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On the Periodic Solution Process to the Stochastic Model of Single Species

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Two stochastic models of single species are studied. A necessary and sufficient condition and a sufficient condition for existence of the periodic solution process are obtained, respectively.

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1. Introduction

As is well known, the logistic model of single species is

$$dN/dt = N(b - cN), \tag{1.1}$$

where N is the population density, b and c are positive numbers, b/c is called carrying capacity. $N = 0$ and $N = b/c$ are equilibrium points of (1.1). The second of them is more important for us as it is asymptotically stable, i.e. every solution with initial value $N(0) > 0$ tends to b/c as $t \to +\infty$. In [5], the author discussed (1.1) with periodic coefficients and obtained a sufficient condition for the existence of a unique periodic solution. In general, the environment where a population lives in possesses random property. In [4], May considered the random environment with a stochastic differential equation model. About this model some valuable remarks were given by [6]. We start out from May's idea (also see [3]) and consider the following stochastic population models :

$$dN(t) = N(t)[b(t) - c(t)N^{\alpha}(t)]dt + a(t)N(t)dW(t), \tag{1.2}$$

$$dN(t) = N(t)[b(t) - c(t)\ln N(t)]dt + a(t)N(t)dW(t), \tag{1.3}$$

where a, b and c are periodic continuous functions with period T, $\alpha > 0$ is a constant. W is a Wiener process with $E\{W(t)\} = 0$, $E\{(dW)^2\} = dt$. (1.2) and (1.3) are first order non-linear Ito's stochastic differential equations.

We define

$$\tau'_M = \inf\{t \geq 0 : N(t) \geq M\}, \quad \tau''_M = \inf\{t \geq 0 : N(t) \leq M^{-1}\}$$

$$\tau' = \lim_{M \to \infty} \tau'_M, \quad \tau'' = \lim_{M \to \infty} \tau''_M.$$

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τ' and τ'' are called *explosion time* and *absorption time*, respectively. For every $N(0) > 0$ the solutions of (1.2) and (1.3) are unique up to the time $\tau' \wedge \tau'' = \min(\tau', \tau'')$.

Definition: A stochastic process $\xi(t) = \xi(t, \omega)$ $(-\infty < t < +\infty)$ with values in E is said to be *periodic* with period T if for every finite sequence of numbers $t_1, \ldots, t_n$ the joint distribution of the random variables $\xi(t_1 + h), \ldots, \xi(t_n + h)$ is independent of h, where $h = kT$ $(k = \pm 1, \pm 2, \ldots)$.

Our purpose is to find conditions for the existence of a periodic solution process of (1.2) and (1.3). We obtained a necessary and sufficient condition for the existence of a periodic solution process of (1.2) and a sufficient condition for the existence and distribution of a periodic solution process of (1.3).

The following Lemma 1 is important in proving our main theorems.

Lemma 1 [1]: *Consider Ito's differential equation of the form*

$$dX(t) = b(X(t), t)\,dt + \sigma(X(t), t)\,dW(t). \tag{1.4}$$

Suppose that the coefficients of this equation are T-periodic in t and satisfy the linear growing condition and the Lipschitz condition in every cylinder $U_R \times [0, \infty)$, for $R > 0$, where $U_R = \{X : |X| < R\}$, and suppose further that there exists a function $V = V(t, X)$ which is twice continuously differentiable with respect to X and once continuously differentiable with respect to t in $E_n \times [0, \infty)$, T-periodic in t and satisfies the conditions

$$\inf_{|X| > R} V(t, X) \to \infty \text{ as } R \to \infty, \qquad \inf_{|X| > R} LV(t, X) \to -\infty \text{ as } R \to \infty.$$

Then there exists a solution of equation (1.4) which is a T-periodic Markov process, where L is the generator for (1.4).

Lemma 2: *Assume that y is a T-periodic process and f is a Borel measurable function. Then $f(y)$ is also a T-periodic process.*

The proof of Lemma 2 is simple, therefore we omit it ∎

2. Main results. We are now in a position to prove the following result.

Theorem 2.1: *In* (1.2), *assume that $c(t) > 0$, a and b are periodic continuous functions. $\alpha > 0$ is a constant. Then, for* (1.2),

$$\int_0^T \left(b(t) - \tfrac{1}{2} a^2(t)\right) dt > 0 \tag{2.1}$$

is a necessary and sufficient condition for the existence of a periodic solution process.

Proof: *Sufficiency.* We take the transformation $X(t) = \alpha \ln N(t)$, which is defined up to the time $\tau' \wedge \tau''$. The explosion time of the process X is $\tau = \liminf_{M \to \infty} \{t \geq 0 : |X(t)| \geq M\}$. Obviously, $\tau = \tau' \wedge \tau''$. The transformation $X = \alpha \ln N$ does not change the periodicity of the process from Lemma 2. Ito's formula shows that

$$dX(t) = \alpha\left[b(t) - \tfrac{1}{2} a^2(t) - c(t)\exp(X(t))\right] dt + \alpha a(t)\,dW(t). \tag{2.2}$$

Let us set

$$f(t) = -\int_0^t (b(s) - \tfrac{1}{2}a^2(s))ds + Bt, \tag{2.3}$$

where $B = \tfrac{1}{T}\int_0^T (b(t) - \tfrac{1}{2}a^2(t))dt$. It is easy to see that f is a T-periodic continuous function. Let $Y(t) = \alpha f(t) + X(t)$. By Ito's formula, we have

$$dY(t) = \alpha\left[B - c(t)\exp(-\alpha f(t) + Y(t))\right]dt + \alpha a(t)dW(t). \tag{2.4}$$

Now we take a Liapunov function $V(Y) = Y^2$. By the definition of generator, we have

$$LV = \alpha\left[B - c(t)\exp(-\alpha f(t) + Y(t))\right]2Y(t) + \alpha^2 a^2(t).$$

Since the periodic continuous functions are bounded, therefore $LV \to -\infty$ as $|Y| \to \infty$. From Lemma 1 we obtain that (2.4) has a periodic solution process, then we get that (1.2) has a periodic solution process. Moreover the explosion time is $\tau = \infty$.

Necessity. We want to prove that (1.2) has no T-periodic solution process if $B \le 0$. We consider the case of $a \neq 0$. In fact, if $a = 0$, (1.2) becomes a deterministic model and obviously the equation has no periodic solution in this case. We consider a comparison equation of (2.4) of the form $d\tilde{Y}(t) = \alpha a(t)dW(t)$. By the comparison theorem [2], the same initial value implies that $Y(t) \le \tilde{Y}(t)$ a.s. But, obviously, $\underline{\lim}_{t\to\infty} \tilde{Y}(t) = -\infty$ a.s, therefore Y can not be a periodic process. We return to (1.2) and note that the transforma tions keep the periodicity of a process. As a consequence, the necessity of the theorem is proved ∎

Corollary: *If* $a \equiv 0$, (1.2) *becomes a deterministic model. From Theorem* 2.1 *we know that if c with* $c(t) > 0$ *and b are* T*-periodic continuous functions, then* $\int_0^T b(t)dt > 0$ *is a necessary and sufficient condition for the existence of a* T*-periodic solution of* (1.2).

Remark: This corollary includes a result of [5].

Theorem 2.2: *Assume in* (1.3) *that a, b and c are continuously periodic functions with period* T, $c(t) > 0$. *Then* (1.3) *has a periodic solution process with period* T, *moreover, we* derive *the distribution of this solution process in the formulas* (2.9) - (2.12).

Proof: We take the transformation $X = \ln N$ which is defined up to $\tau' \wedge \tau''$. By Ito's formula we have

$$dX(t) = \left[b(t) - \tfrac{1}{2}a^2(t) - c(t)X(t)\right]dt + a(t)dW(t). \tag{2.5}$$

Setting $V(X) = X^2$ we know that

$$LV = \left[b(t) - \tfrac{1}{2}a^2(t) - c(t)X\right]2X + a^2(t) \to -\infty \text{ as } |X| \to \infty$$

since $c(t) > 0$. By Lemma 1, (2.5) has a T-periodic solution process. Then we use Lemma 2, therefore (1.3) has a T-periodic solution process. The equation (2.5) is linear, so its solution has an expression of the form (see [7])

$$X(t) = \exp\left(-\int_{t_0}^{t} c(s)\,ds\right) X(t_0) + \int_{t_0}^{t} \exp\left(-\int_{s}^{t} c(u)\,du\right) a(s)\,dW(s) + \int_{t_0}^{t} \exp\left(-\int_{s}^{t} c(u)\,du\right)\left[b(s) - \tfrac{1}{2} a^2(s)\right] ds. \tag{2.6}$$

The mean of the process X is

$$EX(t) = \exp\left(-\int_{t_0}^{t} c(s)\,ds\right) EX(t_0) + \int_{t_0}^{t} \exp\left(-\int_{s}^{t} c(u)\,du\right)\left[b(s) - \tfrac{1}{2} a^2(s)\right] ds \tag{2.7}$$

The covariance function of the process X is

$$\mu(s,t) = \exp\left(-\int_{t_0}^{s} c(u)\,du - \int_{t_0}^{t} c(u)\,du\right) DX(t_0) + 2\int_{t_0}^{s\wedge t} \exp\left(-\int_{u}^{s} c(v)\,dv - \int_{u}^{t} c(v)\,dv\right) a^2(u)\,du.$$

The variance of X is

$$DX(t) = \exp\left(-2\int_{t_0}^{t} c(s)\,ds\right) DX(t_0) + 2\int_{t_0}^{t} \exp\left(-2\int_{s}^{t} c(u)\,du\right) a^2(s)\,ds. \tag{2.8}$$

If we choose the initial value such that

$$EX(t_0) = \left(1 - \exp\left(-\int_{t_0}^{T+t_0} c(s)\,ds\right)\right)^{-1} \times \int_{t_0}^{T+t_0} \exp\left(-\int_{s}^{T+t_0} c(u)\,du\right)\left[b(s) - \tfrac{1}{2} a^2(s)\right] ds,$$

$$DX(t_0) = \left(1 - \exp\left(-2\int_{t_0}^{T+t_0} c(u)\,du\right)\right)^{-1} 2\int_{t_0}^{T+t_0} \exp\left(-2\int_{s}^{T+t_0} c(u)\,du\right) a^2(s)\,ds,$$

then

$$EX(t) = EX(t+T),\quad DX(t) = DX(t+T),\quad \mu(s,t) = \mu(s+T, t+T),$$

and the correlation function is

$$\gamma(s,t) = \mu(s,t)/\sqrt{DX(s)\,DX(t)} = \gamma(s+T, t+T).$$

Now we assume that $X(t_0)$ is of Gaussian type with $N(EX(t_0), DX(t_0))$. Then X is also Gaussian with $N(EX(t), DX(t))$ [7]. The joint distribution of $(X(s), X(t))$ is

$$N(EX(s), EX(t), DX(s), DX(t), \gamma(s,t)).$$

The transition probability density of $(X(t) \mid X(s) = x(s))$ is

$$N\left(EX(t) + r(s,t)\sqrt{DX(t)\,DX(s)}\,(x(s) - EX(s)),\ DX(t)(1 - r^2(s,t))\right).$$

We note the transformation $X(t) = \ln N(t)$, so the distribution of $N(t)$ is logarithmic normal, its density is

$$p(n(t)) = \begin{cases} 0 & , n(t) \le 0 \\ 1/\left(\sqrt{2\pi DX(t)}\, n(t)\right) \exp\left(\dfrac{-(\ln n(t) - EX(t))^2}{2\,DX(t)}\right) & , n(t) > 0 \end{cases} \tag{2.9}$$

and $(N(s), N(t))$ is a two-dimensional Gaussian distribution with the density

$$p(n(s), n(t))$$

$$= 1\Big/\Big(2\pi\sqrt{(1-r^2(s,t)DX(t)DX(s)}\,n(s)n(t)\Big)$$
$$\times \exp\Big(\big(-1/2(1-r^2(s,t))\big)\Big(\big(\ln n(s)-EX(s)\big)^2/DX(s) \qquad (2.10)$$
$$-2r(s,t)\big(\ln n(s)-EX(s)\big)\big(\ln n(t)-EX(t)\big)\Big/\sqrt{DX(t)DX(s)}$$
$$+\big(\ln n(t)-EX(t)\big)^2/DX(t)\Big)\Big)$$

if $n(t), n(s) > 0$ and $p(n(s), n(t)) = 0$ otherwise. The transition probability density of $(N(t) | N(s) = n(s))$ is

$$p(n(t) | n(s)) = p(n(s), n(t))/p(n(s)). \qquad (2.11)$$

Since $N(t)$ is a Markov process, we have thus got the family of the finite-dimensional distributions. Especially, we have

$$\begin{aligned} EN(t) &= \exp\big(EX(t) + 1/2\, DX(t)\big) \\ DN(t) &= \exp\big(2EX(t) + DX(t)\big)\big(\exp(DX(t)) - 1\big). \end{aligned} \qquad (2.12)$$

Due to (2.7) and (2.8), the mean and variance in (2.12) are known ∎

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