and the continuity of $\bar{x}$ the number r can be chosen in such way that $0 < r \leq \varepsilon(\bar{t})$ and

a) $x(t) = \bar{x}(t)$ for each $t \in V := V_{\delta(r)}(\bar{t})$,

b) (LICQ) holds for all $t \in V$ and $\xi \in \bar{U} \cap G(t)$.

Following the line in the proof of Theorem 1 in [5] let us now consider the following auxiliary problem

$\widetilde{P}(t)$ Maximize $f_0(t, \xi)$ subject to $\xi \in \bar{U} \cap G(t), t \in V$,

which possesses for all $t \in V$ at least one global maximizer. On the other hand let be $\Phi(t) = \sup \{f_0(t, \xi) | \xi \in \partial U \cap G(t)\}$ ($-\infty$ if $\partial U \cap G(t) = \emptyset$). Note that Φ is upper semicontinuous in $\bar{t}$. To show this let us consider any sequence $\{t_k\}$ with $t_k \to \bar{t}$. For any k either $\partial U \cap G(t_k) = \emptyset$ and hence $\Phi(t_k) = -\infty$ or there is an element $\xi_k \in \partial U \cap G(t_k)$. If $\Phi(t_k) > -\infty$ only for a finite number of k, then $\overline{\lim}_{k \to \infty} \Phi(t_k) = -\infty \leq \Phi(\bar{t})$. In the other case we have an infinite number of elements ξ_k as above and each accumulation point of this sequence belongs to $\partial U \cap G(\bar{t})$ from which again $\Phi(\bar{t}) \geq \lim_{k \to \infty} f_0(t_k, \xi_k) = \lim_{k \to \infty} \Phi(t_k)$ follows. The relation $r < \varepsilon(\bar{t})$ implies $f_0(\bar{t}, x(\bar{t})) > \Phi(\bar{t})$. Thus, because of the continuity of f_0 and x and the upper semicontinuity of Φ, there is a neighbourhood $V' \subset V$ of $\bar{t}$ with $f_0(t, x(t)) > \Phi(t)$ for $t \in V'$. Therefore any global maximizer $\widetilde{x}(t)$ of $\widetilde{P}(t)$ for $t \in V'$ cannot be situated on the boundary of U and hence $\widetilde{x}(t)$ is even a local maximizer of P(t). Property b) now implies that $\widetilde{x}(t)$ is also a stationary solution of P(t). Thus because of the uniqueness of the stationary solution $x(t)$ in U, we conclude $\widetilde{x}(t) = x(t)$ for $t \in V'$, i.e.

$$f_0(t, x(t)) > f_0(t, \xi) \text{ for each } t \in V' \text{ and } \xi \in U \cap G(t), \xi \neq x(t). \tag{10}$$

If we now suppose that $\bar{\varepsilon} = 0$, then (because of the maximality of $\varepsilon(t_k)$) there is a sequence $\{\xi_k\}, \xi_k \in G(t_k)$, such that $\xi_k \to \bar{\xi}$ and $f_0(t_k, x(t_k)) > f_0(t_k, \xi_k)$ for all k what is a contradiction to (10), hence $\bar{\varepsilon} > 0$ ∎

Remark 1: Our assumptions in Assertion 3 guarantee even the strict inequality in (6) for $\xi = x(t)$.

Remark 2: In Assertion 3 the condition (LICQ) can be replaced by the weaker Mangasarian-Fromovitz Constraint Qualification (MFCQ) which means that there is a $z(t) \in \mathbb{R}^n$ such that grad $f_i(t, x(t))^\tau z(t) < 0$ for all $i \in I_0(t)$. Then the multipliers $\lambda(t)$ are not necessary unique and the sets $I^*(t)$ and $W^*(t)$ are to be replaced by the sets $I^*(t) = \{i \in \{1, ..., l\} | \lambda_i(t) > 0\}$ and $W^*(\lambda(t)) = \{h \in \mathbb{R}^l | h^\tau \text{grad}_\xi f_i(t, x(t)) = 0, i \in I^*(\lambda(t))\}$, respectively. Condition (S) must be fulfilled for each multiplier $\lambda_i(t)$.

4. Statement of the sufficiency theorem

To prove the announced theorem we use the following assumptions (a) - (c) to (P).

(a) Let be given an admissible pair (x, u) to (P) and let x indicate a decomposition of Ω in domains Ω^j with piecewise smooth boundary, where $x \in C^{1,n}(\bar{\Omega}^j)$ $(j = 1,...,\nu)$.

Moreover, with the quadratic statement of S in the dual problem,

$$S^\alpha(t,\xi) = a^\alpha(t) + p^\alpha(t)^\tau(\xi - x(t)) \tag{11}$$
$$+ \tfrac{1}{2}(\xi - x(t))^\tau Q^\alpha(t)(\xi - x(t)),$$

$Q^\alpha \in M^{n \times n}(\bar{\Omega})$, where $M^{n \times n}(\bar{\Omega})$ is the set of all symmetric $n \times n$-matrix functions with components in $C^1(\bar{\Omega}^j) \cap C^0(\bar{\Omega})$, $p^\alpha \in C^{1,n}(\bar{\Omega}^j) \cap C^{0,n}(\bar{\Omega})$ and $a^\alpha \in C^1(\bar{\Omega}^j)$ $(\alpha = 1,...,m)$ let be

$$N(t,\varepsilon,\delta) = \left\{ (\xi,y) \in G(t) \times \mathbf{R}^n \mid \|\xi - x(t)\| < \varepsilon, \|y - p(t)\| < \delta \right\}$$
$$N^j(\varepsilon,\delta) = \left\{ (t,\varepsilon,\delta) \mid t \in \bar{\Omega}^j, (\varepsilon,y) \in N(t,\varepsilon,\delta) \right\} \qquad (\varepsilon,\delta > 0).$$

(b) Let $H(t,\cdot,\cdot) \in C^1(N(t,\varepsilon,\delta))$ for each $t \in \bar{\Omega}^j$, $H(\cdot,\cdot,\cdot)$ and $\mathrm{grad}_{\xi,y} H(\cdot,\cdot,\cdot)$ be continuous on $N^j(\varepsilon,\delta)$, $\mathrm{grad}_{\xi,y} H(t,\cdot,\cdot)$ be locally Lipschitzian on $N(t,\varepsilon,\delta)$, $H(t, x(t), p(t)) < \infty$ for each $t \in \bar{\Omega}^j$, and let the mapping $(t,\xi,y) \to \partial_\xi(\mathrm{grad}_\xi H(t,\xi,y))$ be locally bounded and closed on $N^j(\varepsilon,\delta)$.

(c) Let f_i $(i = 1,...,l)$ belong to the class of functions described in $(8)_1 - (8)_4$ and (LICQ) be fulfilled with respect to $G(t)$ and $\bar{\Omega}$ instead of $\hat{\Omega}$.

Than we can finally show the sufficient locally optimality condition for (P).

Theorem: *Let (x, u) be an admissible pair to (P) satisfying the assumptions $(a) - (c)$. Let be chosen $\lambda_i(t)$ $(i = 1,...,l)$ in such way that for $j = 1,...,\nu$ the conditions*

$$-\sum_{\alpha=1}^{m} p_{t_\alpha}^\alpha(t) = \mathrm{grad}_\xi \left\{ H(t, x(t), p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t, x(t)) \right\}, \tag{12}$$

$$\lambda_i(t) \geq 0, \ \lambda_i(t) f_i(t, x(t)) = 0 \text{ on } \bar{\Omega}^j \ (i = 1,...,l), \tag{13}$$

$$x_{t_\alpha}(t) = \mathrm{grad}_{y_\alpha} H(t, x(t), p(t)) \text{ on } \bar{\Omega}^j \ (\alpha = 1,...,m), \tag{14}$$

$$H(t, x(t), p(t)) = h(t, x(t), u(t), p(t)) \text{ on } \bar{\Omega} \tag{15}$$

are fulfilled and each $M(t)$,

$$M(t) = \sum_{\alpha=1}^{m} \left[Q_{t_\alpha}^\alpha(t) + \partial_{\xi\xi}^2 H(t, x(t), p(t)) \right.$$
$$+ \partial_{\xi y_\alpha}^2 H(t, x(t), p(t)) Q^\alpha(t) + Q^\alpha(t) \partial_{y_\alpha \xi}^2 H(t, x(t), p(t)) \tag{16}$$
$$\left. + \sum_{\beta=1}^{m} Q^\alpha(t) \partial_{y_\alpha y_\beta}^2 H(t, x(t), p(t)) Q^\beta(t) \right] + \sum_{i \in I^+(t)} \lambda_i(t) \partial_{\xi\xi}^2 f_i(t, x(t)),$$

is negative definite on $W^+(t)$. Then the pair (x, u) provides a strong local minimum for (P).

Proof: The idea of the proof is to apply Assertion 2 by using the quadratic statement of S in (11). First we show that the conditions (HJ), (M) and (B) of Assertion 2 are satisfied. Indeed, we can choose $\sum a_{t_\alpha}^\alpha(t)$ in such a way that (HJ) is fulfilled on $\overline{\Omega}$ (because of (b) this expression is well defined), namely

$$a_{t_\alpha}^\alpha(t) = p^\alpha(t)^\tau x_{t_\alpha}(t) - 1/_m H(t, x(t), p(t)) \text{ and } a^\alpha \in C^1(\overline{\Omega}^j).$$

Further, (M) is true according to (15). To show that (B) is true for S in (11) we note only that

$$\inf_{\rho \in Q} \left\{ \sum_{j=1}^{\nu} \int_{\partial\Omega^j} S(s, \zeta(s)) n^j(s) \, do(s) \right\}$$

$$= \sum_{j=1}^{\nu} \int_{\partial\Omega^j} a(s) n^j(s) \, do(s) = \sum_{j=1}^{\nu} \int_{\partial\Omega^j} S(s, \zeta(s)) n^j(s) \, do(s)$$

since $p^\alpha \in C^{0, n}(\overline{\Omega})$ and $Q^\alpha \in M^{n \times n}(\overline{\Omega})$.

Now we shall prove that for some $\varepsilon > 0$, S belongs to $\mathbf{S}_\varepsilon$. Condition 1_ε of Assertion 2 holds because of assumption (a) of the theorem. In order to verify condition 2_ε of Assertion 2 we define f_0 by

$$f_0(t, \xi) = \sum_{\alpha=1}^{m} \left\{ a_{t_\alpha}^\alpha(t) + p_{t_\alpha}^\alpha(t)^\tau (\xi - x(t)) \right.$$

$$- p^\alpha(t) x_{t_\alpha}(t) + 1/_2 (\xi - x(t)) Q_{t_\alpha}^\alpha(t)(\xi - x(t))$$

$$\left. - (\xi - \dot{x}(t))^\tau Q^\alpha(t) x_{t_\alpha}(t) \right\} + H\Big(t, \xi, p(t) + Q(t)(\xi - x(t))\Big).$$

Hence by the special form of a^α it follows

$$f_0(t, \xi) = \sum_{\alpha=1}^{m} \left\{ p_{t_\alpha}^\alpha(t)^\tau (\xi - x(t)) \right.$$

$$+ 1/_2 (\xi - x(t))^\tau Q_{t_\alpha}^\alpha(\xi - x(t)) - (\xi - x(t))^\tau Q^\alpha(t) x_{t_\alpha}(t) \Big\} \tag{17}$$

$$+ H\Big(t, \xi, p(t) + Q(t)(\xi - x(t))\Big) - H(t, x(t), p(t)).$$

Obviously, S fulfils the Hamilton-Jacobi inequality on X^j if x maximizes $f_0(t, \cdot)$ on X_ε^j for some $\varepsilon > 0$. Moreover, this is so if the inequality $f_0(t, \xi) \le f_0(t, x(t))$ holds for $\xi \in G(t) \cap K_\varepsilon(x(t))$, $t \in \overline{\Omega}^j$. We want to use Assertion 3. Therefore we have to ensure that for $\hat{\Omega} = \overline{\Omega}^j$ the functions f_0 and $f_1, \ldots, f_l$ belong to the class defined in $(8)_1 - (8)_4$ ($j = 1, \ldots, \nu$). This is true if the assumptions (b) and (c) are satisfied.

We now choose p in such way that $x(t)$ is a stationary point of the problem

$\mathbf{P}^j(\mathbf{t})$ maximize $f_0(t, \xi)$ subject to $\xi \in G(t)$, $t \in \overline{\Omega}^j$ ($j = 1, \ldots, \nu$).

Therefore $\text{grad}_\xi [f_0(t, \xi) + \sum_{i=1}^{l} \lambda_i(t) f_i(t, \xi)]_{\xi = x(t)}$ must be vanish and (13) must be satisfied on Ω^j ($j = 1, \ldots, \nu$). According to (17) we obtain

$$\text{grad}_\xi \left[f_0(t, \xi) + \sum_{i=1}^{l} \lambda_i(t) f_i(t, \xi) \right]_{\xi = x(t)}$$

$$= \sum_{\alpha=1}^{m} p_{t_\alpha}^\alpha(t) + \text{grad}_\xi \left[H(t, x(t), p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t, x(t)) \right]$$

$$- \sum_{\alpha=1} Q^\alpha(t) \left[x_{t_\alpha}(t) - \text{grad}_{y^\alpha} H(t, x(t), p(t)) \right] = 0$$

if the canonical differential equations (12) and (14) are fulfilled on $\overline{\Omega}^j$. Further on w to verify that the condition (S) for $x(t)$ remains true. From [3, p. 55], the inclusi

$$\partial^2 f_0(t, \xi) \subseteq \sum_{\alpha=1}^{m} \left[Q_{t_\alpha}^\alpha(t) + \partial_{\xi\xi}^2 H(t, x(t), p(t)) \right.$$
$$+ \partial_{\xi y^\alpha} H(t, x(t), p(t)) Q^\alpha(t) + \partial_{y^\alpha \xi} Q^\alpha(t) H(t, x(t), p(t))$$
$$\left. + \sum_{\beta=1}^{m} Q^\alpha(t) \partial_{y^\alpha y^\beta}^2 H(t, x(t), p(t)) Q^\beta(t) \right]$$

holds. If we denote the set on the right-hand side of (18) by $N(t)$ and if for each Λ $N(t) + \sum_{i \in I^+(t)} \lambda_i(t) \partial_{\xi\xi}^2 f_i(t, x(t))$ it follows $h^T M(t) h < 0$ on $W^+(t) \setminus \{0\}$, then ϵ ding to (18) for each $M(t)$ with

$$M(t) \in \partial_{\xi\xi}^2 f_0(t, x(t)) + \sum_{i \in I^+(t)} \lambda_i(t) \partial_{\xi\xi}^2 f_i(t, x(t))$$

it follows $h^T M(t) h < 0$ on $W^+(t) \setminus \{0\}$. By assumption, the inclusion (16) is true, ϵ condition (S) before Assertion 3 is fulfilled. Taking assumption (c) into account, As: 3 can be applied to our situation, which completes the proof ∎

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