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Operator Calculus for Elliptic Boundary Value Problems in Unbounded Domains

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The main result of the paper is the proof of the applicability of hypercomplex methods for elliptic boundary value problems in outer domains, i.e. domains, which lies outside a closed compact surface.

Key words: *operator calculus, elliptic boundary value problems, unbounded domains*
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1. Introduction

In the last years operator principles for the solution of elliptic boundary value problems have got an increasing importance. K. Gürlebeck and W. Sprössig have found in [2] an operator calculus for several boundary value problems in bounded domains which is based on quaternionic analysis. The aim of the present paper is to obtain similar results for unbounded domains.

Let H be the quaternionic algebra with the basis e_0, e_1, e_2, e_3. Every element $a \in H$ has the unique representation $a = \sum_{i=1}^{3} a_i e_i$ with the real coefficients a_i. The conjugate quaternion a is defined by $a = a_0 e_0 - \sum_{i=1}^{3} a_i e_i = \operatorname{Re} a - \operatorname{Im} a$ and the norm by $|a| = (aa)^{1/2}$. Let $G \subset \mathbb{R}^3$ be a domain which lies outside a closed compact, sufficiently smooth surface $\partial G = \Gamma$. An H-valued function f will be written by $f(x) = \sum_{i=0}^{3} f_i(x) e_i$ with $f_i\colon G \to \mathbb{R}^1$. Each point $x \in \mathbb{R}^3$ may be identified with a quaternion of the form $x = \sum_{i=1}^{3} x_i e_i$. The following operator will play an essential role in our considerations: Let $u \in C^1(G) \cap H^\mu(\Gamma)$, then we have

$$Du = \sum_{i=1}^{3} \partial_i e_i u \text{ with } \partial_i = \partial/\partial x_i,\ i = 1, 2, 3,$$

$$(T_G u)(x) = -\int_G e(x-y) u(y)\, dG_y, \text{ with } e(x) = -x/(4\pi|x|^3),$$

$$(F_\Gamma u)(x) = \int_\Gamma e(x-y)\alpha(y) u(y)\, d\Gamma_y,\ x \notin \Gamma,$$

where $\alpha = \sum_{i=1}^{3} \alpha_i e_i$, and $(\alpha_1, \alpha_2, \alpha_3)$ is the unit vector of the inner normal at the point y,

$$(S_\Gamma u)(x) = 2\int_\Gamma e(x-y)\alpha(y) u(y)\, d\Gamma_y,\ x \in \Gamma,$$

where the integral is to be understand in the sense of Cauchy's principal value. The projections $Q_\Gamma = (I - S_\Gamma)/2$ and $P_\Gamma = (I + S_\Gamma)/2$ are defined in [9]. By I we want to denote the identical operator.

The Banach spaces

$$W_{2,H}^{k,[\delta]}(G),\ \overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G) \quad (\delta \in \mathbb{R},\ k \in \mathbb{N}),$$

$$H_H^\mu(\Gamma)\ (\mu \in \mathbb{R}),\ C_{0,H}^k(G),\ C_H^k(G) \quad (0 \le k \le \infty)$$

are defined by their components, which belong to the corresponding scalar-valued spaces. The space $W_{2,H}^{0,[\delta]}(G) = L_{2,H}^{[\delta]}(G)$ may be characterized by the inner product

$$(u,v)_{L_{2,H}^{[\delta]}} = \int_G u\bar{v}(1+|x|^2)^{\delta}\, dG.$$

We briefly write $p_\delta = (1+|x|^2)^{\delta/2}$. Notice that $L_{2,H}^{[0]}(G) = L_{2,H}(G)$. The space $W_{2,H}^{k,[\delta]}$ may be provided with the inner product

$$(u,v)_{W_{2,H}^{k,[\delta]}} = (u,v)_{L_{2,H}^{[\delta]}} + \sum_{1\le|s|\le k} (\partial^s u, \partial^s v)_{L_{2,H}},$$

where

$$\partial^s u = \sum_{i=0}^{3} \partial^s u_i e_i\,,\ \partial^s = \frac{\partial^{s_1+s_2+s_3}}{\partial x_1^{s_1}\partial x_2^{s_2}\partial x_3^{s_3}}\,,\ |s| = s_1+s_2+s_3.$$

A function $u \in C_H^1(G)$ is called *H-regular* if $Du = 0$ and $|u(x)| \to 0$ if $|x| \to \infty$. By $B_r(x_0)$ we denote the open ball with the centre in x_0 and the radius r. We briefly write $B_r(0) = B_r$, $\partial B_r(x) = S_r(x)$ and $G \cup \Gamma = \bar{G}$.

A simple computation shows the following Lemma 1.1 and Lemma 1.2.

Lemma 1.1: *Let u be an H-valued function with $\partial_i u_j \in L_2(G)$. Then*

$$\|Du\|_{L_{2,H}}^2 \le 3\sum_{j=0}^{3}\int_G \sum_{i=1}^{3} |\partial_i u_j|^2 dG.$$

Lemma 1.2: *Let $u \in \overset{\circ}{W}{}_{2,H}^{1,[\delta]}(G)$, $\delta \in \mathbf{R}$. Then*

$$\|Du\|_{L_{2,H}}^2 = \sum_{j=0}^{3}\int_G \sum_{i=1}^{3} |\partial_i u_j|^2 dG.$$

Lemma 1.3 [10]: *Let $u \in \overset{\circ}{W}{}_2^{1,[\delta]}(G)$, $\delta < -1$.* Then

$$\int_G p_\delta^2 |u|^2\, dG \le (2\{|\delta|-1\})^{-1}\int_G \sum_{i=1}^{3} |\partial_i u|^2\, dG.$$

Lemma 1.4 [10]: *The subset* $\{u \in W_2^k(G)$: supp u *bounded in* $G\}$ is dense in $W_2^k(G)$.

In a similar way to [10] the following lemma can be shown.

Lemma 1.5: *The set $C_{0,R,H}^m(G) = \{v \in C_H^m(G): u|_G = v$ for some $u \in C_{0,H}^m(\mathbf{R}^3)\}$ is dense in $W_{2,H}^k(G)$, $m \ge k$, and $C_{0,H}^m(G)$ is dense in $\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)$, $m \ge k$.*

2. The generalized Borel-Pompeiu formula

An important connection between the operators F_G, T_G and D gives the generalized Borel -Pompeiu formula.

Theorem 2.1: *Let $u \in C_{0,R,H}^1(G)$. Then*

$$-F_\Gamma u(x) + T_G Du(x) = \begin{cases} u(x)\,, & x \in G \\ 0\,, & x \in \mathbf{R}^3 \setminus \bar{G}. \end{cases}$$

Proof: Take r such that $\mathbb{R}^3 \setminus G \subset B_r$. Then $G_r = G \cap B_r$ is a 2-manifold connected domain with the boundaries Γ and $\Gamma_r = S_r$. For G_r we have

$$F_{\Gamma_r} u(x) - F_\Gamma u(x) + T_G D u(x) = \begin{cases} u(x), & x \in G_r \\ 0, & x \in \mathbb{R}^3 \setminus \overline{G_r} \end{cases}$$

(cf. [9]). The function $|x - y|^{-2}$ may be developed into the series

$$|x - y|^{-2} = |y|^{-2} \sum_{k=0}^{\infty} C_k^1(t)\{|x|/|y|\}^k, \text{ with } t = \cos\vartheta = |x|/|y|,\ 0 \le \vartheta < \pi,$$

where C_k^1 denotes the Gegenbauer polynomials

$$C_0^1(t) = 1,\ C_1^1(t) = 2t \text{ and } C_{k+1}^1(t) = 2t\, C_k^1(t) - C_{k-1}^1(t)$$

and the series $\sum C_k^1(t)\alpha^k$ converges if $|t| \le 1$ and $0 \le \alpha \le 1$ (see [4]). For every $x \in G$ there exists an $r_0 > 1$ such that $\operatorname{dist}(x, \mathbb{R}^3 \setminus G_{r_0}) > 1$. Then we have for $r > r_0$

$$\begin{aligned} |F_{\Gamma_r} u(x)| &\le \int_{S_r} |e(x-y)|\,|u(y)|\, dS_{r,y} \\ &\le 1/4\pi \int_{S_r} |y|^{-2} \sum_{k=0}^{\infty} C_k^1(|x|/|y|)(|x|/|y|)^k |u(y)|\, dS_{r,y} \\ &\le 1/4\pi \int_{S_r} |y|^{-2} dS_{r,y} \max_{y \in S_r} |u(y)|\, J = J \max_{y \in S_r} |u(y)| \to 0 \text{ as } r \to \infty, \end{aligned}$$

where

$$|x|/|y| \le (r_0 - 1)/r_0 < 1 \quad \text{and} \quad J = \sum_{k=0}^{\infty} C_k^1((r_0 - 1)/r_0)\{(r_0 - 1)/r_0\}^k \ \blacksquare$$

3. The operators T_G, F_Γ, D

In this section we want to state some properties of the operators T_G, F_Γ and D.

Theorem 3.1: *The operator $T_G: L_{2,H}(G) \to L_{2,H}(G \cap B_n)$ is a continuous mapping for all $n > 0$.*

Proof: Let $u \in L_{2,H}(G)$. Then

$$\begin{aligned} 8\pi^2 \|T_G u\|^2_{L_{2,H}(G \cap B_n)} &= 8\pi^2 \int_{G \cap B_n} |Tu|^2 dx \\ &\le \int_{G \cap B_n} \left| \int_{G \cap B_{n+1}} |u(y)|\,|x-y|^{-2} dy \right|^2 dx + \int_{G \cap B_n} \left| \int_{G \setminus B_{n+1}} |u(y)|\,|x-y|^{-2} dy \right|^2 dx \\ &\le \int_{G \cap B_{n+1}} \int_{G \cap B_{n+1}} |x-y|^{-2} dy \int_{G \cap B_{n+1}} |u(y)|^2 |x-y|^{-2} dy\, dx \\ &\quad + \int_{G \cap B_n} \int_{G \setminus B_{n+1}} |u(y)|^2 dy \int_{G \setminus B_{n+1}} |x-y|^{-4} dy\, dx \\ &\le C_1(n)/2 \iint_{G \cap B_{n+1}} |u(y)|^2 |x-y|^{-2} dy\, dx + C_2(n)/2\, \|u\|^2_{L_{2,H}}. \end{aligned}$$

Furthermore we have

$$\iint_{G\cap B_{n+1}} |u(y)|^2 |x-y|^{-2} dx\,dy = \int_{G\cap B_{n+1}} |u(y)|^2 \int_{G\cap B_{n+1}} |x-y|^{-2} dx\,dy \le C_3(n)\|u\|^2_{L_{2,H}}.$$

Hence we get

$$\iint_{G\cap B_{n+1}} |u(y)|^2 |x-y|^{-2} dx\,dy \le C_3(n)\|u\|^2_{L_{2,H}} \text{ and } \|T_G u\|_{L_{2,H}(G\cap B_n)} \le C(n)\|u\|_{L_{2,H}}$$

which shows the assertion ■

Theorem 3.2: *The operator* $T_G : L^{[\gamma]}_{2,H}(G) \to L^{[\gamma+\delta]}_{2,H}(G)$ $(\delta < -1,\ 0 \le \gamma \le 3)$ *is a continuous mapping.*

Proof: We have

$$2\|T_G u\|^2_{L^{[\gamma+\delta]}_{2,H}(G)} = 2\int_G p^2_{\gamma+\delta} \left|\int_G e(x-y)u(y)dG_y\right|^2 dG_x$$

$$\le \int_G p^2_{\gamma+\delta} \left|\int_{G\cap B_1(x)} e(x-y)u(y)dG_y\right|^2 dG_x + \int_G p^2_{\gamma+\delta} \left|\int_{G\setminus B_1(x)} e(x-y)u(y)dG_y\right|^2 dG_x$$

$$\le \int_G p^2_{\gamma+\delta} \int_{G\cap B_1(x)} p_\gamma^{-2} |e(x-y)|\,dG_y \int_{G\cap B_1(x)} |e(x-y)|\,|u(y)|^2 p_\gamma^{-2}\, dG_y\, dG_x$$

$$+\int_G p^2_{\gamma+\delta} \int_{G\setminus B_1(x)} p_\gamma^{-2} |e(x-y)|^{3/2+\eta} dG_y \int_{G\setminus B_1(x)} |e(x-y)|^{1/2-\eta} |u(y)|^2 p_\gamma^2\, dG_y\, dG_x$$

with $\eta + \gamma > 0$, $\eta < 1/2$. Furthermore,

$$\int_{G\cap B_1(x)} p_\gamma^{-2} |e(x-y)|\,dG_y$$

$$\le \begin{cases} \int_{B_1(x)} |x-y|^{-2} dG_y = 1 & \text{if } |x| < 1 \\ 1/(1+(|x|-1)^2)^\gamma \int_{B_1(x)} |x-y|^{-2} dG_y = 1/(1+(|x|-1)^2)^\gamma & \text{if } |x| \ge 1 \end{cases}$$

$$\le c(1+|x|^2)^{-\gamma} = c\,p_\gamma^{-2}$$

and, if $\gamma > 0$,

$$\int_{G\setminus B(x)} p_\gamma^{-2} |e(x-y)|^{3/2+\eta} dG_y \le 1/(4\pi)^{3/2+\eta} \int_{\mathbb{R}^3\setminus B_1(x)} p_\gamma^{-2} |x-y|^{-3-2\eta} dG_y$$

$$\le 1/(4\pi)^{3/2+\eta} \int_{\mathbb{R}^3\setminus B_1(x)} p_\gamma^{-2} |x-y|^{-3} dG_y \le 1/(4\pi)^{3/2+\eta}\, p^{-2}_{\gamma-\omega}$$

with $\gamma > \omega > 0$ (cf. [6]). If $\gamma = 0$ we get the same inequality with $\omega = \gamma - \omega = 0$. Now we obtain

$$1/2\, \|T_G u\|^2_{L^{[\gamma+\delta]}_{2,H}(G)}$$

$$\le \int_G p^2_{\gamma+\delta}\, c(1+|x|^2)^{-\gamma} \int_{G\cap B_1(x)} |e(x-y)|\,|u(y)|^2 p_\gamma^2\, dG_y\, dG_x$$

$$+ 1/(4\pi)^{3/2+\eta} \int_G p^2_{\gamma+\delta}\, p^{-2}_{\gamma-\omega} \int_{G\setminus B_1(x)} |e(x-y)|^{1/2-\eta} |u(y)|^2 p_\gamma^2\, dG_y\, dG_x$$

$$\le {}^1\!/_{4\pi}\int\limits_G\int\limits_G p^2_{\gamma+\delta}(1+|x|^2)^{-\gamma}|x-y|^{-2}|u(y)|^2 p^2_\gamma\, dG_y\, dG_x$$

$$+{}^1\!/_{(4\pi)^2}\int\limits_G\int\limits_G p^2_{\gamma+\omega}|x-y|^{-1+2\eta}|u(y)|^2 p^2_\gamma\, dG_y\, dG_x$$

$$= {}^1\!/_2\int p^2_\gamma |u(y)|^2\Big\{{}^1\!/_{2\pi}\int p^2_{\gamma+\delta}\, c(1+|x|^2)^{-\gamma}|x-y|^{-2}dG_x$$

$$+{}^1\!/_{8\pi^2}\int\limits_G p^2_{\gamma+\omega}|x-y|^{-1+2\eta}\, dG_x\Big\}dG_y \le C\,\|u\|^2_{L^{[\gamma]}_{2,H}(G)}$$

because of Fubini's Theorem and that we can choose η, $\omega > 0$ such that $2(\delta - 1/2 + \eta + \omega) < -3$ and $\delta < -1$ ∎

Theorem 3.3: *The operator* $F_\Gamma: L_{2,H}(\Gamma) \to W^k_{2,H}(G^*)$, $k \in \mathbb{N}$, *is a continuous mapping, where* G^* *is a domain with* $\overline{G^*} \subset G$ *or* $\overline{G^*} \subset \mathbb{R}^3\setminus\overline{G}$. Besides, $F_\Gamma u \in \ker D(\mathbb{R}^3\setminus\Gamma)$.

Proof: We consider the expression

$$\int\limits_{G^*}|D^\beta_x F_\Gamma u|^2\, dG^*_x = {}^1\!/_{(4\pi)^2}\int\limits_{G^*}\Big|\int\limits_\Gamma D^\beta_x e(x-y)\alpha(y)u(y)\,d\Gamma_y\Big|^2 dG^*_x.$$

Because of $\overline{G^*} \subset G$ or $\overline{G^*} \subset \mathbb{R}^3\setminus\overline{G}$ there exists a $\delta > 0$ such that $\operatorname{dist}(G^*,\Gamma) = \delta$ and $|x-y| \ge \delta$ for all $x \in G^*$ and $y \in \Gamma$. Furthermore $e(x-y)$ is a harmonic function in G^* and from [5] we get the estimate

$$|D^\beta_x F_\Gamma u| \le C|x-y|^{-(2+|\beta|)},\ C \text{ a constant.}$$

If mes $G^* < \infty$ we obtain

$$\int\limits_{G^*}|D^\beta_x F_\Gamma u|^2\, dG^* \le \int\limits_{G^*}|D^\beta_x(e(x-y))|^2|\alpha(y)|^2\, d\Gamma_y\int\limits_\Gamma |u(y)|^2 d\Gamma_y\, dG^* \le C_\beta\|u\|^2_{L_{2,H}(G)}$$

using $|\alpha(y)|^2 = 1$, mes $\Gamma < \infty$ and also mes $G^* < \infty$. If G^* is unbounded we decompose

$$G^* = G_1 \cup G_2,\ \text{where } G_1 = \{x \in G^*: |x| \le 2\sup_{y\in\Gamma}|y| + 1\} \text{ and } G_2 = G^*\setminus G_1.$$

Then we have mes $G_1 < \infty$ and analogously we get

$$\int\limits_{G_1}|D^\beta_x F_\Gamma u|^2\, dG_1 \le C_{1,\beta}\|u\|^2_{L_{2,H}(\Gamma)}.$$

In G_2 we have $|D^\beta_x e(x-y)| \le C|x-y|^{-(2+|\beta|)} \le C^*\delta^{-|\beta|}4(1+|x|)^{-2}$ and now it follows the estimate

$$\int\limits_{G_2}|D^\beta_x F_\Gamma u|^2\, dG_2 \le ({}^{4C^*}\!/_{4\pi})^{-2}\int\limits_{G_2}(1+|x|)^{-4}\,\text{mes}\,\Gamma\int\limits_\Gamma |u(y)|^2\, d\Gamma_y\, dG_2$$

$$\le C^{**}\|u\|^2_{L_2(\Gamma)}\int\limits_{G_2}(1+|x|)^{-4}dG_2 = C_{2,\beta}\|u\|^2_{L_{2,H}(\Gamma)}.$$

Thus we have shown that $\|D^\beta F_\Gamma u\|^2_{L_{2,H}(G^*)} \le C_\beta\|u\|^2_{L_{2,H}(\Gamma)}$. Hence we get

$$\|F_\Gamma u\|^2_{W^k_{2,H}(G^*)} = \sum_{|\beta|\le k}\|D^\beta F_\Gamma u\|^2_{L_{2,H}} \le C\,\|u\|^2_{L_{2,H}(\Gamma)},\quad C = \max_{|\beta|\le k} C_\beta.$$

Now we consider

$$
\begin{aligned}
DF_\Gamma u &= \tfrac{1}{4\pi}\sum_{i=1}^{3} e_i\partial_i \sum_{j=1}^{3} e_j \int_\Gamma (x_j - y_j)|x-y|^{-3}\alpha(y)u(y)\,d\Gamma_y \\
&= \tfrac{1}{4\pi}\sum_{i=1}^{3}\sum_{j=1}^{3} e_i e_j \int_\Gamma \partial_i((x_j-y_j)|x-y|^{-3})\alpha(y)u(y)\,d\Gamma_y \\
&= \tfrac{1}{4\pi}\sum_{i,j=1}^{3} e_i e_j \int_\Gamma \left(\partial_{ij}|x-y|^{-3} - 3(x_j-y_j)(x_i-y_i)|x-y|^{-5}\right)\alpha(y)u(y)\,d\Gamma_y = 0
\end{aligned}
$$

in $\mathbb{R}^3\setminus\Gamma$ ∎

Lemma 3.1: *The operator $T_G D$ admits an extension to a continuous map from $W^k_{2,H}(G)$ into $W^k_{2,H}(G^*)$, $k\in\mathbb{N}\setminus\{0\}$, G^* as defined in Theorem 3.3.*

Proof: If $u\in W^k_{2,H}(G)$, then tr $u\in W^{k-1/2}_{2,H}(\Gamma)\subset L_{2,H}(\Gamma)$ is a continuous imbedding. Thus we obtain the continuity of I and $F_\Gamma$tr: $W_{2,H}(G)\to W_{2,H}(G^*)$. Let $u\in C_{0,R,H}(G)$, then by means of the Borel-Pompeiu formula we have $T_G Du = F_\Gamma u + u$ in G and $T_G Du = F_\Gamma u$ in $\mathbb{R}^3\setminus\overline{G}$ and so we get

$$
\|T_G Du\|_{W^k_{2,H}(G^*)} \le \begin{Bmatrix} \|F_\Gamma u\|_{W^k_{2,H}(G^*)} + \|u\|_{W^k_{2,H}(G^*)} \\ \|F_\Gamma u\|_{W^k_{2,H}(G^*)} \end{Bmatrix} \le \|u\|_{W^k_{2,H}(G^*)}.
$$

Because $C_{0,R,H}(G)$ is dense in $W^k_{2,H}(G)$, the operator $T_G D$ admits a continuous extension ∎

Theorem 3.4: *The Borel-Pompeiu formula can be generalized for functions $u\in W^1_{2,H}(G)$.*

Proof: Let $u\in W^1_{2,H}(G)$. Then there exists $u_n\in C^1_{0,R,H}(G)$ such that

$$
\|u_n - u\|_{W^1_{2,H}} \to 0 \text{ if } n\to\infty \quad\text{and}\quad -F_\Gamma u_n + T_G Du_n = \begin{cases} u_n & \text{in } G \\ 0 & \text{in } \mathbb{R}^3\setminus\overline{G}. \end{cases}
$$

Now for every domain G^*, in case $\overline{G^*}\subset G$ and $\overline{G^*}\subset\mathbb{R}^3\setminus G$ we obtain

$$
\begin{array}{ccccc}
-F_\Gamma u_n + T_G Du_n = u_n & & & -F_\Gamma u_n + T_G Du_n = 0 \\
\downarrow \quad\quad \downarrow \quad\quad \downarrow & & \text{and} & \downarrow \quad\quad \downarrow \\
-F_\Gamma u + T_G Du = u & & & -F_\Gamma u + T_G Du = 0,
\end{array}
$$

in the sense of $W^1_{2,H}(G)$, respectively. Because the foregoing formulas are true for every G^*, we get for $u\in W^1_{2,H}(G)$ that $-F_\Gamma u + T_G Du = u$ in G and $-F_\Gamma u + T_G Du = 0$ in $\mathbb{R}^3\setminus\overline{G}$ ∎

Analogously to [6: p. 249] the following lemma may be verified.

Lemma 3.2: *For each $u\in W^k_{2,H}(G)$, $k\in\mathbb{N}$, we have $\partial_k T_G u\in W^k_{2,H}(G)$ and*

$$
\begin{aligned}
\partial_k(T_G u)(x) = &-\int_G \partial_{k,x}\, e(x-y)u(y)\,dG_y \\
&-u(x)\int_{S_1} \sum_{i=1}^{3} e_i(x_i-y_i)|x-y|^{-1}(x_k-y_k)|x-y|^{-1}dS.
\end{aligned}
\tag{3.1}
$$

Proof: The first integral in (3.1) is singular and a simple computation shows the existence of the integral in the sense of Cauchy's principal value. Using [6: p. 316] we get that this integral is bounded in $W^k_{2,H}(G)$. Obviously, the second integral in (3.1) delivers a finite value. For $u \in W^k_{2,H}(G)$ there are $u_n \in C^k_{0,R,H}(G)$ such that $u_n \to u$ in $W^k_{2,H}(G)$. Set $v_n = T_G u_n$. Using calculations carried out in [6: p. 249] we obtain for u_n the expression (3.1). So we have $\partial_k T_G u_n \in W^k_{2,H}(G)$ and $T_G u_n \in W^{k,[\delta]}_{2,H}(G)$, $\delta < -1$, from Theorem 3.2. There exists the limit of the right-hand side for $u_n \to u$ in $W^k_{2,H}(G)$ and is equal to the right-hand side of (3.1). Because of the closure of the operator of differentiation there exists $\partial_k T_G u$ and is equal to the right-hand side of (3.1). Therefore we also get from (3.1) that $\partial_k T_G u \in W^k_{2,H}(G)$ ∎

Corollary 3.1: *The operator T_G maps $W^k_{2,H}(G)$ into $W^{k+1,[\delta]}_{2,H}(G)$, $\delta < -1$, $k \in \mathbb{N}$, and into $W^{k+1}_{2,H}(G \cap B_n)$ for all n such that $G \cap B_n \neq 0$.*

Theorem 3.5: *The operator $\partial_k T$: $W^k_{2,H}(\mathbb{R}^3) \to W^k_{2,H}(\mathbb{R}^3)$ is continuous.*

Proof: Using Lemma 3.2 we get

$$\partial_k Tu = -\tfrac{1}{4\pi} \sum_{i=1}^{3} \sum_{j=0}^{3} e_i e_j \partial_k \int_{\mathbb{R}^3} (x_i - y_i)|x - y|^{-3} u_j(y)\,dy$$

$$= -\tfrac{1}{4\pi} \sum_{i=1}^{3} \sum_{j=0}^{3} e_i e_j \int_{\mathbb{R}^3} \partial_{k,x}((x_i - y_i)|x - y|^{-3})\, u_j(y)\,dy$$

$$- u_j \int_{S_1} (x_i - y_i)|x - y|^{-1}(x_k - y_k)|x - y|^{-1}\,dS.$$

Obviously, the relation

$$- u_j \int_{S_1} (x_i - y_i)|x - y|^{-1}(x_k - y_k)|x - y|^{-1}\,dS = u_j\, C(i,k) \in W^k_{2,H}(\mathbb{R}^3)$$

is valid. Now we consider

$$T_{ik} u_j = \begin{cases} \int_{\mathbb{R}^3} -3(x_i - y_i)(x_k - y_k)|x - y|^{-5} u_j(y)\,dy & \text{if } i \neq k \\ \int_{\mathbb{R}^3} -(|x - y|^{-2} - 3(x_i - y_i)^2)|x - y|^{-5} u_j(y)\,dy & \text{if } i = k \end{cases}$$

$$= \left\{ \begin{matrix} -3x_i x_k |x|^{-5} & \text{if } i \neq k \\ -(|x|^2 - 3x_i)|x|^{-5} & \text{if } i = k \end{matrix} \right\} * u_j = \left\{ \begin{matrix} -\partial_i \partial_k |x|^{-1} \\ -\partial_i \partial_k |x|^{-1} \end{matrix} \right\} * u_j$$

and

$$\|T_{ik} u_j\|^2_{W^k_2} = \int_{\mathbb{R}^3} |\mathcal{F}(T_{ik} u)|^2 (1 + |\xi|^2)^k\,d\xi$$

$$= c \begin{cases} \int_{\mathbb{R}^3} \left|\xi_i \xi_k |\xi|^{-2} \mathcal{F}(u_j)\right|^2 (1 + |\xi|^2)^k\,d\xi & \text{if } i \neq k \\ \int_{\mathbb{R}^3} \left|\xi_i^2 |\xi|^{-2} \mathcal{F}(u_j)\right|^2 (1 + |\xi|^2)^k\,d\xi & \text{if } i = k \end{cases}$$

$$\leq c \int_{\mathbb{R}^3} |\mathcal{F}(u_j)|^2 (1 + |\xi|^2)^k\,d\xi = \|u_j\|^2_{W^k_2}.$$

Therefore $\|\partial_k Tu\|_{W^k_{2,H}} < C^{1/2} \|u\|_{W^k_{2,H}}$, where $\mathcal{F}(u_j)$ denotes the Fourier transformation of the scalar function u_j ∎

Theorem 3.6: *Let $u \in L_{2,H}(G)$. Then $DT_G u = u$ in G and $DT_G u = 0$ in $\mathbb{R}^3\setminus\overline{G}$.*

Proof: Lemma 3.2 shows that $\partial_i T_G u \in L_{2,H}(G)$ exists. At first let $u \in C^1_{0,R,H}(G)$ and $x \in G$. Set $G_\varepsilon = G\setminus B_\varepsilon(x)$. Then we have $DT_{G_\varepsilon}u(x) = 0$ in G_ε and it remains to consider the term in $B_\varepsilon(x)$. Using Lemma 3.2 we obtain

$$\begin{aligned}
&(DT_{B_\varepsilon(x)}u)(x)\\
&= -1/_{4\pi}\sum_{i,j=1}^{3} e_i e_j\, \partial_{i,k}\int_{B_\varepsilon(x)}(x_j - y_j)|x-y|^{-3}u(y)\,dB_{\varepsilon,y}\\
&= -1/_{4\pi}\sum_{i,j=1}^{3} e_i e_j \int_{B_\varepsilon(x)}\partial_{i,k}((x_j - y_j)|x-y|^{-3})u(y)\,dB_{\varepsilon,y}\\
&\quad - u(x)/_{4\pi}\sum_{i,j=1}^{3}\int_{S_1}(x_i - y_i)|x-y|^{-1}(x_j-y_j)|x-y|^{-1}u(y)\,dS_1\\
&= -1/_{4\pi}\sum_{i,j=1}^{3} e_i e_j \int_{B_\varepsilon(x)}(\partial_{ij}|x-y|^{-3} - 3(x_i-y_i)(x_j-y_j)|x-y|^{-5})u(y)\,dB_{\varepsilon,y}\\
&\quad - u(x)/_{4\pi}\sum_{i,j=1}^{3} e_i e_j\int_{S_1}(x_i - y_i)|x-y|^{-1}(x_j-y_j)|x-y|^{-1}\,dS_1.
\end{aligned}$$

Because of $e_i e_j + e_j e_i = 0$ for $i,j \neq 0$ and $i \neq j$ we get

$$\begin{aligned}
(DT_{B_\varepsilon}u)(x) &= -1/_{4\pi}\sum_{i=1}^{3} e_0 \int_{B_\varepsilon(x)}(-|x-y|^{-3} + 3(x_i-y_i)^2|x-y|^{-5})u(y)\,dB_{\varepsilon,y}\\
&\quad + u(x)/_{4\pi}\int_{S_1} e_0 \sum_{i=1}^{3}(x_i - y_i)^2|x-y|^{-2}\,dS_y = u(x).
\end{aligned}$$

If $x \in \mathbb{R}^3\setminus\overline{G}$, then $DT_G u(x) = -\int_G D_x e(x-y)u(y)\,dG = 0$. Since $C^1_{0,R,H}(G)$ is a dense subset of $L_{2,H}(G)$, we have $DT_G u = u$ in G and $DT_G u = 0$ in $\mathbb{R}^3\setminus\overline{G}$, where $u \in L_{2,H}(G)$ ∎

Lemma 3.3: *The operator F_Γ maps $W^{k-1/2}_{2,H}(\Gamma)$ into $W^k_{2,H}(G)$, $k \in \mathbb{N}\setminus\{0\}$.*

Proof: The compactness of Γ guarantees the existence of a fixed n such that $\Gamma \subset B_n$ and $\mathrm{dist}(\Gamma, \mathbb{R}^3\setminus B_n) > 1$. Let $u \in W^{k-1/2}_{2,H}(\Gamma)$. Then there exists a continuous extension $v \in W^k_{2,H}(G)$ with $\mathrm{tr}\, v = u$. If we use the Borel-Pompeiu formula we obtain $F_\Gamma u = T_G Dv - v$. With Corollary 3.1 we obtain $F_\Gamma u \in W^k_{2,H}(G\cap B_n)$ and from Theorem 3.3 we obtain $F_\Gamma u \in W^k_{2,H}(G\setminus B_n)$. Thus $\|F_\Gamma u\|_{W^k_{2,H}(G)} = \|F_\Gamma u\|_{W^k_{2,H}(G\cap B_n)} + \|F_\Gamma u\|_{W^k_{2,H}(G\setminus B_n)}$ ∎

Theorem 3.7: *The operator D: $W^{k+1,[\delta]}_{2,H}(G) \to W^k_{2,H}(G)$, $\delta \le 0$, $k \in \mathbb{N}$, is a continuous mapping.*

Proof: We have

$$\begin{aligned}
\|Du\|^2_{W^k_{2,H}} &= \sum_{|\alpha|\le k}\|\partial^\alpha(Du)\|^2_{L_{2,H}} = \sum_{|\alpha|\le k}\|D\partial^\alpha u\|^2_{L_{2,H}}\\
&\le 3\sum_{|\alpha|\le k}\sum_{j=0}^{3}\int_G\sum_{i=1}^{3}|\partial_i\partial^\alpha u_j|^2\,dx
\end{aligned}$$

$$\le 3\sum_{j=0}^{3}\int_G p_\delta^2|u_j|^2 dG + \sum_{|\alpha|\le k}\int_G \sum_{i=1}^{3}|\partial_i\partial^\alpha u_j|^2 dG$$

$$\le 3\sum_{j=0}^{3}\int_G p_\delta^2|u_j|^2 dG + \sum_{1\le|\alpha|\le k+1}\int_G |\partial^\alpha u_j|^2 dG = 3\|u\|^2_{W_{2,H}^{k+1,[\delta]}}$$

from which the assertion follows ∎

Theorem 3.8: *The operator* $T_G : D(\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)) \to \overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)$, $\delta < -1$, $k \in \mathbb{N}\setminus\{0\}$ *is a continuous* 1-1 - *mapping.*

Proof: For $v \in D(\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G))$ there exists a $u \in \overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)$ such that $Du = v$. Then we get $T_G Du = u$ in G for u from the space $C_{0,H}(G)$ and also from the space $\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)$ because the first space is dense in the second one. That means that T_G is a surjective operator. Furthermore

$$\|T_G v\|^2_{W_{2,H}^{k,[\delta]}} = \|T_G Du\|^2_{W_{2,H}^{k,[\delta]}} = \|u\|^2_{W_{2,H}^{k,[\delta]}}$$

$$= \sum_{j=0}^{3}\left\{\int_G p_\delta^2|u_j|^2 dG + \sum_{1\le|\alpha|\le k}\int_G |\partial^\alpha u_j|^2 dG\right\}$$

$$\le \sum_{j=0}^{3}\left\{ 1/2(|\delta|-1)\int_G \sum_{i=1}^{3}|\partial_i u_j|^2 dG + \sum_{1\le|\alpha|\le k}\int_G |\partial^\alpha u_j|^2 dG\right\}$$

$$\le \max\{1/2(|\delta|-1), 1\}\sum_{j=0}^{3}\sum_{0\le|\alpha|\le k-1}\int_G \sum_{i=1}^{3}|\partial_i\partial^\alpha u_j|^2 dG$$

$$= \max\{1/2(|\delta|-1), 1\}\sum_{|\alpha|\le k-1}\|\partial^\alpha Du\|^2_{L_{2,H}}$$

$$= \max\{1/2(|\delta|-1), 1\}\,\|Du\|^2_{W_{2,H}^{k-1}} = \max\{1/2(|\delta|-1), 1\}\,\|v\|^2_{W_{2,H}^{k-1}}$$

and

$$\|v\|^2_{W_{2,H}^{k-1}} = \|Du\|^2_{W_{2,H}^{k-1}}$$

$$= \sum_{|\alpha|\le k-1}\|\partial^\alpha Du\|^2_{L_{2,H}} = \sum_{|\alpha|\le k-1}\|D\partial^\alpha u\|^2_{L_{2,H}}$$

$$= \sum_{|\alpha|\le k-1}\sum_{j=0}^{3}\int_G \sum_{i=1}^{3}|\partial_i\partial^\alpha u_j|^2 dG$$

$$\le \sum_{j=0}^{3}\left\{\int_G p_\delta^2|u_j|^2 dG + \sum_{|\alpha|\le k}\int_G |\partial^\alpha u_j|^2 dG\right\}$$

$$= \|u\|^2_{W_{2,H}^{k,[\delta]}} = \|T_G Du\|^2_{W_{2,H}^{k,[\delta]}} = \|T_G v\|^2_{W_{2,H}^{k,[\delta]}}.$$

For this reason T_G is a 1-1 - mapping ∎

Corollary 3.2: *Thus* T_G *in the pair of Banach spaces* $\left(D(\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)), \overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)\right)$ *is continuously invertible*, $\delta < -1$ *and* $k \in \mathbb{N}\setminus\{0\}$.

Besides the formulas $T_G D = I$ *in* $\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G)$ *and* $D T_G = I$ *in* $D(\overset{\circ}{W}{}_{2,H}^{k,[\delta]}(G))$ *are valid.*

4. A decomposition formula of $L_{2,H}(G)$

First we will give a representation of harmonic H-valued functions. Then we prove a decomposition formula of $L_{2,H}(G)$, which is important for our applications in Section 5.

Theorem 4.1: *Let $u \in \ker\Delta(G) \cap W_{2,H}^{k}(G)$, $|u| \to 0$ if $|x| \to \infty$, $k \in \mathbb{N}\setminus\{0\}$. Then there exist two unique functions $u_i \in \ker D(G) \cap W_{2,H}^{k+1-i}(G)$, $i = 1,2$, such that $u = T_G u_2 + u_1$.*

Proof: From Theorem 3.4 we obtain $u = -F_\Gamma u + T_G Du$ in G and from Theorem 3.7 and Lemma 3.3 we obtain $F_\Gamma u \in W_{2,H}^{k}(G) \cap \ker D(G)$. Set $u_2 = Du$ and $u_1 = -F_\Gamma u$. Then $u_2 = DT_G u_2 = D(u - u_1) \in W_{2,H}^{k-1}(G)$ and $Du_2 = DDu = -\Delta u = 0$, i.e. $u_2 \in \ker D(G)$. Suppose $u = T_G u_{21} + u_{11} = T_G u_{22} + u_{21}$. Then $Du = u_{21} = u_{22}$ and $T_G u_{21} - T_G u_{22} = T_G(u_{21} - u_{22}) = u_{21} - u_{11} = 0$. Hence the representation is unique ∎

Lemma 4.2: *The operator $\operatorname{tr} T_G F_\Gamma : \operatorname{im} P_\Gamma \cap W_{2,H}^{k-1/2}(\Gamma) \to \operatorname{im} Q_\Gamma \cap W_{2,H}^{k+1/2}(\Gamma)$, $k \in \mathbb{N}\setminus\{0\}$, is a bijective mapping.*

Proof: We consider the bijective sequence

$$\operatorname{im} P_\Gamma \cap W_{2,H}^{k-1/2}(\Gamma) \xrightarrow{F_\Gamma} W_{2,H}^{k}(G) \cap \ker D(G)$$

$$\xrightarrow{T_G} W_{2,H}^{k+1}(G) \cap \ker\Delta(G) \cap \ker D(\mathbb{R}^3\setminus\overline{G}) \xrightarrow{\operatorname{tr}} \operatorname{im} Q_\Gamma \cap W_{2,H}^{k+1/2}(\Gamma).$$

Now we want to show that $\operatorname{tr} T_G F_\Gamma u = 0$ implies $u = 0$. Let $T_G F_\Gamma u$ be the solution of the equation $-\Delta(T_G F_\Gamma u) = 0$ in G and $\operatorname{tr} T_G F_\Gamma u = 0$ on Γ. Because Dirichlet's problem has at most one solution it follows that $T_G F_\Gamma u = 0$ and $F_\Gamma u = 0$ and thus $u = 0$ on Γ. Now we show that the mapping is surjective. Let $h \in \operatorname{im} Q_\Gamma \cap W_{2,H}^{k+1/2}(\Gamma)$. Then there is the solution w for the problem $-\Delta w = 0$ in G, $\operatorname{tr} w = h$ on Γ, $|w| \to 0$ if $|x| \to \infty$ (cf. [5]), and from Theorem 4.1 we get $w = T_G w_2 + w_1$, whith $w_i \in \ker D(G) \cap W_{2,H}^{k+2-i}(G)$ and $w_1 = F_\Gamma \operatorname{tr} w$, $i = 1, 2$. Using Plemelj-Sokhotzki's formula (cf. [9]) we get $-F_\Gamma \operatorname{tr} w \to P_\Gamma(\operatorname{tr} w)$ if $x \to x_0 \in \Gamma$ ($x \in G$), and it follows $w_1 = -F_\Gamma P_\Gamma \operatorname{tr} w = -F_\Gamma P_\Gamma h = 0$, as $h \in \operatorname{im} Q_\Gamma$. This means $w = T_G Dw$ $T_G w_2 = -T_G F_\Gamma w_2$ and $\operatorname{tr} w = h = \operatorname{tr} T_G F_\Gamma(-w_2)$. Altogether for any $h \in \operatorname{im} Q_\Gamma \cap W_{2,H}^{k+1/2}(\Gamma)$ there is one and only one $v \in \operatorname{im} P_\Gamma \cap W_{2,H}^{k-1/2}(\Gamma)$ such that $h = \operatorname{tr} T_G F_\Gamma v$ ∎

Theorem 4.2: *The operator $\mathbf{P} = F_\Gamma(\operatorname{tr} T_G F_\Gamma)^{-1} \operatorname{tr} T_G$ is a projection onto the subspace $\ker D(G) \cap L_{2,H}(G)$ and $\mathbf{Q} = I - \mathbf{P}$ is a projection onto the complementary subspace $D\big(\overset{\circ}{W}{}_{2,H}^{1,[\delta]}(G)\big)$, $\delta < -1$.*

Proof: We consider the sequence ($\delta < -1$)

$$W_{2,H}^{1}(G) \xrightarrow{T_G} W_{2,H}^{2,[\delta]}(G) \xrightarrow{\operatorname{tr}} W_{2,H}^{3/2}(\Gamma) \xrightarrow{(\operatorname{tr} T_G F_\Gamma)^{-1}} W_{2,H}^{1/2}(\Gamma) \xrightarrow{F_\Gamma} W_{2,H}^{1}(G),$$

i.e. $\mathbf{P}$ and $\mathbf{Q}$ map $W_{2,H}^{1}(G)$ into itself.

Now we prove that $\mathbf{P}u = u$ iff $u \in \ker D(G) \cap W_{2,H}^{1}(G)$. Let $u \in \ker D(G) \cap W_{2,H}(G)$. Then we have $u = -F_\Gamma u$ and $\mathbf{P}u = -\mathbf{P}F_\Gamma u = -F_\Gamma(\operatorname{tr} T_G F_\Gamma)\operatorname{tr} T_G F_\Gamma u = -F_\Gamma u = u$. Conversely, let $u \in W_{2,H}^{1}(G)$ and $\mathbf{P}u = u$. Then we have $u = F_\Gamma(\operatorname{tr} T_G F_\Gamma)^{-1} \operatorname{tr} T_G u \in \ker D(G) \cap W_{2,H}(G)$ because of $(\operatorname{tr} T_G F_\Gamma)^{-1} \operatorname{tr} T_G u \in \operatorname{im} P_\Gamma \cap W_{2,H}(\Gamma)$.

Now we prove that $\mathbf{Q}u = u$ iff $u \in D(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)) \cap W^1_{2,H}(G)$, $\delta < -1$. Let u belong to that intersection. Then a $v \in \overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)$ exists such that $u = Dv$ and $\mathrm{tr}T_G u = \mathrm{tr}T_G Dv = \mathrm{tr}\, v = 0$. This means $\mathbf{P}u = 0$ and we get $\mathbf{Q}u = (I - \mathbf{P})u = u$. Conversely, let $u \in W^1_{2,H}(G)$ and $\mathbf{Q}u = u$. Then we have $T_G u \in W^{1,[\delta]}_{2,H}(G)$ and $\mathrm{tr}T_G u = \mathrm{tr}T_G \mathbf{Q}u = (\mathrm{tr}T_G - \mathrm{tr}T_G\mathbf{P})u = (\mathrm{tr}T_G - \mathrm{tr}T_G F_\Gamma(\mathrm{tr}T_G F_\Gamma)^{-1}\mathrm{tr}T_G)u = 0$, i.e. $T_G u \in \overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)$ and $u = DT_G u$, $u \in D(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)) \cap W_{2,H}(G)$, $\delta < -1$.

Because of the density of $W^1_{2,H}(G)$ in $L_{2,H}(G)$ there is a continuous extension of $\mathbf{P}$ onto the subspace $\ker D(G) \cap L_{2,H}(G)$ and of $\mathbf{Q}$ onto the subspace $D(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G))$. Furthermore,

$$\mathbf{P}^2 u = F_\Gamma(\mathrm{tr}T_G F_\Gamma)^{-1}\mathrm{tr}T_G F_\Gamma(\mathrm{tr}T_G F_\Gamma)^{-1}\mathrm{tr}T_G u = F_\Gamma(\mathrm{tr}T_G F_\Gamma)^{-1}\mathrm{tr}T_G u = \mathbf{P}u,$$

$$\mathbf{Q}^2 u = (I - \mathbf{P})(I - \mathbf{P})u = (I - \mathbf{P} - \mathbf{P} + \mathbf{P}^2)u = (I - \mathbf{P})u = \mathbf{Q}u,$$

$$\mathbf{QP}u = (I - \mathbf{P})\mathbf{P}u = (\mathbf{P} - \mathbf{P}^2)u = 0, \quad \mathbf{PQ}u = \mathbf{P}(I - \mathbf{P})u = (\mathbf{P} - \mathbf{P}^2)u = 0,$$

which show that the statements of the lemma are true ∎

Now we have shown that $\mathbf{P}$ and $\mathbf{Q}$ are projections. In the following theorem we will show that $\mathbf{P}$ and $\mathbf{Q}$ are even orthoprojections.

Theorem 4.3 (Decomposition Theorem): *We have the decomposition*

$$L_{2,H}(G) = \ker D(G) \cap L_{2,H}(G) \oplus D(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)), \ \delta < -1,$$

where $\oplus$ *denotes the orthogonal sum with the inner product of* $L_{2,H}$. *The orthoprojections are* $\mathbf{P}$ *and* $\mathbf{Q}$, *respectively.*

Proof: Take $X_1 = \ker D(G) \cap L_{2,H}(G)$ and $X_2 = D(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G))$. Let $u \in L_{2,H}(G)$. Then $\mathbf{P}u + \mathbf{Q}u = \mathbf{P}u + u - \mathbf{P}u = u$, i.e. $L_{2,H}(G) \subset X_1 + X_2$. Obviously $X_1 + X_2 \subset L_{2,H}(G)$. Let $u \in X_1 \cap W^1_{2,H}(G)$ and $v \in X_2$. Then there exists a $w \in \overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)$ such that $v = Dw$ and on account of the Gauß-Ostrogradski formula (cf. [9]) we get

$$\int_G u\bar{v}\, dG = \int_G u\overline{Dw}\, dG = \int_G Du\bar{w}\, dG + \int_\Gamma u\bar{\alpha}\bar{w}\, d\Gamma = 0,$$

as $u \in X_1$, i.e. $Du = 0$, and $\mathrm{tr}\, w = 0$. Since the subspace $X_1 \cap W^1_{2,H}(G)$ is dense in X_1 we also get $\int_G u\bar{v}\, dG = 0$ for $u \in X_1$ and $v \in X_2$. As X_1 is closed in $L_{2,H}(G)$, the subspace $X_2 = L_{2,H}(G) \ominus X_1$ is also closed in $L_{2,H}(G)$ ∎

5. Applications

Now we want to apply our operator calculus to the Dirichlet problem and the Stokes problem.

Theorem 5.1: *The Dirichlet problem for outer domains*

$$-\Delta u = f \ in\ G \ \left(f \in L^{[m]}_{2,H}(G),\ m > 1\right), \quad \mathrm{tr}\, u = g \ on\ \Gamma \ \left(g \in W^{3/2}_{2,H}(\Gamma)\right)$$

has the unique solution

$$u = -F_\Gamma P_\Gamma g + T_G F_\Gamma(\mathrm{tr}T_G F_\Gamma)^{-1}Q_\Gamma g + T_G\mathbf{Q}T_G f \in W^{2,[l]}_{2,H}(G),\ l < -1.$$

Proof: We consider the sequences ($l < -1$)

$$W_{2,H}^{3/2}(\Gamma)\cap \operatorname{im} Q_\Gamma \xrightarrow{(\operatorname{tr} T_G F_\Gamma)^{-1}} W_{2,H}^{1/2}(\Gamma)\cap \operatorname{im} P_\Gamma \xrightarrow{F_\Gamma} W_{2,H}^{1}(\Gamma) \xrightarrow{T_G} W_{2,H}^{2,[l]}(G),$$

$$L_{2,H}^{[m]}(G) \xrightarrow{T_G} W_{2,H}^{1}(\Gamma) \xrightarrow{\mathbf{Q}} W_{2,H}^{1}(\Gamma) \xrightarrow{T_G} W_{2,H}^{2,[l]}(G),$$

$$W_{2,H}^{3/2}(\Gamma)\cap \operatorname{im} P_\Gamma \xrightarrow{F_\Gamma} W_{2,H}^{2}(G)$$

(cf. Section 3), i.e. $u \in W_{2,H}^{2,[l]}(G)$. Furthermore,

$$-\Delta u = DDu = D\left\{D(-F_\Gamma P_\Gamma g) + DT_G F_\Gamma(\operatorname{tr} T_G F_\Gamma)^{-1} Q_\Gamma g + DT_G \mathbf{Q} T_G f\right\}$$

$$= DF_\Gamma(\operatorname{tr} T_G F_\Gamma)^{-1} Q_\Gamma g + D\mathbf{Q} T_G f = DT_G f - D\mathbf{P} T_G f = f$$

and

$$\operatorname{tr}\left\{-F_\Gamma P_\Gamma g + T_G F_\Gamma(\operatorname{tr} T_G F_\Gamma)^{-1} Q_\Gamma g + T_G \mathbf{Q} T_G f\right\} = P_\Gamma g + Q_\Gamma g = \operatorname{tr} g,$$

because of $\mathbf{Q} T_G f \in D\left(\overset{\circ}{W}{}_{2,H}^{1,[l]}(G)\right)$ and so $\operatorname{tr} T_G \mathbf{Q} T_G f = 0$ holds ∎

Now we consider the Stokes problem for outer domains

$$-\Delta u + (1/\eta)\operatorname{grad} p = (\rho/\eta) f \quad \text{in } G,\ \operatorname{div} u = 0 \ \text{ in } G,\ u = 0 \text{ on } \Gamma$$

and also the problem

$$u + 1/\eta\, T_G \mathbf{Q} T_G p = \rho/\eta\, T_G \mathbf{Q} T_G f \ \text{ in } G,\ \ 1/\eta \operatorname{Re} \mathbf{Q} p = \rho/\eta \operatorname{Re} \mathbf{Q} T_G f \ \text{ in } G, \tag{5.1}$$

with $u = (0,u_1,u_2,u_3)$, $p = (p_0,0,0,0)$, $f = (0,f_1,f_2,f_3)$, $f \in L_{2,H}^{[m]}(G)$, $m > 1$. Let Γ be a compact C^∞-surface; ρ, η shall be certain physical positive constants.

Remark 5.1: Our proofs of the Stokes problem are based on the methods given in [1] for bounded domains.

Remark 5.2: It is possible to show in an analogous way to [1] that the problem (5.1) is equivalent to the weak Stokes' problem

$$\sum_{i=1}^{3}\left(\operatorname{grad} u_i, \operatorname{grad} v_i\right)_{L_2} = \rho/\eta\,(f,v)_{L_2} \text{ for all } \tilde{v} \in V, \tag{5.2}$$

with $\tilde{u} = (0,u_1,u_2,u_3)$, $\tilde{v} = (0,v_1,v_2,v_3)$ and $V = \left\{\tilde{u} \in W_{2,H}^{2,[l]}(G),\ l < -1,\ \operatorname{div}\tilde{u} = 0\right\}$ in the following sense: If u, p is a solution of (5.1), then with $\tilde{u} = \operatorname{Im} u$ equation (5.2) is fulfilled. If $\tilde{u}$ is a solution of (5.2), then there is a real function p such that $\tilde{u}, p$ is a solution of (5.1).

Lemma 5.1: *If* $u \in \overset{\circ}{W}{}_{2,H}^{1,[l]}(G) \cap \ker\operatorname{div}(G)$, $l < -1$, $p \in L_2(G)$, *then* $\operatorname{Re}(Du, \mathbf{Q}p)_{L_{2,H}(G)} = 0$.

Proof: We get $\operatorname{Re}(Du,\mathbf{Q}p)_{L_{2,H}(G)} = \operatorname{Re}(Du, p - \mathbf{P}p)_{L_{2,H}(G)} = \operatorname{Re}(Du, p)_{L_{2,H}(G)} = \sum_{i=1}^{3}\int_G \partial_i u_i p\, dG = 0$ ∎

Theorem 5.2 (a priori estimation): *If* $u \in \overset{\circ}{W}{}_{2,H}^{[l]}(G) \cap \ker\operatorname{div}(G)$, $l < -1$, $p \in L_2(G)$, *then*

$$\left\{1 + \frac{1}{2(|l|-1)}\right\}^{-1} \|u\|_{W_{2,H}^{1,[l]}}^2 + \frac{1}{\eta^2}\|\mathbf{Q}p\|_{L_{2,H}}^2 \le \frac{\rho^2}{\eta^2}\|T_G f\|_{L_{2,H}}^2 .$$

Proof: From (5.1) we get $Du + 1/\eta\, DT_G \mathbf{Q} T_G p = \rho/\eta\, DT_G \mathbf{Q} T_G f$, i.e.

$$Du + {}^1\!/_\eta \mathbf{Q} T_G p = {}^\rho\!/_\eta \mathbf{Q} T_G f \text{ and } \|Du + {}^1\!/_\eta \mathbf{Q} p\|^2_{L_{2,H}} = \|{}^\rho\!/_\eta \mathbf{Q} T_G f\|^2_{L_{2,H}}.$$

We now consider $(u,v)_{L_{2,H,\mathrm{Re}}} = \mathrm{Re}(u,v)_{L_{2,H}}$. Then we get $\|u\|^2_{L_{2,H,\mathrm{Re}}} = \|u\|^2_{L_{2,H}}$ because of the definition. For that reason we have

$$\begin{aligned}\|Du + {}^1\!/_\eta \mathbf{Q} p\|^2_{L_{2,H}} &= \|Du + {}^1\!/_\eta \mathbf{Q} p\|^2_{L_{2,H,\mathrm{Re}}} = \mathrm{Re}(Du + {}^1\!/_\eta \mathbf{Q} p, Du + {}^1\!/_\eta \mathbf{Q} p)_{L_{2,H}} \\ &= \mathrm{Re}(Du,Du)_{L_{2,H}} + ({}^1\!/_\eta)^2 \mathrm{Re}(\mathbf{Q}p,\mathbf{Q}p)_{L_{2,H}} + {}^2\!/_\eta \mathrm{Re}(Du,\mathbf{Q}p)_{L_{2,H}} \\ &= \|Du\|^2_{L_{2,H}} + ({}^1\!/_\eta)^2 \|\mathbf{Q}p\|^2_{L_{2,H}} = ({}^\rho\!/_\eta)^2 \|\mathbf{Q} T_G f\|^2_{L_{2,H}} \le ({}^\rho\!/_\eta)^2 \|T_G f\|^2_{L_{2,H}}.\end{aligned}$$

Because of $u \in \overset{\circ}{W}{}^{1,[l]}_{2,H}(G)$ and Lemma 1.2 it follows

$$\begin{aligned}\|Du\|^2_{L_{2,H}} &= \sum_{j=1}^{3} \int_G \sum_{i=1}^{3} |\partial_i u_j|^2 dG \\ &\ge \sum_{j=1}^{3} \left\{1 + \frac{1}{2(|l|-1)}\right\}^{-1} \|u_j\|^2_{W^{1,[l]}_{2,H}} = \left\{1 + \frac{1}{2(|l|-1)}\right\}^{-1} \|u\|^2_{W^{1,[l]}_{2,H}}.\end{aligned}$$

This leads to the desired inequality ■

Lemma 5.2: *Let* $\tilde{L}_{2,H}(G) = \{u \in L_{2,H}(G) : \mathrm{Re}\,u = u e_0 = 0\}$ *with the inner product* $(u,v)_{\tilde{L}_{2,H}} = \mathrm{Re}(u,v)_{L_{2,H}} = \sum_{i=1}^{3} \int_G u_i v_i \, dG$. *Then we have the decomposition*

$$\begin{aligned}\tilde{L}_{2,H}(G) &= \mathrm{grad}\, W_2^{1,[\delta]}(G) \cap \ker\Delta \oplus_{\tilde{L}_2} D\big(\overset{\circ}{W}{}^{1,[l]}_{2,H}(G) \cap \ker \mathrm{div}\,\mathrm{Im}(G)\big), \\ &= \qquad H^{\perp} \qquad \oplus_{\tilde{L}_2} \qquad H\end{aligned}$$

$(\delta < -1)$, *where* $\oplus_{\tilde{L}_2}$ *denotes the orthogonal sum with the inner product* $(\cdot,\cdot)_{\tilde{L}_{2,H}}$.

Proof: By the means of Theorem 4.3 we obtain

$$L_{2,H}(G) \cap \tilde{L}_{2,H}(G) = \ker D \cap \tilde{L}_{2,H}(G) \oplus D\big(\overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)\big) \cap \tilde{L}_{2,H}(G) = X_1 \oplus X_2,$$

where $\oplus$ is the orthogonal sum with the inner product of $L_{2,H}$. Thus for every element $u \in \tilde{L}_{2,H}(G)$ there exist $u_1 \in X_1$ and $u_2 \in X_2$ such that $u = u_1 + u_2$ and $(u_1,u_2)_{L_{2,H}} = 0$ and so we have $\mathrm{Re}(u_1,u_2)_{L_{2,H}} = (u_1,u_2)_{\tilde{L}_{2,H}} = 0$. Now we want to describe the subsets X_1 and X_2. We have $X_1 = \{u \in \tilde{L}_{2,H}(G) : Du = 0\}$, i.e. $\mathrm{curl}\,u = 0$ and $\mathrm{div}\,u = 0$ and thus $u = \mathrm{grad}\,q$, $q \in W^{1,[\delta]}_{2,H}(G)$, and $D(\mathrm{grad}\,q) = DDq = -\Delta q = 0$ and $X_1 = H^{\perp}$. Besides we have $X_2 = \{u \in \tilde{L}_{2,H}(G) : u = Dw,\ w \in \overset{\circ}{W}{}^{1,[\delta]}_{2,H}(G)\}$, i.e. $\mathrm{Re}\,u = \mathrm{Re}\,Dw = \mathrm{div}(\mathrm{Im}\,u) = 0$ and $X_2 = H$ ■

Theorem 5.3: *There exist a unique solution* $u \in \overset{\circ}{W}{}^{1,[l]}_{2,H}(G) \cap \ker\mathrm{div}(G)$, $l < -1$, $p \in L_2(G)$ *of* (5.1).

Proof: We consider the system

$$\begin{aligned}&\mathrm{Re}(Du, {}^1\!/_\eta \mathbf{Q} T_G h)_{L_{2,H}} = 0 \text{ for all } u \in \overset{\circ}{W}{}^{1,[l]}_{2,H}(G) \cap \ker\mathrm{div}(G), \\ &\mathrm{Re}(\mathbf{Q}p, {}^1\!/_\eta \mathbf{Q} T_G h)_{L_{2,H}} = 0 \text{ for all } p \in L_{2,H}(G),\ \mathrm{Im}\,p = 0,\end{aligned} \tag{5.3}$$

where $h \in L_{2,H}^{[m]}(G)$, $m > 1$, $h = (0, h_1, h_2, h_3)$. Hence as

$$\mathrm{Re}(Du, {}^1\!/_\eta \mathbf{Q} T_G h)_{L_{2,H}} = \mathrm{Re}(Du, {}^1\!/_\eta \mathrm{Im}\,\mathbf{Q} T_G h)_{L_{2,H}} = \mathrm{Re}(Du, {}^1\!/_\eta \mathrm{Im}\,\mathbf{Q} T_G h)_{L_{2,H}}$$

and $Du \in H$ we obtain

$$\mathrm{Im}\,\mathbf{Q} T_G h \in H^\perp, \mathrm{Im}\,\mathbf{Q} T_G h = \mathrm{grad}\, q,\ q \in W_2^{1,[8]}(G),\ \Delta q = 0.$$

Furthermore, we have

$$\mathrm{Re}(\mathbf{Q}p, {}^1\!/_\eta \mathbf{Q} T_G h)_{L_{2,H}} = \mathrm{Re}(p, {}^1\!/_\eta \mathbf{Q} T_G)_{L_{2,H}} = {}^1\!/_\eta \int_G p \,\mathrm{Re}(\mathbf{Q} T_G h)\, dG = 0 \ \forall\, p \in L_2(G)$$

and we obtain with the Lemma of Du Bois Reymond $\mathrm{Re}(\mathbf{Q} T_G h) = 0$. This means $h = D\mathbf{Q} T_G h = D(\mathrm{Im}\,\mathbf{Q} T_G h) = D(\mathrm{grad}\, q) = -\Delta q = 0$. So we obtain the existence of a solution of

$$Du + {}^1\!/_\eta \mathbf{Q} p = {}^\rho\!/_\eta \mathbf{Q} T_G f \ \left(u \in \overset{\circ}{W}{}_{2,H}^{1,[l]}(G) \cap \ker \mathrm{div}(G),\ l < -1,\ p \in L_2(G)\right) \forall\, f \in L_{2,H}^{[m]}(G).$$

Now we get from (5.3) $T_G Du + {}^1\!/_\eta T_G \mathbf{Q} p = {}^\rho\!/_\eta T_G \mathbf{Q} T_G f$ and because of $u \in \overset{\circ}{W}{}_{2,H}^{1,[l]}(G)$ we obtain $u + {}^1\!/_\eta T_G \mathbf{Q} p = {}^\rho\!/_\eta T_G \mathbf{Q} T_G f$. Furthermore, the solution of (5.1) is uniquely determined. This immediately follows from Theorem 5.2 and the fact that, if $D(p_1 - p_2) = 0$ and $p_1, p_2 \in L_2(G)$, we get $p_1 = p_2$ ∎

Remark 5.3 : We also want to mention the interesting papers [7] and [8].

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