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A Fixed Point Theorem for Non-Expansive Mappings on Star-Shaped Domains

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It is shown that some modification of the fixed point theorem of Browder, Göhde and Kirk remains valid on star-shaped domains as well. Additionally, we will give an iteration scheme for the approximation of some fixed point of the mapping under consideration. Finally, in connection with the result above, two characterizations of inner product spaces will be obtained.

Key words: *fixed points, star-shaped sets, nonexpansive and pseudocontractive mappings, characterizations of inner product spaces*
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0. Introduction

In the fixed point theory of non-expansive mappings the related domains are often supposed to be convex. The used methods, however, mostly don't carry over to not necessarily convex domains like, e.g., star-shaped sets. A well-known access for star-shaped domains consists in trying to find almost fixed points and to ensure their convergence to an actual fixed point. Besides results of Dotson [16,17] and Guseman and Peters [25], who assumed the domain A to be compact, which is rather restrictive, we call attention to the works of Göhde [23,24], Crandall and Pazy [10] and Reinermann [37], where A is a closed bounded and star-shaped subset of a Hilbert space. Motivated by these works the question occurs whether the famous Browder-Göhde-Kirk Theorem [5,22,31] ("every non-expansive selfmapping of a closed, bounded and convex subset of a uniformly convex Banach space has at least one fixed point") remains true, if the domain is assumed to be merely star-shaped. A positive answer was given by Müller and Reinermann [33] in case of a reflexive Banach space admitting a weakly sequentially continuous duality mapping. For further fixed point results on non-convex domains see, e.g., [9,19,27,38].

In Section 1, we will give another version of the Browder-Göhde-Kirk Theorem, holding true on star-shaped subsets of a reflexive Kadec-Klee space, by sharpening the assumptions made on the operator (Theorem 1.16). After a short discussion of the necessity of the several assumptions (Section 2), we present some applications of the results derived in Section 1 (Section 3). In Section 4 we deal with an explicit iteration scheme due to Halpern [26] and finally, in Section 5, we examine the relations between *non-expansive, pseudo-contractive* and the new defined (see Section 1) *nearly pseudo-contractive* mappings.

Conventions: Throughout this paper all normed spaces are assumed to be real Banach spaces. Let $(E, \|\cdot\|)$ be a normed space, $A \subset E$, $(x_n) \in E^{\mathbf{N}}$, $x, x_0 \in E$, $r > 0$ and $T : A \to E$. We denote by $(E^*, \|\cdot\|)$ the strong dual space of E equipped with the usual operator norm, $\operatorname{conv} A$, $\overline{\operatorname{conv}}\, A$, $\overline{A}$, ∂A stand for the convex hull, the closed convex hull, the closure and the boundary of A, respectively. The weak and strong convergence of (x_n) to x is indicated by $x_n \rightharpoonup x$ and $x_n \to x$, respectively, and if we just say that (x_n) converges to x we will always mean that $x_n \to x$. We use the abbreviation $\operatorname{Fix} T$ for the fixed point set of T and denote the closed ball of radius r around x_0 by $\overline{B}(x_0, r)$. Finally, we call A *star-shaped* if there exists an $x_0 \in A$ such that for all $\lambda \in [0,1]$ and all $x \in A$ it follows that $\lambda x + (1-\lambda)x_0 \in A$.

1. A fixed point theorem for non-expansive nearly pseudo-contractive mappings

Before stating our main results we recall some definitions needed in the sequel.

Definition 1.1 (see, e.g., [13: pages 111-113, 15: pages 21,23,32 and 36]): A normed space $(E, \|\cdot\|)$ is called

(1) *strictly convex* if for all $x, y \in \overline{B}(0,1)$ it follows from $x \neq y$ that $\|x+y\| < 2$;

(2) *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in \overline{B}(0,1)$ with $\|x - y\| \geq \epsilon$ it follows that $\|x+y\| \leq 2(1-\delta)$;

(3) *(uniformly) smooth* if its norm is (uniformly) Gâteaux differentiable on $\partial\overline{B}(0,1)$;

(4) *Kadec-Klee space* if for each sequence $(x_n) \in E^{\mathbf{N}}$ and each point $x \in E$ with both $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ it follows that $x_n \to x$.

For a discussion of these and other concepts see, e.g., [2: Part 3, 11,18,30,42].

Definition 1.2 (see, e.g., [6]): Let $(E, \|\cdot\|)$ be a normed space and $J : E \to E^*$.

(1) The (normalized) *duality mapping* $J_E : E \to 2^{E^*}$ is given by $J_E(0) = \{0\}$ and

$$J_E(x) = \{\, u \in E^* \mid u(x) = \|u\|\|x\| \text{ and } \|u\| = \|x\| \,\} \text{ for all } x \neq 0.$$

(2) J is called a (normalized) *duality mapping* if $J(x) \in J_E(x)$ for all $x \in E$.

(3) J is said to be *weakly sequentially continuous* if for each sequence $(x_n) \in E^{\mathbf{N}}$ and each point $x \in E$ it follows from $x_n \rightharpoonup x$ that $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

(4) The *modulus of convexity* is given by

$$\delta(\epsilon) = \inf\{\, 1 - \|x+y\|/2 \mid x, y \in \overline{B}(0,1) \text{ and } \|x-y\| \geq \epsilon \,\} \text{ for all } \epsilon \in [0,2].$$

Remark 1.3: Note that by the Hahn-Banach Theorem $J_E(x) \neq \emptyset$ for all $x \in E$ and that $(E, \|\cdot\|)$ is smooth if and only if $|J_E(x)| = 1$ for all $x \in E$ (see, e.g., [2: Part 3, Chapter I, §2, Proposition 2] taking into account the definition of smoothness given in [2: page 177]). In the latter case we regard J_E as a mapping from E to E^*. If there is no fear of confusion we simply write J for J_E. Recall that for a uniformly convex space $(E, \|\cdot\|)$ the mapping δ is continuous with $\delta(2) = 1$ and that in arbitrary normed spaces for each $\epsilon > 0$ and each $M \geq \epsilon/2$ and all $x, y \in \overline{B}(0, M)$ with $\|x - y\| \geq \epsilon$ it follows that $\|x+y\|/2 \leq (1 - \delta(\epsilon/M))M$.

Definition 1.4 [8: page 211]: Let $(E, \|\cdot\|)$ be a normed space, $x_0 \in E$ and $r > 0$. We define the *radial projection* $R[x_0, r]$ by

$$R[x_0,r](x) = \begin{cases} x & , \|x - x_0\| \leq r \\ x_0 + (r/\|x - x_0\|)(x - x_0) & , \text{otherwise} \end{cases}$$

Obviously $R[x_0, r]|_{\overline{B}(x_0,r)} = \mathrm{id}|_{\overline{B}(x_0,r)}$ and $R[x_0, r](E \setminus \overline{B}(x_0, r)) \subset \partial\overline{B}(x_0, r)$.

First of all we need two lemmas of geometrical nature.

Lemma 1.5: *Let $(E, \|\cdot\|)$ be a normed space, $x_0 \in E$, $r > 0$ and $R = R[x_0, r]$. Then $\|y\| \geq \|R(0)\|$ for all $y \in \overline{B}(x_0, r)$.*

Proof: If $0 \in \overline{B}(x_0, r)$ we have $R(0) = 0$ and there is nothing to show. Otherwise $R(0) = x_0(1 - r/\|x_0\|)$ and $1 - r/\|x_0\| > 0$. Therefore, $\|R(0)\| = \|x_0\| - r$ and so, for $y \in \overline{B}(x_0, r)$, we have $r \geq \|y - x_0\| \geq \|x_0\| - \|y\|$. Hence $\|y\| \geq \|x_0\| - r = \|R(0)\|$ ∎

Lemma 1.6: *Let $(E, \|\cdot\|)$ be a normed space, $(x_n) \in (E \setminus \{0\})^{\mathbb{N}}$ and $(\mu_n) \in (0, \infty)^{\mathbb{N}}$ strictly decreasing. For $n, m \in \mathbb{N}$ we define*

$$z_{nm} = (1/2)((\mu_n/\mu_m) + 1)x_n, \quad r_{nm} = (1/2)((\mu_n/\mu_m) - 1)\|x_n\|$$

and

$$\Omega_{nm} = \overline{B}(z_{nm}, r_{nm}).$$

Assume furthermore that $x_m \in \Omega_{nm}$ for all $m > n$. Then $(\|x_n\|)$ is non-decreasing.

We will keep to the abbreviations above throughout the whole paper.

Proof: For $m > n$ we have $\mu_m < \mu_n$ and so $r_{nm} > 0$. Since $\|z_{nm}\| > r_{nm}$, it follows that $0 \notin \Omega_{nm}$, and denoting $R[z_{nm}, r_{nm}]$ by R we have $R(0) = z_{nm}(1 - r_{nm}/\|z_{nm}\|)$, where $r_{nm}/\|z_{nm}\| = (\mu_n - \mu_m)/(\mu_n + \mu_m)$. Hence we conclude that

$$R(0) = (1/2)((\mu_n/\mu_m) + 1)x_n(1 - (\mu_n - \mu_m)/(\mu_n + \mu_m)) = x_n.$$

Since $x_m \in \Omega_{nm}$ for all $m > n$, it follows from Lemma 1.5 that $\|x_m\| \geq \|x_n\|$ for all $m > n$ ∎

Now we are able to prove

Theorem 1.7: *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space, $(\mu_n) \in (0, \infty)^{\mathbb{N}}$ strictly decreasing and $(x_n) \in E^{\mathbb{N}}$ bounded with $x_m \in \overline{B}(z_{nm}, r_{nm})$ for all $m > n$. Then (x_n) converges.*

Proof: If there is $n \in \mathbb{N}$ such that $x_n = 0$, then $x_m \in \Omega_{nm} = \{0\}$ for all $m > n$ and so $(x_m) \to 0$. We now assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. From Lemma 1.6 and the boundedness of (x_n) we know that $(\|x_n\|)$ converges to some $\alpha \geq 0$ and that $\|x_n\| \leq \alpha$ for all $n \in \mathbb{N}$. Fix $m > n$ now. Then $\|x_n\|, \|x_m\| \leq \alpha$ and $\epsilon := 2\|x_n\| > 0$. Since $x_n, x_m \in \Omega_{nm}$, we have $(x_n + x_m)/2 \in \Omega_{nm}$, and since $R[z_{nm}, r_{nm}](0) = x_n$ (see above), it follows from Lemma 1.5 that $\|x_n + x_m\| \geq 2\|x_n\| = \epsilon$. Hence (see Remark 1.3) $\|x_n - x_m\| \leq 2(1 - \delta(2\|x_n\|/\alpha))\alpha$ for all $m > n$. Since the right side is independent of m and tends to $2(1 - \delta(2))\alpha = 0$ for $n \to \infty$, we conclude that (x_n) is a Cauchy sequence and the result follows ∎

Remark 1.8: Let $(E, (\cdot,\cdot))$ be a Hilbert space and consider the following conditions.
(a) $\|(1+\mu_n)x_n - (1+\mu_m)x_m\| \le \|x_n - x_m\|$ for all $n, m \in \mathbb{N}$.
(b) $(x_n - x_m, \mu_n x_n - \mu_m x_m) \le 0$ for all $n, m \in \mathbb{N}$.
(c) $x_m \in \overline{B}(z_{nm}, r_{nm})$ for all $m > n$.
From elementary calculations (cf. [23]) we obtain the relations (a) $\Longrightarrow$ (b) and (b) $\Longleftrightarrow$ (c). Thus, Theorem 1.7 yields as a special case the convergence lemmas given by Crandall and Pazy in [10: Lemma 2.4(b)] and Göhde in [23] (in the course of the proof of Theorem 1) dealing with (b),(a), respectively. For a similar result see also [4: Lemma 1.4].

Next, we wish to show that, in case of a reflexive Kadec-Klee space, it is possible to ensure the convergence of a subsequence, although we can't guarantee that the whole sequence converges.

Lemma 1.9: *Let $(E, \|\cdot\|)$ be a normed space, $(\mu_n) \in (0,\infty)^{\mathbb{N}}$ strictly decreasing, $(x_n) \in (E \setminus \{0\})^{\mathbb{N}}$ bounded and $x \in E$ such that $x_n \rightharpoonup x$ and $x_m \in \overline{B}(z_{nm}, r_{nm})$ for all $m > n$. Then $\|x_n\| \to \|x\|$.*

Proof: Analogously to the proof of Theorem 1.7 it follows that $(\|x_n\|)$ converges to some $\alpha \ge 0$. Since $x_n \rightharpoonup x$, we have $\|x\| \le \liminf \|x_n\| = \alpha$. Fix $n \in \mathbb{N}$ now. For $m > n$ we have $\Omega_{nm} \subset \Omega_{n,m+1}$, because for $m > n$ and $y \in \Omega_{nm}$ we have $\mu_n/\mu_{m+1} - \mu_n/\mu_m > 0$ and

$$\begin{aligned}\|y - z_{n,m+1}\| &\le \|y - z_{nm}\| + \|z_{nm} - z_{n,m+1}\| \\ &\le r_{nm} + (\mu_n/\mu_{m+1} - \mu_n/\mu_m)\|x_n\|/2 \\ &= (\mu_n/\mu_{m+1} - 1)\|x_n\|/2 = r_{n,m+1},\end{aligned}$$

and therefore $y \in \Omega_{n,m+1}$. It follows that $x_m \in \Omega_{nm} \subset \Omega_{ni}$ for all $i > n$ and all $m \in \{n+1, \ldots, i\}$. Taking into account the convexity of Ω_{ni}, we conclude $\overline{\mathrm{conv}}\{x_m \mid n < m \le i\} \subset \Omega_{ni}$ for $i > n$. So, for $i > n$ and $z \in \overline{\mathrm{conv}}\{x_m \mid n < m \le i\}$, we have $\|z - z_{ni}\| \le r_{ni}$ and therefore $\|z\| \ge \|z_{ni}\| - \|z - z_{ni}\| \ge \|z_{ni}\| - r_{ni} = \|x_n\|$. Since $(x_m)_{m>n} \rightharpoonup x$, we know that $x \in \overline{\mathrm{conv}}\{x_m | m > n\}$, and from the considerations above it follows $\|x\| \ge \|x_n\|$ (letting i tend to infinity). Since n was arbitrary, we conclude $\|x\| \ge \alpha$ and thus $\|x\| = \alpha$. Hence $\|x_n\| \to \|x\|$. ∎

Theorem 1.10: *Let $(E, \|\cdot\|)$ be a reflexive Kadec-Klee space, $(\mu_n) \in (0,\infty)^{\mathbb{N}}$ strictly decreasing and $(x_n) \in E^{\mathbb{N}}$ bounded, with $x_m \in \overline{B}(z_{nm}, r_{nm})$ for all $m > n$. Then (x_n) possesses a convergent subsequence (x_{φ_n}).*

Proof: As already shown in the proof of Theorem 1.7, we may assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. Since $(E, \|\cdot\|)$ is reflexive and (x_n) is bounded, there exists an $x \in E$ and some subsequence (x_{φ_n}) of (x_n) such that $x_{\varphi_n} \rightharpoonup x$ (Pettis' theorem). We may assume φ to be strictly increasing, so that $\|x_{\varphi_n}\| \to \|x\|$ by Lemma 1.9. Since $(E, \|\cdot\|)$ is a Kadec-Klee space, the result follows ∎

The next theorem states some examples of reflexive Kadec-Klee spaces. We refer to [13: pages 112-113] for the definition of *local uniform convexity, k-rotundity* and *property (K)* and to [28: page 744] for the definition of *nearly uniform convexity*.

Theorem 1.11 [13,28,34,41]: *The following spaces are reflexive Kadec-Klee spaces:*
(1) *a space of finite dimension,*
(2) *a reflexive space which is locally uniformly convex,*
(3) *a uniformly convex space,*
(4) *a k-rotund space with* $k \geq 2$*,*
(5) *a space having property (K),*
(6) *a nearly uniformly convex space,*
(7) *a reflexive space admitting a weakly sequentially continuous duality mapping,*
(8) *a space for which its operator norm is Fréchet differentiable on* $E^* \setminus \{0\}$.

Note that in [28] there is given the following example of an infinite-dimensional reflexive Kadec-Klee space $(E, \|\cdot\|)$ which is not uniformly convex:

$$E = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} E_n \mid \sum_{n=1}^{\infty} \|x_n\|_n e_n \in F \right\} \text{ normed by } \|x\| = \left\| \sum_{n=1}^{\infty} \|x_n\|_n e_n \right\|_F,$$

where $(E_n, \|.\|_n)$ denotes $\mathbb{R}^n$ with the usual l^n-norm, $(F, \|.\|_F)$ stands for the sequence space l^2 with its l^2-norm and $\{ e_n \mid n \in \mathbb{N} \}$ is the standard Schauder basis of l^2.

Let us now recall some definitions and introduce a new property which we will call "nearly pseudo-contractive".

Definition 1.12 (see, e.g., [8: page 198]): Let $(E, \|\cdot\|)$ be a normed space and $\emptyset \neq A \subset E$. A mapping $T : A \to E$ is called

(1) *non-expansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in A$,

(2) *pseudo-contractive* if for all $x, y \in A$ there exists some $u \in J_E(x - y)$ such that $u(Tx - Ty) \leq \|x - y\|^2$,

(3) *nearly pseudo-contractive* if $\|Tx - Ty - (1-\lambda)(x-y)\| \leq \|Tx - Ty - (1+\lambda)(x-y)\|$ for all $x, y \in A$ and all $\lambda \geq 0$,

(4) *strongly pseudo-contractive* if there is a $k \in [0,1)$ such that for all $x, y \in A$ there exists some $u \in J_E(x - y)$ with $u(Tx - Ty) \leq k\|x - y\|^2$.

Using Lemma 1.13 (existence of almost fixed points) and Theorem 1.14 we will be able to state our first fixed point result for non-expansive nearly pseudo-contractive mappings (Theorem 1.16).

Lemma 1.13: *Let* $(E, \|\cdot\|)$ *be a Banach space,* $\emptyset \neq A \subset E$ *closed and star-shaped with respect to 0,* $(\lambda_n) \in [0,1)^{\mathbb{N}}$ *and* $T : A \to E$ *non-expansive with* $T(\partial A) \subset A$. *Then for each* $n \in \mathbb{N}$ *there is exactly one* $x_n \in A$ *such that* $x_n = \lambda_n T x_n$.

Proof: For $n \in \mathbb{N}$ define $T_n = \lambda_n T : A \to E$. Then

$$T_n(\partial A) = \lambda_n T(\partial A) \subset \lambda_n A = \lambda_n A + (1 - \lambda_n)\{0\} \subset A$$

because of the star-shapedness of A. Since the mapping T is non-expansive, we conclude $\|T_n x - T_n y\| \leq \lambda_n \|x - y\|$ for all $x, y \in A$, and it follows from the classical contraction principle in the form of Assad [1] that T_n has exactly one fixed point $x_n \in A$ ∎

Theorem 1.14: *Let $(E, \|\cdot\|)$ be a reflexive Kadec-Klee space, $\emptyset \neq A \subset E$ closed, $T : A \to E$ continuous and nearly pseudo-contractive, $(\lambda_n) \in (0,1)^{\mathbb{N}}$ strictly increasing with $\lambda_n \to 1$ and $(x_n) \in A^{\mathbb{N}}$ bounded such that $x_n = \lambda_n T x_n$ for all $n \in \mathbb{N}$. Then* $\operatorname{Fix} T \neq \emptyset$.

Proof: Set $\mu_n = 1/\lambda_n - 1 > 0$ for all $n \in \mathbb{N}$. Then (μ_n) is strictly decreasing. Since T is nearly pseudo-contractive, it follows for all $m > n$ that

$$\|Tx_m - Tx_n - (1-\mu_m)(x_m - x_n)\| \leq \|Tx_m - Tx_n - (1+\mu_m)(x_m - x_n)\|.$$

Taking into account $\mu_j x_j = Tx_j - x_j$ for all $j \in \mathbb{N}$ and $\mu_n - \mu_m > 0$, we have

$$\|\mu_m x_m - \mu_n x_n + \mu_m x_m - \mu_m x_n\| \leq \|\mu_m x_m - \mu_n x_n - \mu_m x_m + \mu_m x_n\|,$$

hence $\|2\mu_m x_m - (\mu_n + \mu_m)x_n\| \leq (\mu_n - \mu_m)\|x_n\|$. Dividing by $2\mu_m$ we see that $x_m \in \overline{B}(z_{nm}, r_{nm})$ and so, by Theorem 1.10, there exists an $x \in E$ and some subsequence (x_{φ_n}) of (x_n) converging to x. Since T is continuous and $Tx_{\varphi_n} = x_{\varphi_n}/\lambda_{\varphi_n}$ it follows that $Tx = x$ ∎

Remark 1.15: From Theorem 1.11 we know that Theorem 1.14 especially applies to reflexive normed spaces having a weakly sequentially continuous duality mapping. Nevertheless, in this case Lemma 2.7 of [33] (Müller and Reinermann) tells us that we can replace the assumption "T nearly pseudo-contractive" by the weaker one (see Lemma 2.1) "T pseudo-contractive".

Theorem 1.16: *Let $(E, \|\cdot\|)$ be a reflexive Kadec-Klee space, $\emptyset \neq A \subset E$ closed, bounded and star-shaped and $T : A \to E$ non-expansive and nearly pseudo-contractive with $T(\partial A) \subset A$. Then* $\operatorname{Fix} T \neq \emptyset$.

Proof: Without loss of generality we may assume that Θ is a star point of A. It follows from Lemma 1.13 that for each $n \in \mathbb{N}$ there is exactly one $x_n \in A$ such that $x_n = \lambda_n T x_n$, where (e.g.) $\lambda_n = 1 - 1/n$. Since A is bounded, (x_n) is bounded too and applying Theorem 1.14 we are done ∎

Remark 1.17: In case of a uniformly convex Banach space and a convex subset A of E it is a consequence of the Browder-Göhde-Kirk Theorem (see Introduction), that the assumption "T nearly pseudo-contractive" may be dropped.

2. Necessity of the assumptions made in Theorem 1.16

The question occurs whether we may weaken the assumptions made in Theorem 1.16. With the help of simple counter-examples (cf. [36: page 67/68]) one easily sees that we can't dispense with any of the assumptions "$T(\partial A) \subset A$", "A bounded", "A closed" and "A star-shaped". We also can't drop the property that $(E, \|\cdot\|)$ is a reflexive Kadec-Klee space, as the following example due to Göhde [22] will show. Let

$$(E, \|\cdot\|) = \big(C([0,1], \mathbb{R}), \|\cdot\|_\infty\big)$$

and

$$A = \big\{ f \in E \mid f(0) = 0 \text{ and } f(x) \in [0,1] \text{ for all } x \in [0,1] \big\}.$$

Define $T : A \to E$ as follows:

$$T(f)(x) = \begin{cases} 0 & , x = 0 \\ (1-x)f(x) + (1/2)x(1+\sin(1/x)) & , x \neq 0 \end{cases}.$$

From [22] we know that A is closed, bounded and convex, $T(A) \subset A$, T non-expansive and $\operatorname{Fix} T = \emptyset$. It remains to show that T is nearly pseudo-contractive. But this can be easily seen, observing $|x - \lambda| \leq |x + \lambda|$ for all $\lambda \geq 0$ and all $x \geq 0$ and

$$\left|T(f)(x) - T(g)(x) - (1 \pm \lambda)(f(x) - g(x))\right| = |\lambda \pm x|\,|f(x) - g(x)|$$

for all $\lambda \geq 0$, all $x \in [0,1]$ and all $f, g \in A$. The question, whether we can dispense with T being nearly pseudo-contractive, was already raised in [38] (Reinermann and Stallbohm) in case of a uniformly convex Banach space. This problem still seems to be open.

Finally, we will show that the non-expansiveness of T may be replaced merely by the continuity of the operator, if we demand A to be convex instead of being just star-shaped.

Lemma 2.1: *Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq A \subset E$ and $T : A \to E$ nearly pseudo-contractive. Then T is pseudo-contractive.*

Proof: Taking into account that T is nearly pseudo-contractive, it follows that for arbitrary $x, y \in A$ and all $\lambda > 0$

$$\begin{aligned} &\|\lambda(Tx - Ty) - (1+\lambda)(x-y)\| \\ &= \lambda\left\|(Tx - Ty) - (1 + 1/\lambda)(x - y)\right\| \geq \lambda\left\|(Tx - Ty) - (1 - 1/\lambda)(x-y)\right\| \\ &= \left\|\lambda(Tx - Ty) - (\lambda - 1)(x-y)\right\| = \left\|\lambda(Tx - Ty) - (1+\lambda)(x-y) + 2(x-y)\right\| \\ &\geq 2\|x - y\| - \left\|\lambda(Tx - Ty) - (1+\lambda)(x-y)\right\| \end{aligned}$$

and consequently

$$\|\lambda(Tx - Ty) - (1+\lambda)(x-y)\| \geq \|x - y\|,$$

i.e.

$$\|x - y\| \leq \|(x-y) + \lambda\big((\mathrm{id} - T)x - (\mathrm{id} - T)y\big)\|.$$

By a lemma of Schöneberg [40: page 24] this means that T is pseudo-contractive ∎

The following result of Deimling [14] is also needed.

Theorem 2.2: *Let $(E, \|\cdot\|)$ be a Banach space, let $\emptyset \neq A \subset E$ be closed and $T : A \to E$ continuous and strongly pseudo-contractive such that for all $x \in A$*

$$(1/\lambda)\operatorname{dist}((1-\lambda)x + \lambda Tx, A) \to 0 \text{ if } \lambda \to 0+.$$

Then T has exactly one fixed point.

Theorem 2.3: *Let $(E, \|\cdot\|)$ be a reflexive Kadec-Klee space, let $\emptyset \neq A \subset E$ be closed, bounded and convex and $T : A \to A$ continuous and nearly pseudo-contractive. Then* $\operatorname{Fix} T \neq \emptyset$.

Proof: Assuming $0 \in A$ (without loss of generality) and setting $\lambda_n = 1 - 1/n$, we observe that $(\lambda_n T)(A) \subset A$ (cf. proof of Lemma 1.13). Since T is pseudo-contractive by Lemma 2.1, $\lambda_n T$ is clearly strongly pseudo-contractive. Finally, because A is convex, $\operatorname{dist}((1-\lambda)x + \lambda\lambda_n Tx, A) = 0$ for all $x \in A$ and all $\lambda \in (0,1)$. Thus it follows from Theorem 2.2 that for each $n \in \mathbb{N}$ there exists a unique $x_n \in A$ such that $x_n = \lambda_n T x_n$. The result follows by Theorem 1.14 ∎

3. Some applications of Theorem 1.14

We begin with a generalization of a result of Goebel and Kuczumow [20: Theorem 2]. It was originally proved for non-expansive mappings on a closed convex subset of a Hilbert space. Note that A is not necessarily bounded.

Theorem 3.1: *Let $(E, \|\cdot\|)$ be a reflexive Banach space, let $\emptyset \neq A \subset E$ be closed and star-shaped with respect to some $z \in A$ and $T : A \to E$ non-expansive with $T(\partial A) \subset A$. Suppose that the set $G = \{\, y \in T(A) \mid u(Tz - z) > 0$ for all $u \in J_E(y - z) \,\}$ is bounded and assume furthermore that (a) or (b) holds, where*

(a) $(E, \|\cdot\|)$ *possesses a weakly sequentially continuous duality mapping*

(b) $(E, \|\cdot\|)$ *is a reflexive Kadec-Klee space and T is nearly pseudo-contractive.*

Then $\operatorname{Fix} T \neq \emptyset$.

Proof: Without loss of generality we may assume that $z = 0$ and $T0 \neq 0$. It follows from Lemma 1.13 that for each $n \in \mathbb{N}$ there exists a unique $x_n \in A$ such that $x_n = \lambda_n T x_n$, where $\lambda_n = 1 - 1/n$. Then, for $u \in J(Tx_n)$, we have

$$\begin{aligned} \|x_n\|^2 = \lambda_n^2 \|Tx_n\|^2 = \lambda_n^2 u(Tx_n) &= \lambda_n^2 u(T0) + \lambda_n^2 u(Tx_n - T0) \\ &\leq \lambda_n^2 u(T0) + \lambda_n^2 \|Tx_n\| \, \|Tx_n - T0\| \\ &\leq \lambda_n^2 u(T0) + \lambda_n^2 \|Tx_n\| \, \|x_n\| = \lambda_n^2 u(T0) + \lambda_n \|x_n\|^2. \end{aligned}$$

Hence $(1 - \lambda_n)\|x_n\|^2 \leq \lambda_n^2 u(T0)$. If $x_n \neq 0$, we have $u(T0) \geq (1-\lambda_n)\|x_n\|^2/\lambda_n^2 > 0$. Otherwise $u \in J(T0)$ and so $u(T0) = \|T0\|^2 > 0$. This shows that $Tx_n \in G$ for all $n \in \mathbb{N}$, and so (Tx_n) is bounded. Since $x_n = \lambda_n T x_n$ for all $n \in \mathbb{N}$, it follows that (x_n) is bounded, too, and applying Lemma 2.7 of [33] in case (a) and Theorem 1.14 in case (b) we are done ∎

Note that all those results of [33] and [32: Chapter 3] which were derived from Lemma 2.7 of [33] with the help of almost fixed points (see Section 1) carry over to our situation (T nearly pseudo-contractive, $(E, \|\cdot\|)$ a reflexive Kadec-Klee space) immediately (just use Theorem 1.14 instead of [33: Lemma 2.7]). Exemplarily, we state the following result, which is an analogue to Theorem 3.14 of [32: page 38].

Theorem 3.2: *Let $(E, \|\cdot\|)$ be a reflexive Kadec-Klee space, let $\emptyset \neq A \subset E$ be open and bounded, $0 \in A$ and $T : \overline{A} \to E$ be non-expansive and nearly pseudo-contractive. Additionally suppose that T satisfies the following Leray-Schauder condition: for all $x \in \partial A$ and all $\lambda \geq 0$ with $Tx = \lambda x$ it follows that $\lambda \leq 1$. Then* $\operatorname{Fix} T \neq \emptyset$.

4. On an iteration scheme due to Halpern

Let us introduce some abbreviations (see Halpern [26]). A sequence (λ_n) is said to fulfill condition *(Hal)* if

(1) $(\lambda_n) \in (0,1)^{\mathbb{N}}$ is non-decreasing with $\lambda_n \to 1$,

(2) there exists some non-decreasing sequence $(\beta_n) \in \mathbb{N}^{\mathbb{N}}$ such that $\beta_n(1-\lambda_n) \to \infty$ and $(1-\lambda_{n+\beta_n})/(1-\lambda_n) \to 1$.

Halpern gave the following example of such a sequence: $\lambda_n = 1-n^{-\alpha}$, where $\alpha \in (0,1)$. In the course of the proof of [26: Theorem 3] he actually showed that the following holds.

Lemma 4.1: *Let $(E, \|\cdot\|)$ be a normed space, let $\emptyset \neq A \subset E$ be bounded and star-shaped with respect to 0, $T : A \to A$ non-expansive, (λ_n) a sequence which fulfils (Hal) and let $(x_n) \in A^{\mathbb{N}}$ be such that $x_n = \lambda_n T x_n$ for all $n \in \mathbb{N}$. Define $z_{n+1} = \lambda_{n+1} T z_n$ for all $n \in \mathbb{N}_0$, where z_0 is an arbitrary point in A. Assume furthermore that (x_n) converges to some $q \in E$. Then (z_n) converges to q as well.*

Note that (z_n) is well-defined, because $T(A) \subset A$ and A is star-shaped with respect to 0. In analogy to [26: Theorem 1] (E Hilbert space, $T : \overline{B}(0,1) \to \overline{B}(0,1)$) we will show next

Theorem 4.2: *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space, $\emptyset \neq A \subset E$ closed, bounded and star-shaped with respect to 0, $T : A \to E$ non-expansive and nearly pseudo-contractive with $T(\partial A) \subset A$ and $(\lambda_n) \in (0,1)^{\mathbb{N}}$ strictly increasing with $\lambda_n \to 1$. Then*

(1) *for each $n \in \mathbb{N}$ there is exactly one $x_n \in A$ such that $x_n = \lambda_n T x_n$,*

(2) *(x_n) converges to some $q \in \operatorname{Fix} T$,*

(3) $\|q\| = \min\{\|z\| \mid z \in \operatorname{Fix} T\}$.

Proof: Set $\mu_n = 1/\lambda_n - 1$ for all $n \in \mathbb{N}$. From Section 1 (see Lemma 1.13, Theorem 1.14, Theorem 1.7) we already know that (1) and (2) hold. Since $\mu_n > 0$ and T is nearly pseudo-contractive, we conclude that for all $z \in \operatorname{Fix} T$

$$\|Tx_n - Tz - (1-\mu_n)(x_n - z)\| \leq \|Tx_n - Tz - (1+\mu_n)(x_n - z)\|.$$

Since $\mu_n x_n = Tx_n - x_n$ and $Tz = z$, it follows that

$$\|\mu_n x_n + \mu_n(x_n - z)\| \leq \|\mu_n x_n - \mu_n(x_n - z)\|,$$

hence $\mu_n\|2x_n - z\| \leq \mu_n\|z\|$ and therefore $\|z\| \geq \|2x_n - z\| \geq 2\|x_n\| - \|z\|$, which implies that $\|z\| \geq \|x_n\|$. Letting n tend to infinity, it follows that $\|z\| \geq \|q\|$. ∎

Contrary to Theorem 1.7, Theorem 1.10 just supplies a convergent subsequence. Therefore we can't expect that the theorem above is still true in arbitrary reflexive Kadec-Klee spaces. Nevertheless, we have

Theorem 4.3: *Let $(E, \|\cdot\|)$ be a reflexive and strictly convex Kadec-Klee space, $\emptyset \neq A \subset E$ closed, bounded and convex, $0 \in A$, $T : A \to E$ non-expansive and nearly pseudo-contractive with $T(\partial A) \subset A$ and $(\lambda_n) \in (0,1)^{\mathbb{N}}$ strictly increasing with $\lambda_n \to 1$. Then the assertion of Theorem 4.2 holds.*

Proof: Since A is closed and convex, $(E, \|\cdot\|)$ is strictly convex and T is non-expansive, we know from [7] (Browder) that $\operatorname{Fix} T$ is closed and convex. Additionally, $\operatorname{Fix} T \neq \emptyset$ by Theorem 1.16. Hence (see, e.g., [21: page 12]) there is exactly one $p \in \operatorname{Fix} T$ such that $\|p\| = \min\{\, \|z\| \mid z \in \operatorname{Fix} T\,\}$. Consider an arbitrary subsequence (x'_n) of (x_n) now, where (x_n) is choosen according to Lemma 1.13. Following the proofs of Theorem 1.14 and Theorem 4.2 we obtain some $q \in E$ and some subsequence (x''_n) of (x'_n) such that $x''_n \to q$, $Tq = q$ and $\|q\| = \min\{\, \|z\| \mid z \in \operatorname{Fix} T\,\}$. From the uniqueness part of this relation it follows that $q = p$. Therefore $x_n \to p$ ∎

Combining Theorems 4.1-4.3 we obtain

Theorem 4.4: *Let $(E, \|\cdot\|)$ be a reflexive Banach space and $\emptyset \neq A \subset E$ closed and bounded. Suppose that (a) or (b) holds, where*

(a) *$(E, \|\cdot\|)$ is uniformly convex and A is star-shaped with respect to 0,*

(b) *$(E, \|\cdot\|)$ is a strictly convex Kadec-Klee space and A is convex.*

Assume furthermore that $T : A \to A$ is non-expansive and nearly pseudo-contractive, that (λ_n) is a sequence which fulfils (Hal) and that (z_n) is given by $z_0 \in A$ and $z_{n+1} = \lambda_{n+1} T z_n$ for all $n \in \mathbb{N}_0$. Then (z_n) converges to some $y \in \operatorname{Fix} T$ such that $\|y\| = \min\{\, \|z\| \mid z \in \operatorname{Fix} T\,\}$.

Remark 4.5: **(1)** $(E, \|\cdot\|)$ is a reflexive and strictly convex Kadec-Klee space if and only if its operator norm is Fréchet-differentiable on $E^* \setminus \{0\}$ (see, e.g., [34]). **(2)** In case of a nearly pseudo-contractive T, Theorem 4.4(a) improves a result contained in [35: Theorem 3.1] (Reich), where A was additionally demanded to be convex and $(E, \|\cdot\|)$ had to be a smooth normed space possessing a duality mapping which is weakly sequentially continuous at 0. **(3)** For further fixed point iterations on star-shaped domains see, e.g., [37].

5. Comparison of the properties "non-expansive" and "pseudo-contractive" with "nearly pseudo-contractive"

The following theorem shows that the terms *pseudo-contractive* and *nearly pseudo-contractive* coincide in case of a Hilbert space. Therefore Theorem 1.16 contains, as a special case, the results of Göhde [24], Crandall and Pazy [10] and Reinermann [37], which were already mentioned in the introduction.

Theorem 5.1: *Let $(E, (\cdot, \cdot))$ be a Hilbert space, $\emptyset \neq A \subset E$ and $T : A \to E$. Then T is pseudo-contractive if and only if T is nearly pseudo-contractive.*

The easy proof ($\|\cdot\|$ is a Hilbert space norm !) is omitted ∎

Actually, it is characterizing for Hilbert spaces that the properties above coincide, as we will show in Theorem 5.5. But first we have to give a further definition and to state some lemmas.

Definition 5.2 (cf. [3,12,29,39]): Let $(E, \|\cdot\|)$ be a normed space and $x, y \in E$. We say that

(1) x is *orthogonal* to y in the sense of *Roberts* ($x \perp_R y$) if $\|x - \lambda y\| = \|x + \lambda y\| \,\forall \lambda \in \mathbb{R}$,

(2) x is *orthogonal* to y in the sense of *Birkhoff* ($x \perp_B y$) if $\|x\| \leq \|x + \lambda y\| \,\forall \lambda \in \mathbb{R}$.

Lemma 5.3 [40: page 11]: *Let $(E, \|\cdot\|)$ be a normed space and $x, y \in E$. Then there exists an $u \in J_E(x)$ such that $u(y) \geq 0$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \geq 0$.*

Lemma 5.4: *Let $(E, \|\cdot\|)$ be a normed space which satisfies condition*

(**) *for all $x, y \in E$ it follows from $x \perp_B y$ that $x \perp_R y$.*

Then $(E, \|\cdot\|)$ is a Hilbert space.

Proof: According to a result of James [29: Corollary 4.7], it suffices to show that for all two-dimensional subspaces F of E and for all $x \in F$ there exists a $y \in F \setminus \{0\}$ such that $x \perp_R y$. Therefore suppose $F = \Sigma\{e_1, e_2\}$, $x = x_1 e_1 + x_2 e_2 \in F$ and let u be an arbitrary element of $J_E(x)$. Without loss of generality we may assume that $u(e_2) \neq 0$, so that $y := e_1 - \big(u(e_1)/u(e_2)\big)e_2 \in F \setminus \{0\}$ is well-defined. Clearly, $u(y) = 0$ and so, by Lemma 5.3, $x \perp_B y$. Since (**) implies that $x \perp_R y$, we are done ∎

Theorem 5.5: *Let $(E, \|\cdot\|)$ be a normed space which satisfies the following condition: for each $\emptyset \neq A \subset E$ with $|A| = 2$ and for each $T : A \to E$ which is pseudo-contractive it follows that T is nearly pseudo-contractive, too. Then $(E, \|\cdot\|)$ is a Hilbert space.*

Proof: Fix $x, y \in E$ such that $x \perp_B y$. We may assume that $x \neq 0$. Define $A = \{x, 0\}$ and $T0 = y$ as well as $Tx = x$ and observe that $x \perp_B y$ implies $\|x - 0\| \leq \|(1+\lambda)(x-0) - \lambda(Tx - T0)\|$ for all $\lambda \geq 0$. Hence T is pseudo-contractive (see above) and from our assumption also nearly pseudo-contractive. Therefore, for $\lambda \geq 0$,

$$\|Tx - T0 - (1-\lambda)(x-0)\| \leq \|Tx - T0 - (1+\lambda)(x-0)\|,$$

i.e. $\|y - \lambda x\| \leq \|y + \lambda x\|$. If we define $\tilde{T}0 = -y$ and $\tilde{T}x = x$, it follows in the same manner that the inequality above holds for $\lambda \leq 0$, too. Finally, because $-\mathbb{R} = \mathbb{R}$, it follows that $\|y - \lambda x\| = \|y + \lambda x\|$ for all $\lambda \in \mathbb{R}$, i.e. $y \perp_R x$. It's not difficult to see that this implies $x \perp_R y$. Applying Lemma 5.4 we are done ∎

With regard to Theorem 1.16 there also arises the question whether there are spaces in which every non-expansive mapping is nearly pseudo-contractive, too. At least in smooth spaces of dimension strictly greater than two we will see that this property yields a further characterization of Hilbert spaces (Theorem 5.10). First of all we need several lemmas.

Lemma 5.6: *Let $(E, \|\cdot\|)$ be a normed space. Then the following properties are equivalent:*

(a) *for all $x, y \in E$ with $\|x\| \leq \|y\|$ it follows that $\|x-(1-\lambda)y\| \leq \|x-(1+\lambda)y\|$ for all $\lambda \geq 0$*

(b) *for each $\emptyset \neq A \subset E$ with $|A| = 2$ and for each $T : A \to E$ which is non-expansive it follows that T is nearly pseudo-contractive, too.*

The proof uses a method similar to the one used in the proof of Theorem 5.5 ∎

Lemma 5.7: *Let $(E, \|\cdot\|)$ be a smooth normed space, $x, y \in E$ and $\lambda_0 > 0$ such that*

$$J_E(x-(1-\lambda)y)(y) + J_E(x-(1+\lambda)y)(y) > 0 \text{ for all } \lambda \in [0, \lambda_0].$$

Then, for all $\lambda \in (0, \lambda_0)$, we have $\|x-(1-\lambda)y\| > \|x-(1+\lambda)y\|$.

Proof: Define $g(z) = \|z\|^2/2$ and $f(\lambda) = g(x-(1-\lambda)y) - g(x-(1+\lambda)y)$ for all $z \in E$ and $\lambda \geq 0$, respectively. Since $D_w g|_z = J(z)(w)$ for all $z, w \in E$, one easily verifies that

$$f'(\lambda) = J(x-(1-\lambda)y)(y) + J(x-(1+\lambda)y)(y) > 0$$

for all $\lambda \in (0, \lambda_0)$. Thus f is strictly increasing in $(0, \lambda_0)$. Since f is continuous, this implies that $f(\lambda) > f(0) = 0$ for all $\lambda \in (0, \lambda_0)$, from which the result follows ∎

Lemma 5.8: *Let $(E, \|\cdot\|)$ be a smooth normed space and $x, y \in E$ such that*

$$\|x-(1-\lambda)y\| \leq \|x-(1+\lambda)y\| \text{ for all } \lambda \geq 0.$$

Then $J_E(x-y)(y) \leq 0$.

Proof: Suppose $J(x-y)(y) > 0$. Since $(E, \|\cdot\|)$ is smooth, J is strong-weak*-continuous (see, e.g., [15: Chapter 2, §1, Theorem 1]) and therefore

$$J(x-(1-\lambda)y)(y) + J(x-(1+\lambda)y)(y) \to 2J(x-y)(y) > 0 \text{ if } \lambda \to 0+.$$

Thus there exists a $\lambda_0 > 0$ such that $J(x-(1-\lambda)y)(y) + J(x-(1+\lambda)y)(y) > 0$ for all $\lambda \in [0, \lambda_0]$. Using Lemma 5.7 we derive a contradiction to our assumption ∎

Lemma 5.9: *Let $(E, \|\cdot\|)$ be a smooth normed space with* $\dim E \geq 3$ *which satisfies the condition*

$(***)$ *for all $x, y \in E$ it follows from $\|x\| \leq \|y\|$ that $J_E(x-y)(y) \leq 0$.*

Then $(E, \|\cdot\|)$ is a Hilbert space.

Proof: From [12: Theorem 6.4] (Day) we know, that $(E, \|\cdot\|)$ is a Hilbert space if and only if $\perp_B$ is symmetric. Fix $x, y \in E$ such that $x \perp_B y$. Then $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in I\!R$, and so it follows from $(***)$ that $J(x - (x + \lambda y))(x + \lambda y) \leq 0$, i.e. $\lambda J(y)(x) + \lambda^2 \|y\|^2 \geq 0$. For $\lambda > 0$ ($\lambda < 0$) this implies that $J(y)(x) + \lambda \|y\|^2 \geq 0$ (≤ 0), from which we conclude that $J(y)(x) \geq 0$ (≤ 0) letting $\lambda \to 0+$ ($\lambda \to 0-$). Hence $J(y)(x) = 0$ and so $y \perp_B x$ by Lemma 5.3. We have shown that $\perp_B$ is symmetric and thus the result follows ∎

Combining the lemmas above we obtain at once

Theorem 5.10: *Let $(E, \|\cdot\|)$ be a smooth normed space with* $\dim E \geq 3$ *which satisfies the following condition : for each $\emptyset \neq A \subset E$ with $|A| = 2$ and for each $T : A \to E$ which is non-expansive it follows that T is nearly pseudo-contractive, too. Then $(E, \|\cdot\|)$ is a Hilbert space.*

Remark 5.11: With the help of explicit counter-examples and in view of Lemma 2.1, we see that the following relations hold in general: (a) "T nearly pseudo-contractive" implies "T pseudo-contractive", but the converse implication does not hold; (b) "T non-expansive" implies "T pseudo-contractive", but the converse implication does not hold; (c) "T non-expansive" and "T nearly pseudo-contractive" are independent.

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Book review

B.-W. SCHULZE and H. TRIEBEL (eds.): **Symposium "Partial Differential Equations"**, Holzhau 1988 (Teubner-Texte zur Mathematik: Vol. 112). Leipzig: B.G. Teubner Verlagsges. 1989, 316 S.

The volume contains part of the papers contributed to the Conference "Partial Differential Equations" held in Holzhau in April 1988 and organized by the Karl-Weierstrass-Institut of Mathematics of the Academy of Sciences of the GDR. The majority of the papers included in this volume deals with spectral and scattering theory for linear operators, in particular Schrödinger operators and, as a whole, they offer a useful overview of recent trends in the current research in this field. Both the analytic point of view of differential and pseudodifferential operators and the one related to diffusion processes and functional integration are represented, as some aspects of spectral theory on manifolds. Another group of articles is concerned with various problems in the theory of pseudodifferential operators, in particular on non-smooth manifolds, and with other aspects of differential geometry. A third group deals with some non-linear equations. Most of these papers are pleasant to read and this is an additional motivation to recommend the volume to everyone interested in these fields. We can only sketch the content of each contribution, by grouping them for convenience as indicated above.

E. Balslev and *E. Skibsted* consider Schrödinger operators with short-range potentials in $\mathbb{R}^N$ and study the analytic continuation of the resolvent operator and the S-matrix in the half planes. *Ph. Briet, J.M. Combes* and *P. Duclos* describe some spectral stability properties of a Schrödinger operator with a many-well potential in $\mathbb{R}^N$ in terms of the one-well operators associated with the truncated potentials in each connected component of the classically allowed region in $\mathbb{R}^N$. *V. Enss* gives a new proof of the asymptotic completeness for the scattering in two- and three-particle systems, for a class of short range potentials. *P. Exner* and *P. Seba* discuss the existence of bound states for sufficiently thin strips, locally deformed by bends or protrusions, in connection with various physical models of classical and quantum waveguides and layered semiconductors. *M.* and *T. Hoffmann-Osterhof* describe the asymptotic behaviour