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Determination of a Real Parameter in the Coefficient of a Quasilinear Elliptic Differential Equation

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We study the quasilinear elliptic differential equation $\mathrm{div}((b(u - u_0)^l + c)\,\mathrm{grad}\,u) = 0$ with boundary value conditions of Dirichlet type in an n-dimensional cylinder. Provided we know the numbers u_0, l and c, the solution u on a plane which is parallel to the basic area of the cylinder, an eigenvalue and an eigenfunction of a certain eigenvalue problem, we give an explicit formula for the calculation of the real parameter b.

Key words: *Inverse problems, boundary value problems, quasilinear elliptic differential equations*
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1. Introduction

Let D be a bounded region of the n-dimensional Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary ∂D. Points of $\mathbb{R}^{n-1}$ are denoted by $x' = (x_1, \ldots, x_{n-1})$, those of $\mathbb{R}^n$ by $x = (x', x_n)$. We consider the quasilinear elliptic boundary value problem

$$Lu(x) = \mathrm{div}(a(u(x))\,\mathrm{grad}\,u(x)) = 0 \quad \text{in } D \tag{1.1}$$

$$u(x) = g(x) \quad \text{on } \partial D, \tag{1.2}$$

where a is a function of the real function $u = u(x)$ which is strictly positive, continuously differentiable and satisfies the condition

$$u \in C(\bar{D}) \cap C^2(D). \tag{1.3}$$

We prove the validity of a weak maximum principle in the following sense.

Definition 1 [3]: Let $a(u)$ be strictly positive and continuously differentiable, $u \in C^1(D)$. Then, in a generalized sense, u is said to satisfy the relation $Lu = 0$ (≥ 0, ≤ 0) in D, in dependence on whether $\int_D a(u(x))\,\mathrm{grad}\,u(x)\,\mathrm{grad}\,\varphi(x)\,dx = 0$ (≤ 0, ≥ 0) for all non-negative functions $\varphi \in C_0^1(D)$.

Theorem 1: *Let $a(u)$ be strictly positive and continuously differentiable, u satisfy the condition* (1.3) *and suppose*

$$\int_D a(u(x))\,\mathrm{grad}\,u(x)\,\mathrm{grad}\,\varphi(x)\,dx = 0 \tag{1.4}$$

for all non-negative functions $\varphi \in C_0^1(D)$. Then

$$\inf_{\partial D} u \leq \inf_D u \tag{1.5}$$

and

$$\sup_D u \leq \sup_{\partial D} u. \tag{1.6}$$

Proof: We prove (1.6) supposing, contrary to the assertion, that $\sup_D u > \sup_{\partial D} u = u_1$. Then, for some constant $c > 0$, there is a subdomain $D' \subset D$ in which $v = u - u_1 - c > 0$ and $v = 0$ on $\partial D'$. The relation (1.4) remains true with u replaced by v, and with $\varphi = v$ on D', $\varphi = 0$ elsewhere. We have $\varphi \notin C_0^1(D)$, but (1.4) can be seen to hold by approximating φ with functions in $C_0^1(D)$. It follows $\int_{D'} a(\operatorname{grad} v)^2 dx = 0$ and hence, since a is a positive function, we infer $\operatorname{grad} v = 0$ in D'. On account of $v = 0$ on $\partial D'$ we have $v = 0$ in D', which contradicts the definition of the function v. Analogously the inequality (1.5) can be proved ∎

We set $u_0 = \inf_{\partial D} u$ and suppose $u_0 < u_1$. Then we obtain from (1.5) and (1.6) $u_0 \le u(x) \le u_1$ for all $x \in \bar{D}$. In the following the coefficient $a(u)$ in the equation (1.1) may be given in the form

$$a(u) = b(u - u_0)^l + c \quad \text{for all } [u_0, u_1], \tag{1.7}$$

where l is a natural number and b, c are real constants, with $c > 0$ and $b > -c(u_1 - u_0)^{-l}$. This implies $a(u) > 0$ for all $u \in [u_0, u_1]$. Further we assume $c = a(u_0)$ and l to be known. Then $a(u)$ is available completely if the constant b is known. Finally, in addition to the assumptions stated, suppose that for every function $a(u)$ of the form (1.7) (i.e. for every $b > -c(u_1 - u_0)^{-l}$) there exists a unique solution $u = u(x,b)$ of the boundary value problem (1.1) - (1.2). For brevity we write $u(x) = u(x,b)$.

The inverse problem to the boundary value problem (1.1) - (1.2) consists in the determination of the coefficient $a(u)$ (i.e. of the real parameter b) under some additional information on u. In the second section we describe (under some restrictions on the domain D) a procedure for the calculation of the number b, which also implies its existence and uniqueness.

The third section contains three special cases and one example.

A quasilinear parabolic equation with a coefficient of the form (1.7) is investigated by MEYER [4,5]. CANNON [2] has considered the equation (1.1) with the Neumann boudary condition $a(u)\partial u/\partial\nu = g$ on ∂D and the additional condition $u = f$ on C, where C denotes a smooth curve on a portion of ∂D. Some results on inverse problems in quasilinear equations are given by ANGER [1].

2. A method for the determination of the number b

We use the following notations:

$D_{n-1} \subset \mathbb{R}^{n-1}$ is a bounded region with sufficiently smooth boundary ∂D_{n-1},
(d_1, d_2) is an interval on the x_n-axis,
$D = D_{n-1} \times (d_1, d_2)$, $\bar{B} = D_{n-1} \times [d_1, d_2]$,
Δ_k is the Laplacian in the k-dimensional space.

The boundary value problem (1.1)-(1.2) is considered with a coefficient $a(u)$ of the form (1.7) and the special boundary conditions

$$u(x', d_1) = p_1(x') \text{ for } x' \in D_{n-1},\ x_n = d_1 \tag{2.1}$$

$$u(x', d_2) = p_2(x') \text{ for } x' \in D_{n-1},\ x_n = d_2 \tag{2.2}$$

$$u(x) = q(x) \quad \text{for } x \in \bar{B}, \tag{2.3}$$

where p_1, p_2 and q are given functions satisfying the conditions

$$p_1, p_2 \in C(\overline{D_{n-1}}),\ q \in C(\bar{B}) \tag{2.4}$$

$$p_i(x') = q(x', d_i) \text{ for } x' \in D_{n-1} \text{ and } i = 1, 2. \tag{2.5}$$

From (2.4) and Theorem 1 it follows

$$u_0 = \min\left\{ \min_{x' \in \overline{D_{n-1}}} p_i(x')\ (i = 1, 2),\ \min_{x \in \bar{B}} q(x) \right\},$$

$$u_1 = \max\left\{ \max_{x' \in \overline{D_{n-1}}} p_i(x')\ (i = 1, 2),\ \max_{x \in \bar{B}} q(x) \right\}.$$

Moreover, suppose that the following additional information on the solution $u(x', x_n)$ of the direct problem is available:

$$u(x', x_n^0) = f(x') \quad \text{for } x' \in D_{n-1}, \tag{2.6}$$

where x_n^0 is a fixed point in the interval (d_1, d_2) and

$$f \in C(\overline{D_{n-1}}). \tag{2.7}$$

Further, we consider the eigenvalue problem

$$\Delta_{n-1} y(x') = -\lambda y(x'),\ x' \in D_{n-1} \quad \text{and} \quad y(x') = 0,\ x' \in \partial D_{n-1}. \tag{2.8}$$

It is known [6] that for every n and for a bounded domain with piecewise smooth boundary the set of eigenvalues of this problem is discrete and all eigenvalues are positive. We assume that we know one of the eigenvalues $\lambda > 0$ and the corresponding eigenfunction $y = y(x')$ of the problem (2.8). Setting

$$v(x) = v(x', x_n) = \int_{u_0}^{u(x', x_n)} a(s)\, ds = \int_{u_0}^{u(x', x_n)} (b(s - u_0)^l + c)\, ds \tag{2.9}$$

we obtain from (1.1) and (2.1)-(2.3) the linear boundary value problem

$$\Delta_n v = \Delta_{n-1} v + v_{x_n x_n} = 0, \qquad (x \in D) \tag{2.10}$$

$$v(x', d_1) = \int_{u_0}^{p_1(x')} (b(s - u_0)^l + c)\, ds = b g_1(x') + c g_2(x') \qquad (x' \in D_{n-1}) \tag{2.11}$$

$$v(x', d_2) = \int_{u_0}^{p_2(x')} (b(s - u_0)^l + c)\, ds = b g_3(x') + c g_4(x') \qquad (x' \in D_{n-1}) \tag{2.12}$$

$$v(x', x_n) = \int_{u_0}^{q(x', x_n)} (b(s - u_0)^l + c)\, ds = b h_1(x', x_n) + c h_2(x', x_n) \qquad (x \in \bar{B}). \tag{2.13}$$

From the additional information (2.6) and (2.9) we receive

$$v(x', x_n^0) = \int_{u_0}^{f(x')} (b(s - u_0)^l + c)\, ds = b r_1(x') + c r_2(x') \qquad (x' \in D_{n-1}). \tag{2.14}$$

We use the following notations ($i = 1, 2$; $j = 1, 2, 3, 4$):

$$A_i(x_n) = \int_{\partial D_{n-1}} (\partial y(x')/\partial \nu)\, h_i(x', x_n)\, d\sigma \tag{2.15}$$

$$B_j = \int_{D_{n-1}} g_j(x') y(x')\, dx' \tag{2.16}$$

$$E_i(x_n) = -\tfrac{1}{2\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}x_n}\int_{d_1}^{x_n} e^{\sqrt{\lambda}t}A_i(t)\,dt - e^{\sqrt{\lambda}x_n}\int_{d_1}^{x_n} e^{-\sqrt{\lambda}t}A_i(t)\,dt\right) \tag{2.17}$$

$$F_i = \int_{D_{n-1}} r_i(x')y(x')\,dx', \tag{2.18}$$

where y denotes the known eigenfunction of the problem (2.8) and $\partial/\partial\nu$ the normal derivative in the $(n-1)$-dimensional space. Further, we set $w = e^{\sqrt{\lambda}(d_1-d_2)} - e^{-\sqrt{\lambda}(d_1-d_2)}$.

Theorem 2: *Let p_1, p_2, q and f be given functions satisfying* (2.4), (2.5) *and* (2.7), *respectively. Suppose*

$$\begin{aligned} F_1 - \Big[&\Big(B_1 e^{-\sqrt{\lambda}d_2} - B_3 e^{\sqrt{\lambda}d_1} + E_1(d_2)e^{-\sqrt{\lambda}d_1}\Big)e^{\sqrt{\lambda}x_n^0} \\ &+\Big(B_3 e^{\sqrt{\lambda}d_1} - B_1 e^{\sqrt{\lambda}d_2} - E_1(d_2)e^{\sqrt{\lambda}d_1}\Big)e^{-\sqrt{\lambda}x_n^0}\Big]w^{-1} - E_1(x_n^0) \neq 0, \end{aligned} \tag{2.19}$$

where λ denotes an eigenvalue of the problem (2.8). *Then the inverse problem* (1.1), (2.1) -(2.3), (2.6) *of the determination of the real parameter b has a unique solution, which can be found explicitely.*

Proof: Multiplication of the formula (2.10) by $y(x')$ and integration with respect to x' supplies

$$\int_{D_{n-1}}\left(\Delta_{n-1}v(x',x_n) + v_{x_nx_n}(x',x_n)\right)y(x')\,dx' = 0. \tag{2.20}$$

We put

$$z(x_n) = \int_{D_{n-1}} v(x',x_n)y(x')\,dx'. \tag{2.21}$$

Then we have

$$z''(x_n) = z_{x_nx_n}(x_n) = \int_{D_{n-1}} v_{x_nx_n}(x',x_n)y(x')\,dx'. \tag{2.22}$$

Applying (2.8) and (2.13) we obtain by partial integration

$$\begin{aligned} &\int_{D_{n-1}} \Delta_{n-1}v(x',x_n)y(x')\,dx' \\ &= \int_{D_{n-1}} \Delta_{n-1}y(x')v(x',x_n)\,dx + \int_{\partial D_{n-1}}(\partial v(x',x_n)/\partial\nu)y(x') - (\partial y(x')/\partial\nu)v(x',x_n)\,d\sigma \\ &= -\lambda\int_{D_{n-1}} v(x',x_n)y(x')\,dx' - \int_{\partial D_{n-1}}(\partial y(x')/\partial\nu)\big[b\,h_1(x',x_n) + c\,h_2(x',x_n)\big]d\sigma. \end{aligned} \tag{2.23}$$

Using (2.21)-(2.23) we receive from (2.20) an inhomogeneous linear ordinary differential equation of second order with respect to $z(x_n)$:

$$z''(x_n) - \lambda z(x_n) = s(x_n), \tag{2.24}$$

with right-hand side

$$s(x_n) = \int_{\partial D_{n-1}}(\partial y(x')/\partial\nu)\big[b\,h_1(x',x_n) + c\,h_2(x',x_n)\big]d\sigma = bA_1(x_n) + cA_2(x_n),$$

where $A_i(x_n)$ $(i = 1,2)$ given by (2.15) are known functions independent of b and c. Moreover, we obtain from (2.11) and (2.12) the boundary conditions

$$z(d_1) = \int_{D_{n-1}} v(x',d_1)y(x')dx = bB_1 + cB_2 \tag{2.25}$$

$$z(d_2) = \int_{D_{n-1}} v(x',d_2)y(x')dx = bB_3 + cB_4, \tag{2.26}$$

where B_j $(j = 1,2,3,4)$ given by (2.16) are known constants independent of b and c. Because $-\lambda < 0$ we conclude that the boundary value problem (2.24)-(2.26) is always uniquely solvable [7]. The general solution of the equation (2.24) can be written in the form

$$z(x_n) = C_1 e^{\sqrt{\lambda} x_n} + C_2 e^{-\sqrt{\lambda} x_n} + z_s(x_n), \tag{2.27}$$

where C_i $(i = 1,2)$ are arbitrary constants and

$$z_s(x_n) = -1/2\sqrt{\lambda}\left(e^{-\sqrt{\lambda} x_n}\int_{d_1}^{x_n} e^{\sqrt{\lambda} t} s(t)\,dt - e^{\sqrt{\lambda} x_n}\int_{d_1}^{x_n} e^{-\sqrt{\lambda} t} s(t)\,dt\right).$$

is a particular solution of the inhomogeneous equation (2.24). For brevity we write $z_s(x_n) = bE_1(x_n) + cE_2(x_n)$, where the functions $E_i(x_n)$ $(i = 1,2)$ given by (2.17) are independent of b and c. From (2.17) it follows $z_s(d_1) = 0$. Inserting (2.27) into (2.25) and (2.26) we obtain a particular solution

$$\begin{aligned} z(x_n) = b&\Big\{\Big[\Big(B_1 e^{-\sqrt{\lambda} d_2} - B_3 e^{-\sqrt{\lambda} d_1} + E_1(d_2)e^{-\sqrt{\lambda} d_1}\Big)e^{\sqrt{\lambda} x_n} \\ &+\Big(B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2} - E_1(d_2)e^{\sqrt{\lambda} d_1}\Big)e^{-\sqrt{\lambda} x_n}\Big]w^{-1} + E_1(x_n)\Big\} \\ &+ c\Big\{\Big[\Big(B_2 e^{-\sqrt{\lambda} d_2} - B_4 e^{-\sqrt{\lambda} d_1} + E_2(d_2)e^{-\sqrt{\lambda} d_1}\Big)e^{\sqrt{\lambda} x_n} \\ &+\Big(B_4 e^{\sqrt{\lambda} d_1} - B_2 e^{\sqrt{\lambda} d_2} - E_2(d_2)e^{\sqrt{\lambda} d_1}\Big)e^{-\sqrt{\lambda} x_n}\Big]w^{-1} + E_2(x_n)\Big\} \end{aligned} \tag{2.28}$$

with $w = e^{\sqrt{\lambda}(d_1 - d_2)} - e^{-\sqrt{\lambda}(d_1 - d_2)}$. By means of (2.14) we calculate $z(x_n^o)$ as

$$z(x_n^o) = \int_{D_{n-1}} v(x',x_n^o)y(x')dx' = bF_1 + cF_2, \tag{2.29}$$

where the constants F_i $(i = 1,2)$ given by (2.18) are known and independent of b and c. Replacing $z(x_n)$ by $z(x_n^o)$ in the solution (2.28) and using (2.29) we arrive at an algebraic equation for the determination of the real parameter b:

$$\begin{aligned} b = c&\Big\{\Big[\Big(B_2 e^{-\sqrt{\lambda} d_2} - B_4 e^{-\sqrt{\lambda} d_1} + E_2(d_2)e^{-\sqrt{\lambda} d_1}\Big)e^{\sqrt{\lambda} x_n^o} \\ &+\Big(B_4 e^{\sqrt{\lambda} d_1} - B_2 e^{\sqrt{\lambda} d_2} - E_2(d_2)e^{\sqrt{\lambda} d_1}\Big)e^{-\sqrt{\lambda} x_n^o}\Big]w^{-1} + E_2(x_n^o) - F_2\Big\} \\ &\times\Big\{F_1 - \Big[\Big(B_1 e^{-\sqrt{\lambda} d_2} - B_3 e^{\sqrt{\lambda} d_1} + E_1(d_2)e^{-\sqrt{\lambda} d_1}\Big)e^{\sqrt{\lambda} x_n^o} \\ &+\Big(B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2} - E_1(d_2)e^{\sqrt{\lambda} d_1}\Big)e^{-\sqrt{\lambda} x_n^o}\Big]w^{-1} - E_1(x_n^o)\Big\}^{-1}. \end{aligned}$$

From (2.19) the uniqueness of the parameter b follows ∎

3. Special cases

Throughout what follows the compatibility conditions are assumed to be fulfilled.

3.1 We consider the boundary value problem

$$\begin{aligned}
\operatorname{div}((b(u-u_0)^l + c)\operatorname{grad} u(x)) &= 0 && \text{for } x \in D \\
u(x', d_1) &= p_1(x') && \text{for } x' \in D_{n-1}, x_n = d_1 \\
u(x', d_2) &= p_2(x') && \text{for } x' \in D_{n-1}, x_n = d_2 \\
u(x) &= 0 && \text{for } x \in \overline{B} \\
u(x', x_n^0) &= f(x') && \text{for } x' \in D_{n-1},\ x_n^0 \in (d_1, d_2),
\end{aligned} \tag{3.1}$$

where $f \in C(\overline{D_{n-1}})$ satisfies the condition

$$F_1 - \Big[\Big(B_1 e^{-\sqrt{\lambda} d_2} - B_3 e^{\sqrt{\lambda} d_1}\Big) e^{\sqrt{\lambda} x_n^0} + \Big(B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2}\Big) e^{-\sqrt{\lambda} x_n^0}\Big] W^{-1} \neq 0. \tag{3.2}$$

The inverse problem (3.1) of the determination of b can be written as an inverse problem for the homogeneous ordinary differential equation of the variable x_n:

$$z''(x_n) - \lambda z(x_n) = 0, \tag{3.3}$$

with the inhomogeneous boundary condition

$$\begin{aligned}
z(d_1) &= bB_1 + cB_2 \\
z(d_2) &= bB_3 + cB_4
\end{aligned} \tag{3.4}$$

and the additional assumption

$$z(x_n^0) = bF_1 + cF_2, \tag{3.5}$$

where the constants B_j ($j = 1,2,3,4$) and F_i ($i = 1,2$) are independent of b and c and are given by (2.16) and (2.18). The solution of the direct problem (3.3)-(3.4) has the form

$$\begin{aligned}
z(x_n) = {} & b\Big[\Big(B_1 e^{-\sqrt{\lambda} d_2} - B_3 e^{-\sqrt{\lambda} d_1}\Big) e^{\sqrt{\lambda} x_n} + \Big(B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2}\Big) e^{-\sqrt{\lambda} x_n}\Big] W^{-1} \\
& + c\Big[\Big(B_2 e^{-\sqrt{\lambda} d_2} - B_4 e^{-\sqrt{\lambda} d_1}\Big) e^{\sqrt{\lambda} x_n} + \Big(B_4 e^{\sqrt{\lambda} d_1} - B_2 e^{\sqrt{\lambda} d_2}\Big) e^{-\sqrt{\lambda} x_n}\Big] W^{-1}.
\end{aligned} \tag{3.6}$$

Because of (3.2) the inverse problem (3.3)-(3.5) is uniquely solvable:

$$\begin{aligned}
b = c\Big\{ & \Big[\Big(B_2 e^{-\sqrt{\lambda} d_2} - B_4 e^{-\sqrt{\lambda} d_1}\Big) e^{\sqrt{\lambda} x_n^0} + \Big(B_4 e^{\sqrt{\lambda} d_1} - B_2 e^{\sqrt{\lambda} d_2}\Big) e^{-\sqrt{\lambda} x_n^0}\Big] W^{-1} - F_2\Big\} \\
& \times \Big\{-\Big[\Big(B_1 e^{-\sqrt{\lambda} d_2} - B_3 e^{-\sqrt{\lambda} d_1}\Big) e^{\sqrt{\lambda} x_n^0} + \Big(B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2}\Big) e^{-\sqrt{\lambda} x_n^0}\Big] W^{-1} + F_1\Big\}
\end{aligned}$$

From this we deduce the uniqueness of the parameter b in the problem (3.1).

3.2 We consider the boundary value problem

$$\begin{aligned}
\operatorname{div}((b(u-u_0)^l + c)\operatorname{grad} u(x)) &= 0 && \text{for } x \in D \\
u(x', d_1) &= 0 && \text{for } x' \in D_{n-1}, x_n = d_1 \\
u(x', d_2) &= 0 && \text{for } x' \in D_{n-1}, x_n = d_2 \\
u(x) &= q(x) && \text{for } x \in \overline{B} \\
u(x', x_n^0) &= f(x') && \text{for } x' \in D_{n-1},\ x_n^0 \in (d_1, d_2),
\end{aligned} \tag{3.7}$$

where $f \in C(\overline{D_{n-1}})$ satisfies the condition

$$F_1 - E_1(d_2)\left[e^{\sqrt{\lambda}(x_n^0 - d_1)} - e^{-\sqrt{\lambda}(x_n^0 - d_1)}\right]W^{-1} - E_1(x_n^0) \neq 0. \tag{3.8}$$

The inverse problem (3.7) of the determination of b can be written as an inverse problem for the inhomogeneous ordinary differential equation of the variable x_n:

$$z''(x_n) - \lambda z(x_n) = bA_1(x_n) + cA_2(x_n), \tag{3.9}$$

with the homogeneous boundary conditions

$$z(d_i) = 0 \quad (i = 1,2) \tag{3.10}$$

and the additional assumption

$$z(x_n^0) = bF_1 + cF_2, \tag{3.11}$$

where the functions $A_i(x_n)$ and the constants F_i $(i = 1,2)$ are independent of b and c and are given by (2.15) and (2.18). The solution of the direct problem (3.9)-(3.10) has the form

$$z(x_n) = b\left[E_1(d_2)\left(e^{\sqrt{\lambda}(x_n - d_1)} - e^{-\sqrt{\lambda}(x_n - d_1)}\right)W^{-1} + E_2(x_n)\right] + c\left[E_2(d_2)\left(\ldots\right)W^{-1} + E_2(x_n)\right].$$

Because of (3.8) the inverse problem (3.9)-(3.11) is uniquely solvable:

$$b = c\left[E_2(d_2)\left(e^{\sqrt{\lambda}(x_n^0 - d_1)} - e^{-\sqrt{\lambda}(x_n^0 - d_1)}\right)W^{-1} + E_2(x_n^0) - F_2\right]\left[-E_1(d_2)\left(\ldots\right)W^{-1} - E_1(x_n^0) + F_1\right]^{-1}.$$

From this we deduce the uniqueness of the parameter b in the problem (3.7).

3. Let $n = 1$. Then we have the boundary value problem

$$\begin{aligned} ((b(u - u_0)^l + c)u')' &= 0 && \text{for } x \in (d_1, d_2) \\ u(d_i) &= g_i && \text{for } i = 1,2 \end{aligned} \tag{3.12}$$

$$u(x_0) = h \qquad \text{for a fixed } x_0 \in (d_1, d_2). \tag{3.13}$$

Let $g_1 \neq g_2$. It is easily seen that the solution u is a strictly monotone increasing function if $g_1 < g_2$ and a strictly monotone decreasing function if $g_1 > g_2$. Without loss of generality we suppose $g_1 < g_2$. Then we have $u_0 = \min\{g_1, g_2\} = g_1$ and $u_1 = \max\{g_1, g_2\} = g_2$. Using the transformation

$$v(x) = \int_{g_1}^{u(x)} a(s)\,ds = \int_{g_1}^{u(x)} (b(u - u_0)^l + c)\,ds$$

we obtain from (3.12)

$$v''(x) = 0 \text{ and } v(d_1) = 0,\ v(d_2) = \int_{g_1}^{g_2} (b(s - g_1)^l + c)\,ds$$

with the solution $v(x) = (x - d_1)(d_2 - d_1)^{-1}v(d_2)$. We replace v by u to obtain

$$\int_{g_1}^{u(x)} (b(s - g_1)^l + c)\,ds = (x - d_1)(d_2 - d_1)^{-1}\int_{g_1}^{g_2} (b(s - g_1)^l + c)\,ds$$

and then receive the algebraic equation

$$\begin{aligned} b &= c(l+1)\left[(x - d_1)(d_1 - d_2)^{-1}(g_2 - g_1) - (u(x) - g_1)\right] \\ &\quad \times \left[-(x - d_1)(d_2 - d_1)^{-1}(g_2 - g_1)^{l+1} + (u(x) - g_1)^{l+1}\right] \end{aligned}$$

for the determination of the parameter b. Because of (3.13), b is uniquely determined.

4. Now an example for $n = 2$ and $l = 1$ follows. Let $D = \{(x_1, x_2) | \; 0 < x_1, x_2 < 1\}$. We consider the boundary value problem

$$\begin{aligned} \operatorname{div}((b(u - u_0) + c) \operatorname{grad} u(x)) &= 0 && \text{for } (x_1, x_2) \in D \\ u(x_1, 0) = u(x_1, 1) &= x_1(1 - x_1) && \text{for } 0 < x_1 < 1 \\ u(0, x_2) = u(1, x_2) &= 0 && \text{for } 0 < x_2 < 1 \\ u(x_1, x_2^0) &= f(x_1) && \text{for } 0 < x_1 < 1, \; x_2^0 \in (0,1), \end{aligned}$$

where $f \in C[0,1]$ satisfies the condition

$$F_1 - R(x_2^0) B_1 \neq 0 \tag{3.14}$$

with $R(x_2^0) = \left(e^{\pi(x_2^0 - 1)} - e^{\pi x_2^0} + e^{-\pi x_2^0} - e^{-\pi(x_2^0 - 1)}\right)\left(e^{-\pi} - e^{+\pi}\right)^{-1}$ and B_1 and F_1 given by (2.16), (2.18). The solutions of the eigenvalue problem $y''(x_1) + \lambda y(x_1) = 0$, $y(0) = y(1) = 0$ are known: $\lambda_k = k^2 \pi^2$, $y_k(x_1) = \sin(k \pi x_1)$, $k \in \mathbb{N}$. For the following calculations we choose $\lambda_1 = \pi^2$, $y_1(x_1) = \sin(\pi x_1)$. By virtue of Subsection 3.1, we have

$$z''(x_2) - \pi^2 z(x_2) = 0$$

$$\begin{aligned} z(0) = z(1) &= b\int_0^1 (x_1^3 - x_1^4) \sin(\pi x_1)\, dx_1 \\ &\quad + c\int_0^1 (x_1 - x_1^2) \sin(\pi x_1)\, dx_1 = bB_1 + cB_2 \end{aligned} \tag{3.15}$$

$$\begin{aligned} z(x_2^0) &= \tfrac{1}{2} b \int_0^1 (f(x_1))^2 \sin(\pi x_1)\, dx_1 \\ &\quad + c\int_0^1 f(x_1) \sin(\pi x_1)\, dx_1 = bF_1 + cF_2 . \end{aligned} \tag{3.16}$$

Setting in (3.6) $B_1 = B_3$ and $B_2 = B_4$ we get the solution of the direct problem (3.15) in the form $z(x_2) = (bB_1 + cB_2) R(x_2)$. Putting $x_2 = x_2^0$ and using (3.14),(3.16) it finally results that $b = 2c(R(x_2^0)B_2 - F_2)(F_1 - R(x_2^0)B_1)^{-1}$ is uniquely determined.

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