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On an Application of a Modification of the Zincenko Method to the Approximation of Implicit Functions

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We use the Zincenko iteration to approximate implicit functions in a Banach space by solving a linear algebraic system of finite order. The non-linear equations involved contain a non-differentiable term. Our hypotheses are more general than Zabrejko and Nguen's [10], in this case.

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1. Introduction

Let E, Λ be Banach spaces and denote by $U(x_0, R)$ the closed ball with center $x_0 \in E$ and of radius R in E. We will use the same symbol for the norm $\|\cdot\|$ in both spaces. Let P be a linear projection operator ($P^2 = P$) which projects E on its subspace E_P and set $Q = I - P$. Suppose that the non-linear operators $F(x, \lambda)$ and $G(x, \lambda)$ with values in E are defined for $x \in D$, where D is some convex subset of E containing $U(x_0, R)$ and $\lambda \in U(\lambda_0, S)$. For each fixed $\lambda \in U(\lambda_0, S)$ the operator $PF(z, \lambda)$ will be assumed to be Fréchet differentiable for all $z \in D$. Then $PF'(x, \lambda)$ will denote the Fréchet derivative of the operator $PF(z, \lambda)$ with respect to the argument z at $z = x$. Moreover, we assume that $(PF'(x_0, \lambda_0))^{-1}$ exists and

$$\left\|(PF'(x_0, \lambda_0))^{-1}(PF'(x, \lambda) - PF'(y, \lambda))\right\| \le k_1(r, s)\|x - y\|, \tag{1}$$

$$\left\|(PF'(x_0, \lambda_0))^{-1}(PF'(x_0, \lambda) - PF'(x_0, \lambda_0))\right\| \le k_2(s)\|\lambda - \lambda_0\|, \tag{2}$$

$$\left\|(PF'(x_0, \lambda_0))^{-1}\left[(QF(x, \lambda) + G(x, \lambda)) - (QF(y, \lambda) + G(y, \lambda))\right]\right\| \le k_3(r, s)\|x - y\|, \tag{3}$$

for all $x, y \in U(x_0, r) \subset U(x_0, R)$ and $\lambda \in U(\lambda_0, s) \subset U(\lambda_0, S)$. Here k_1, k_2 and k_3 are non-decreasing functions on the intervals $[0, R] \times [0, S]$, $[0, R]$ and $[0, R] \times [0, S]$, respectively. We use a modification of the Zincenko iteration [11]

$$x_{n+1}(\lambda) = x_n(\lambda) - (PF'(x_n(\lambda), \lambda))^{-1}(F(x_n(\lambda), \lambda) + G(x_n(\lambda), \lambda)) \quad (n \ge 0) \tag{4}$$

to approximate a solution $x^* = x^*(\lambda)$ of the equation

$$F(x, \lambda) + G(x, \lambda) = 0. \tag{5}$$

By x_0 we mean $x_0(\lambda)$. That is x_0 depends on the λ used in (4).

It can easily be shown by induction on n that under the above hypotheses $F(x_n(\lambda),\lambda) + G(x_n(\lambda),\lambda)$ belongs to the domain of $PF'(x_n(\lambda),\lambda)^{-1}$ for all $n \geq 0$. Therefore, if the inverses exist (at it will be shown later in Theorem 1), then the iterates x_n can be computed for all $n \geq 0$.

Our assumptions (1) - (3) generalize the ones made by Zabrejko and Nguen [10], Yamamoto [9] and (for $G = 0$) Potra and Pták [6]. The iterates generated by the above authors cannot be easily computed in infinite dimensional spaces since the inverses of the linear operators involved ($P = I$, then) may be to difficult or impossible to find. It is easy to see, however, that the solution of equations (4) reduces to solving certain operator equations in the space E_P. If, moreover, E_P is a finite-dimensional space of dimension N, we obtain a system of linear algebraic equations of at most order N. Furthermore, several authors have treated the case when $G = 0$, $P \neq I$ provided that k_1 and k_2 are constants (or not) [1,2,4 - 6].

We provide sufficient conditions for the convergence of iteration (4) to a locally unique solution $x^*(\lambda)$ of equation (5) as well as several error bounds on the distances $\|x_{n+1}(\lambda) - x_n(\lambda)\|$ and $\|x_n(\lambda) - x^*(\lambda)\|$.

We need to define the functions

$$a_s = k(s)\left\|(PF'(x_0,\lambda_0))^{-1}(F(x_0,\lambda) + G(x_0,\lambda))\right\| \quad (s = 0 \text{ if } \lambda = \lambda_0),$$

$$\omega_s(r) = \int_0^r k_1(t,s)\,ds\,, \quad k_4(s) = \int_0^s k_2(t)\,dt\,, \quad k(s) = (1 - k_4(s))^{-1}$$

provided that

$$k_4(S) < 1, \quad \varphi_s(r) = a_s + k(s)\int_0^r \omega_s(t)\,dt - r,$$

$$\psi_s(r) = k(s)\int_0^r k_3(t,s)\,dt\,, \quad \chi_s(r) = \varphi_s(r) + \psi_s(r)$$

and the iteration ($y_0 = x_0$, $n \geq 0$)

$$y_{n+1}(\lambda) = y_n(\lambda) - (PF'(x_0,\lambda_0))^{-1}(F(y_n(\lambda),\lambda) + G(y_n(\lambda),\lambda)). \tag{6}$$

2. Convergence results

We can now formulate the following result.

Theorem 1: *Suppose that the function $\chi_s = \chi_s(r)$ has a unique zero $\rho^* = \rho_s^*$ in $[o,R]$ and $\chi_s(R) \leq 0$. Then the following statements are true.*

(a) *Equation (5) has a unique solution $x^* = x^*(\lambda) \in U(x_0,R)$ with $x^*(\lambda) \in U(x_0,\rho^*)$.*

(b) *The estimates*

$$\|y_{n+1}(\lambda) - y_n(\lambda)\| \leq v_{n+1} - v_n \tag{7}$$

and

$$\|y_n(\lambda) - x^*(\lambda)\| \leq \rho^* - v_n \tag{8}$$

are true where the scalar sequence $\{v_n\}_{n\geq 0}$ is monotonically increasing and convergent to ρ^* with

$$v_{n+1} = d_s(v_n) \quad (n \geq 0,\ v_0 = 0) \quad \text{and} \quad d_s(r) = r + \chi_s(r) \tag{9}$$

Proof: It is a simple calculus to show that the sequence $\{v_n\}$ is monotonically increasing and convergent to ρ^* (see also, [10: p. 675]). Using induction to n we will show that the estimate (7) is true, from which (8) will follow immediately. From (6) for $n = 0$, we get

$$\|y_1(\lambda) - y_0(\lambda)\| \le \|(PF'(x_0,\lambda))^{-1}(F(x_0,\lambda) + G(x_0,\lambda))\| \le a_s = d_s(0) = v_1 - v_0.$$

That is, the estimate (7) is true for $n = 0$. Let us assume that (7) is true for $n < k$. Then by (6), (1), (3), [10: p. 674] and the induction hypothesis we get

$$\begin{aligned}
&\|y_{n+1}(\lambda) - y_k(\lambda)\| \\
&\le \|(y_k(\lambda) - y_{k-1}(\lambda)) - (PF'(x_0,\lambda_0))^{-1}(PF(y_k(\lambda),\lambda) - PF(y_{k-1}(\lambda),\lambda))\| \\
&\quad + \|(PF'(x_0,\lambda_0))^{-1}\{(QF(y_k(\lambda),\lambda) + G(y_k(\lambda),\lambda)) - (QF(y_{k-1}(\lambda),\lambda) + G(y_{k-1}(\lambda),\lambda))\}\| \\
&\le \int_0^1 \|(PF'(x_0,\lambda_0))^{-1}\{PF'((1-t)y_{k-1}(\lambda) + t y_k(\lambda)) - PF'(x_0,\lambda_0)\}\| \|y_k(\lambda) - y_{k-1}(\lambda)\| dt \\
&\quad + \|(PF'(x_0,\lambda_0))^{-1}\{(QF(y_k(\lambda),\lambda) + G(y_k(\lambda),\lambda)) - (QF(y_{k-1}(\lambda),\lambda) + G(y_{k-1}(\lambda),\lambda))\}\| \\
&\le \int_0^1 \omega((1-t)v_{k-1} + t v_k)(v_k - v_{k-1}) dt + \int_{v_{k-1}}^{v_k} k_3(t,s) dt \\
&\le k(s)\left[\int_{v_{k-1}}^{v_k} \omega_s(t) dt + \int_{v_{k-1}}^{v_k} k_3(t,s) dt\right] = d_s(v_k) - d_s(v_{k-1}) = v_{k+1} - v_k.
\end{aligned}$$

That is, the estimate (7) is true for $n = k$. Hence, $\{y_n(\lambda)\}$ is a cauchy sequence in a Banach space and as such converges to some $x^*(\lambda) \in U(x_0,\rho^*) \subset U(x_0,R)$. By letting $n \to \infty$ in (6) we deduce that $x^*(\lambda)$ is a solution of equation (5).

We will now show that $x^*(\lambda)$ is the unique solution of equation (5) in $U(x_0,R)$, by considering the sequences given by ($n \ge 0$; $z_0 \in U(x_0,R)$ and $w_0 = R$)

$$z_{n+1}(\lambda) = z_n(\lambda) - (PF'(x_0,\lambda_0))^{-1}(F(z_n(\lambda),\lambda) + G(z_n(\lambda),\lambda)), \tag{10}$$

and

$$w_{n+1} = d_s(w_n). \tag{11}$$

It is enough to show that

$$\|y_n(\lambda) - z_n(\lambda)\| \le w_n - v_n, \; n \ge 0. \tag{12}$$

It is a simple calculus to show that the scalar sequence given by (11) is monotonically convergent to ρ^*. Hence, if for z_0 we choose the second solution $y^*(\lambda) \in U(x_0,r)$ of equation (5), then, by (12), $\|x^*(\lambda) - y^*(\lambda)\| \le w_n - v_n$. That is, $x^*(\lambda) = y^*(\lambda)$.

For $n = 0$, (12) becomes $\|y_0 - x_0\| \le R - 0 = R$. Hence, (12) is true for $n = 0$. Let us assume that (12) holds for $n \le k$. Then by (6), (10) as before we get

$$\begin{aligned}
&\|y_{k+1}(\lambda) - z_{k+1}(\lambda)\| \\
&\le \|(z_k(\lambda) - y_k(\lambda)) - (PF'(x_0,\lambda_0))^{-1}(PF(z_k(\lambda),\lambda) - PF(y_k(\lambda),\lambda))\| \\
&\quad + \|(PF'(x_0,\lambda_0))^{-1}\{(QF(z_k(\lambda),\lambda) + G(z_k(\lambda),\lambda)) - (QF(y_k(\lambda),\lambda) + G(y_k(\lambda),\lambda))\}\| \\
&\le \int_0^1 \|(PF'(x_0,\lambda_0))^{-1}\{PF'((1-t)y_k(\lambda) + t z_k(\lambda)) - PF'(x_0,\lambda_0)\}\| \|z_k(\lambda) - y_k(\lambda)\| dt
\end{aligned}$$

$$+\int_{v_k}^{w_k} k_3(t,s)dt$$

$$\leq \int_0^1 \omega_s((1-t)v_k + t w_k)(w_k - v_k)dt + \int_{v_k}^{w_k} k_3(t,s)dt$$

$$\leq k(s)\left[\int_{v_k}^{w_k} \omega(s)dt + \int_{v_k}^{w_k} k_3(t,s)dt\right] = d_s(w_k) - d_s(v_k) = w_{k+1} - v_{k+1}.$$

That completes the proof of the theorem ∎

We can now formulate the main result.

Theorem 2: *Suppose that the hypotheses of Theorem 1 are satisfied. Then the following statements are true.*

(a) *The sequence* (ρ_n) *given by*

$$\rho_{n+1} = \rho_n + u_s(\rho_n) \quad (\rho_0 = 0) \text{ with } u_s(r) = -\chi_s(r)/\varphi_s'(r)$$

is monotonically increasing and converges to ρ^*.

(b) *The iterates generated by* (4) *are well defined for all* $n \geq 0$ *and remain in* $U(x_0, \rho^*)$.

(c) *Moreover, the estimates*

$$\|x_{n+1}(\lambda) - x_n(\lambda)\| \leq \rho_{n+1} - \rho_n \quad (n \geq I) \tag{13}$$

and

$$\|x_{n+1}(\lambda) - x^*(\lambda)\| \leq \rho^* - \rho_n \quad (n \geq 0) \tag{14}$$

are true.

Proof: Part (a) can be shown exactly as in Proposition 3 in [10: p. 677]. We will only show (13) since (14) will follow then from it immediately. For $n = 0$ we get $\|x_1(\lambda) - x_0(\lambda)\| \leq a_s = \rho_1 - \rho_0$. That is, (13) is true for $n = 0$. Let us assume that (13) is true for $n < k$. By the induction hypothesis

$$\|x_k(\lambda) - x_0\| \leq \sum_{j=1}^{k} \|x_j(\lambda) - x_{j-1}(\lambda)\| \leq \sum_{j=1}^{k} (\rho_j - \rho_{j-1}) = \rho_k,$$

The Banach lemma on invertible operators, (2) and the estimate

$$\|(PF'(x_0,\lambda_0))^{-1}(PF'(x_k(\lambda),\lambda) - PF'(x_0,\lambda_0))\| \leq k(s)\omega_s(\rho_k) \leq k(s)\omega_s(\rho^*) = \varphi_s'(\rho^*) + 1 \leq 1,$$

it follows that $PF'(x,\lambda)$ is invertible for all $(x,\lambda) \in U(x_0,R) \times U(\lambda_0,S)$ and

$$\begin{aligned} &\|(PF'(x_k(\lambda),\lambda))^{-1} PF'(x_0,\lambda_0)\| \\ &\leq \|\{I + (PF'(x_0,\lambda))^{-1}(PF'(x,\lambda) - PF'(x_0,\lambda_0))\}^{-1}\| \, \|(PF'(x_0,\lambda))^{-1} PF'(x_0,\lambda_0)\| \\ &\leq -k(s)/\varphi_s'(\rho_k). \end{aligned} \tag{15}$$

Then by (4), (1) - (3), (15) and the induction Hypothesis we get

$$\begin{aligned} &\|x_{k+1}(\lambda) - x_k(\lambda)\| \\ &= \|(PF'(x_k(\lambda),\lambda))^{-1}(F(x_k(\lambda),\lambda) + G(x_k(\lambda),\lambda))\| \\ &\leq \|(PF'(x_k(\lambda),\lambda))^{-1}\{F(x_k(\lambda),\lambda) - F(x_{k-1}(\lambda),\lambda) \end{aligned}$$

$$-PF'(x_{k-1}(\lambda),\lambda)(x_k(\lambda) - x_{k-1}(\lambda)) + G(x_k(\lambda),\lambda) - G(x_{k-1}(\lambda),\lambda)\}\|$$

$$\leq \|(PF'(x_k(\lambda),\lambda))^{-1}PF'(x_0,\lambda_0)\|\Big[\int_0^1 \|(PF'(x_0,\lambda_0))^{-1}$$

$$\times\{(PF'(1-t)x_{k-1}(\lambda) + t x_k(\lambda)) - PF'(x_{k-1}(\lambda))\}\|\,\|x_k(\lambda) - x_{k-1}(\lambda)\|\,dt$$

$$+\|(PF'(x_0,\lambda_0))^{-1}\{(QF(x_k(\lambda),\lambda) + G(x_k(\lambda),\lambda)) - (QF(x_{k-1}(\lambda),\lambda) + G(x_{k-1}(\lambda),\lambda))\}\|\Big]$$

$$\leq -\frac{k(s)}{\varphi_s'(\rho_k)}\int_0^1\{\omega_s((1-t)\rho_{k-1} + t\rho_k) - \omega_s(\rho_{k-1})\}(\rho_k - \rho_{k-1})\,dt - \frac{1}{\varphi_s'(\rho_k)}(\psi_s(\rho_k) - \psi_s(\rho_{k-1}))$$

$$\leq \frac{\varphi_s(\rho_k) - \varphi_s(\rho_{k-1}) - \varphi_s'(\rho_{k-1})(\rho_k - \rho_{k-1}) + \psi_s(\rho_k) - \psi_s(\rho_{k-1})}{\varphi_s'(\rho_k)}$$

$$= \rho_{k+1} - \rho_k.$$

Hence, (13) is true for $n = k$ ∎

We will now derive some a posteriori error bounds for iteration (4). Let

$r_{n,s} = r_n = \|x_n(\lambda) - x_0\|$, $q_{n,s}(r) = q_n(r) = k_1(r_n + r, s)$,

$f_{n,s}(r) = f_n(r) = k_3(r_n + r, s)$ for $r \in [0, R - r_n]$

and set

$$a_{n,s} = a_n = \|x_{n+1}(\lambda) - x_n(\lambda)\|,\ b_{n,s} = b_n = k(s)(1 - k(s)\omega_s(r_n))^{-1}.$$

Without loss of generality we assume that $a_n > 0$. Then exactly as in Theorem 2 in [9: p. 989] we can show

Theorem 3: *Suppose that the hypotheses of Theorem 1 are satisfied. Then the following statements are true.*

(a) *The equation*

$$r = a_n + b_n\int_0^r((r - t)q_n(t) + f_n(t))\,dt$$

has a unique positive zero $\rho_{n,s}^* = \rho_n^*$ *in the interval* $[0, R - r_n]$, $n \geq 0$ *and* $\rho_0^* = \rho^*$.

(b) *The estimates*

$$\|x_n(\lambda) - x^*(\lambda)\| \leq \rho_n^* \leq \begin{cases} (\rho - \rho_n)a_n/\Delta\rho_n & \text{for } n \geq 0 \\ (\rho^* - \rho_n)a_{n-1}/\Delta\rho_{n-1} & \text{for } n \geq 1 \\ \rho^* - \rho_n & \text{for } n \geq 0 \end{cases} \quad (16)$$

are true, where $\Delta\rho_n = \rho_{n+1} - \rho_n$. *That is, our bound* (16) *is sharper than Miel-type bounds* [3, 7] *and more general than the corresponding one in* [9: p. 989] (*for* $P = I$).

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