Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 3, 415-416

# A Strengthening of a Lemma on Continuous Families of Closed Convex Sets

R. KLÖTZLER

Let $X$ be a normed linear space and $E(\cdot)$ a family of set-valued mappings on a metric space $T$. It will be shown the following strengthening of a result by S. Rolewicz. If $E(\cdot)$ is lower semi-continuous in $t_0$ and $0 \in \operatorname{int} E(t_0)$, then $0 \in \operatorname{int} E(t)$ holds for sufficiently neighbouring $t$ of $t_0$ too.

Key words: *Semi-continuous set-valued mapping, interior points*
AMS subject classification: 54 C 60

In the book [2: p.174] by S. ROLEWICZ the following useful lemma is stated.

**Lemma 1**: *Let $X$ be a normed vector space, $T$ a metric space with the metric $\rho$, as well as $E(t) \subseteq X$, $t \in T$, a family of closed convex sets, which is continuous in $t_0 \in T$. We assume $\operatorname{int} E(t_0) \neq \Phi$. Then there is a number $\eta > 0$ such that $\operatorname{int} E(t) \neq \Phi$ holds for every $t \in T$ with $\rho(t_0, t) < \eta$.*

Now we shall prove the following strengthening of this lemma.

**Theorem**: *Let $X$ be a normed vector space, $T$ a metric space with the metric $\rho$, as well as $E(t) \subseteq X$, $t \in T$, a family of closed convex sets, which is lower semi-continuous in $t_0 \in T$. We assume the zero element $0 \in \operatorname{int} E(t_0)$. Then there is a number $\eta > 0$ such that $0 \in \operatorname{int} E(t)$ for every $t \in T$ with $\rho(t_0, t) < \eta$.*

Proving of theorem we shall use the following lemma by H. RADSTRÖM [1].

**Lemma 2**: *Let $A, B, C$ be subsets of a normed vector space $X$. If $A + C \subseteq B + C$ holds for a convex and closed $B$ and bounded $C$, then the inclusion $A \subseteq B$ follows.*

Further we introduce the following denotations:

$V_\varepsilon = B(0, \varepsilon)$ is the closed ball in $X$ with the radius $\varepsilon$ and the center $0$. (1)

$M_{-\varepsilon} = \{ z \in M \mid z + V_\varepsilon \subseteq M \}$ for any set $M \subseteq X$. (2)

**Assertion 1**: *For each convex and closed set $M \subseteq X$ we have the equation* $(M + V_\varepsilon)_{-\varepsilon} = M$.

**Proof**: Per definitionem (2) the relation

$$(M + V_\varepsilon)_{-\varepsilon} = \{ z \in M + V_\varepsilon \mid z + V_\varepsilon \subseteq M + V_\varepsilon \} \supseteq M \tag{3}$$

is evident. We shall prove that even $(M + V_\varepsilon)_{-\varepsilon} = M$ holds. For this end we assume the contrary. Thus there is an element $z_0 \in (M + V_\varepsilon)_{-\varepsilon}$ which does not belong to $M$. On the other hand, since (3), $z_0 + V_\varepsilon \subseteq M + V_\varepsilon$ follows and in consequence of Lemma 2 $z_0 \in M$ in contradiction to $z_0 \notin M$ ∎

**Assertion 2:** *From $M_1 \supseteq M_2$ follows $(M_1)_{-\varepsilon} \supseteq (M_2)_{-\varepsilon}$.*

**Proof:** We have

$$(M_2)_{-\varepsilon} = \{ z \in M_2 \mid z + V_\varepsilon \subseteq M_2 \}$$
$$\subseteq \{ z \in M_2 \mid z + V_\varepsilon \subseteq M_1 \} \subseteq \{ z \in M_1 \mid z + V_\varepsilon \subseteq M_1 \} = (M_1)_{-\varepsilon} \blacksquare$$

**Proof of the Theorem:** On account of the lower semi-continuity of the family of set-valued mappings $E(t)$ in $t_0$, for any $\varepsilon > 0$ there is a number $\eta(\varepsilon) > 0$ such that

$$E(t) + V_\varepsilon \supseteq E(t_0) \text{ for } \rho(t, t_0) < \eta(\varepsilon) \tag{4}$$

holds. Since $0 \in \operatorname{int} E(t_0)$ we can choose $\varepsilon$ so very small that $E(t_0)_{-\varepsilon} \neq \Phi$ and even an $\varepsilon_1 > 0$ exists with the property

$$0 \in V_{\varepsilon_1} \subseteq E(t_0)_{-\varepsilon}. \tag{5}$$

By Assertion 2 the conditions (4) and (5) together lead to the inclusion

$$0 \in V_{\varepsilon_1} \subseteq E(t_0)_{-\varepsilon} \subseteq (E(t) + V_\varepsilon)_{-\varepsilon}$$

and, because of Assertion 1, to the result $0 \in V_\varepsilon \subseteq (E(t) + V_\varepsilon) = E(t)$, i.e. to the inclusion $0 \in \operatorname{int} E(t) \blacksquare$

## REFERENCES

[1] RÅDSTRÖM, H.: *An embedding theorem for spaces of convex sets.* Proc. Amer. Math. **3** (1952), 165 - 169.

[2] ROLEWICZ, S.: *Funktionalanalysis und Steuerungstheorie.* Berlin - Heidelberg - New York 1976.

Received 10. 12. 1990

Author's address:

Prof. Dr. R. KLÖTZLER
Sektion Mathematik der Universität Leipzig
Augustusplatz 10
D (Ost) - 7010 Leipzig