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Regularity of Solutions of the Weak Floating Beam Problem

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This paper describes a weak formulation of the time-harmonic two-dimensional floating beam problem in a fluid domain of finite depth. This is a simplified version of the floating body problem which was investigated by F. John in his classic papers [9,10] in 1950. Contrary to the integral equation approach of F. John we use a Hilbert-space method based on the investigation of a not necessarily positive definite sesquilinear form. Interior as well as boundary regularity are the main concerns of this paper. Especially we show that the solutions lie in a weighted H^2-space in the neighbourhood of the endpoints of the beam.

Key words: Weighted regularity, floating beam problem

AMS subject classification: 35 D 10, 35 J 25, 76 B 99

1. Introduction and statement of the problem

1.1. We denote by $p = (x, y)$ the elements in the two-dimensional Euclidean space $\mathbf{R}^2$. Let $\Omega_S := \mathbf{R}\times]-h,0[$ be the undisturbed fluid domain with the bottom surface $S_B := \mathbf{R} \times \{-h\}$. Furthermore, we define for a smooth convex bounded domain F in $\{y > -h\}$ with $\Omega_S \cap F \neq \emptyset$ the disturbed (unbounded) fluid domain Ω by

$$\Omega = \Omega_S \setminus \overline{F}.$$

We call

$$S_I = \overline{\Omega} \cap \partial F$$

the immersed *ship hull* and

$$S_F = \partial\Omega \setminus (\overline{S_I} \cup S_B)$$

the (unbounded) *free fluid surface*. Finally, write

$$\{p_1, p_2\} = \partial S_I.$$

A basic problem in linear hydrodynamics consists in determining the velocity potential of an inviscid incompressible stationary fluid flow in Ω. It is formulated as follows (cf. F. John [9,10], M. Simon/F. Ursell [18]):

Problem A. *(Classical formulation of the floating body problem.) Find all $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that*

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u_n = \begin{cases} 0 & \text{on } S_B \\ f & \text{on } S_I \\ \lambda u & \text{on } S_F, \end{cases}$$

where u_n denotes the outer normal derivative of u to the domain Ω, f is a given function on the ship hull S_I and $\lambda \in \mathbf{C}$ is the wave-number with $\operatorname{Im}(\lambda) \geq 0$ (cf. Subsection 1.3).

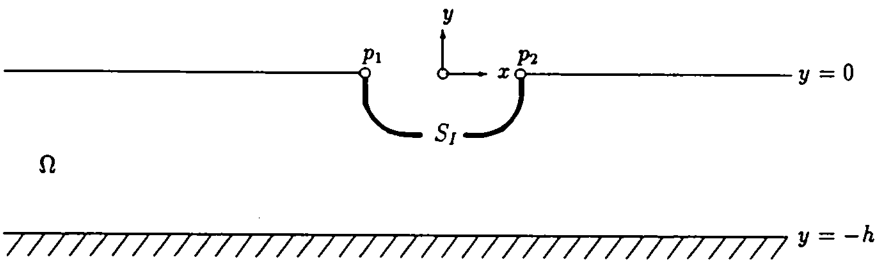

Fig. 1

1.2. In his paper F. John [10], p.50, expressed the idea that the discovery of a variational formulation of Problem A could facilitate the existence proof and also permit the construction of approximate expressions for the solutions. Therefore in K. Doppel [1], K. Doppel/G. C. Hsiao [2] the following weak formulation of Problem A was given. Consider the sesquilinear form

$$(u,v)_E = \int_\Omega \nabla u \cdot \nabla \bar{v} dp + \int_{S_F} u\bar{v} ds$$

on the Sobolev space $H^1(\Omega)$ where ds denotes the line element on S_F. It can be shown that $(\cdot,\cdot)_E$ is a well-defined inner product on $H^1(\Omega)$ which is equivalent to the usual one $(\cdot,\cdot)_1$ (cf. [2], Section 3, [3] and [6]). Then $a_\lambda(\cdot,\cdot)$ with

$$a_\lambda(u,v) = \int_\Omega \nabla u \cdot \nabla \bar{v} dp - \lambda \int_{S_F} u\bar{v} ds \tag{1.1}$$

is a continuous sesquilinear form on $H^1(\Omega)$. Furthermore, for $f \in L^2(S_I)$ define (cf. Section 4 in [2]) the bounded anti-linear form l_f on $H^1(\Omega)$ by

$$l_f(v) = \int_{S_I} f\bar{v} ds. \tag{1.2}$$

Problem B. *(Hilbert space formulation of the floating body problem.) For given $f \in L^2(S_I)$ and $\lambda \in \mathbf{C}$ find all $u \in H^1(\Omega)$ (weak solutions of Problem A) such that $a_\lambda(u,v) = l_f(v)$ holds for all $v \in H^1(\Omega)$.*

Take $\mathbf{R}_0^+ = [0,\infty[$ and note that Problem B is uniquely solvable for $\lambda \in \mathbf{C} \setminus \mathbf{R}_0^+$ (cf. K. Doppel/G. C. Hsiao [2], Theorem 12). In the case $\lambda = 0$ one has to restrict the functions f to be in the smaller space $L^2_{\text{ad}}(\Omega)$ and to replace $H^1(\Omega)$ by $F^1_E(\Omega)$ which is obtained by the completion of the Schwartz space $\mathcal{S}(\Omega)$ (cf. M. Schechter [17]) with respect to the norm $|\phi|_1 = (\int_\Omega |\nabla\phi|^2 dp)^{1/2}$ (for details cf. K. Doppel/G. C. Hsiao [2], Theorem 13).

1.3. In this paper we shall study the regularity of weak solutions of Problem B. Since the corners p_1, p_2 are extremely unpleasant we assume in this note as a first step that the unbounded fluid domain Ω has a smooth boundary. To do so we reformulate Problem A in the following way and call this the floating beam problem. Let Ω_S be defined as before and fix an open bounded interval $]p_1,p_2[\subset \mathbf{R}\times\{0\}$

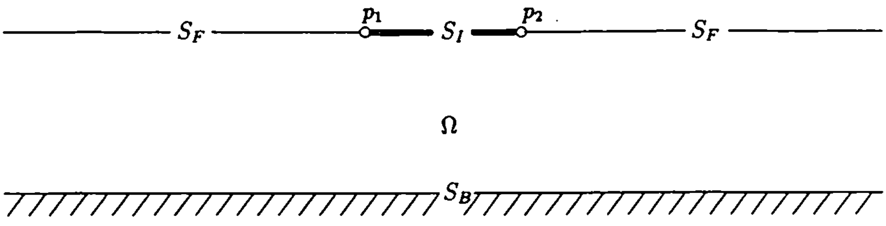

Fig. 2

and call it the floating *beam*. If $S_I =]p_1, p_2[$, then $\Omega = \Omega_S$ and Problem A reduces to

Problem A′. *Find all $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that*

$$(1.3) \qquad \Delta u = 0 \quad \text{in } \Omega, \qquad u_n = \begin{cases} 0 & \text{on } S_B \\ f & \text{on } S_I =]p_1, p_2[\\ \lambda u & \text{on } S_F, \end{cases}$$

where now f is a given function on the floating beam S_I.

Analogously to Problem B we pose

Problem C. *(Hilbert space formulation of the floating beam problem.) For given $f \in L^2(S_I)$, $S_I =]p_1, p_2[$ and $\lambda \in \mathbf{C}$ find all $u \in H^1(\Omega)$ (weak solutions of Problem A′) such that*

$$(1.4) \qquad a_\lambda(u, v) = l_f(v)$$

holds for all $v \in H^1(\Omega)$.

1.4. In Section 2 we will investigate the regularity of weak solutions of Problem C away from the corners p_1, p_2. Especially, using a *bootstrap argument* we will show (cf. Theorem 2.9):

The solutions u of Problem C are of class C^∞ away from the floating beam.

Furthermore we have (cf. Lemma 2.14 and Theorem 2.15)

The solutions u of Problem C lie in $H^{3/2}(\Omega \cap B)$ for each open bounded $B \subset \mathbf{R}^2$. Especially the restrictions $u|_{\partial\Omega}$ lie in $H^1_{\text{loc}}(\partial\Omega)$.

1.5. In Section 3 we will study the weighted H^2-regularity of solutions of Problem C in the corners p_1, p_2. We remark that in this field the pioneering work was done by V. A. Kondrat'ev [12] in the sixtees. He has attracted the interest of quite a number of authors to study the singularities of weak solutions of (homogeneous) mixed boundary value problems at the boundary, especially at corners (P. Grisvard [7], A. Kufner/A.-M. Sãndig [14], W. L. Wendland et al. [20], V. G. Maz'ja/J.

Rossmann [16], B. Kawohl [11], J. Weisel [19]). As far as we know inhomogeneous mixed boundary value problems, especially of the Robin–Neumann type as Problem C, have not been investigated in the literature.

To be more precise, define the weighted Sobolev space $H^2(\Omega;\rho)$ as follows. Let α be a multi-index, i.e. $\alpha = (\alpha_1,\alpha_2), \alpha_i \in \mathbf{N}_0$, $|\alpha| = \alpha_1 + \alpha_2$. For a nonnegative (positive almost everywhere) measurable function ρ on Ω let us denote by $H^2(\Omega;\rho)$ the space of all functions $u \in H^1(\Omega)$ such that $\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u(p)|^2 \rho(p)dp < \infty$.

Now take the special weight function ρ_ϵ, $\epsilon > 0$, defined by

$$\rho_\epsilon(p) = \operatorname{dist}(p, \{p_1, p_2\})^{2+\epsilon} |\phi(p)|^2, \tag{1.5}$$

where $\phi \in C_0^\infty(\mathbf{R}^2)$ is an arbitrary but fixed test function such that $\{p_1,p_2\} \subset \operatorname{supp} \phi$. Then we will use the results stated in Subsection 1.4 to prove the following weighted H^2-regularity (cf. Section 3):

If $f \in H^{3/2}(S_I)$, then each solution of Problem C lies in $H^2(\Omega;\rho_\epsilon)$ for all $\epsilon > 0$.

2. Regularity away from the corners

2.1. First we note that it was shown in K. Doppel/G. C. Hsiao [2], Theorem 9, that Weyl's Lemma implies the following interior regularity result.

Lemma 2.1. *Each weak solution of Problem C lies in $C^\infty(\Omega)$.*

2.2. Let us fix some notations. Let $\mathbf{R}^2_- = \{p = (x,y) |\ x \in \mathbf{R}, y < 0\}$ be the lower halfspace and $\Gamma = \{(x,y) \mid y = 0\}$ be its boundary. For a point $p_0 \in \Gamma$ and $R > 0$ we define

$$B(p_0;R) = \{p \in \mathbf{R}^2 \mid |p - p_0| < R\}, \quad B^-(p_0;R) = B(p_0;R) \cap \mathbf{R}^2_-$$

and

$$\Gamma(p_0;R) = \Gamma \cap B(p_0;R).$$

Furthermore, let

$$X_R(p_0) := \{v \in H^1(B^-(p_0;R)) \mid \operatorname{dist}(\operatorname{supp} v, \partial B(p_0;R)) > 0\}.$$

Note that $X_R(p_0) \cap C^\infty(\mathbf{R}^2)$ lies dense in $X_R(p_0)$ with respect to $\|\cdot\|_1$. In the following we need the well-known trace theorem.

Lemma 2.2. *Let $G \subset \mathbf{R}^2$ be a bounded domain with boundary ∂G of class C^2. Then for each $s \in \mathbf{N}$ there exists a linear bounded trace operator $T_0 : H^s(G) \to H^{s-\frac{1}{2}}(\partial G)$ which is onto and fulfils $T_0\phi = \phi|_{\partial G}$ for all $\phi \in C^\infty(\overline{G})$. Furthermore, if $s > 1$ there is a linear bounded surjection $T_1 : H^s(G) \to H^{s-\frac{3}{2}}(\partial G)$ with $T_1\phi = \phi_n|_{\partial G}$ for all $\phi \in C^\infty(\overline{G})$.*

For the proof see J. Wloka [21], Theorems 8.7 and 8.8.

2.3. The following lemmas hold.

Lemma 2.3. *Let $R > 0$, $u \in H^1(B^-(p_0; R))$ and $f \in H^s(B^-(p_0; R))$, $s \in \mathbf{N}_0$, such that*

$$\int_{B^-(p_0;R)} \nabla u \cdot \nabla \bar{v} dp = \int_{B^-(p_0;R)} f \bar{v} dp \quad \forall v \in X_R(p_0) \cap C^\infty(\mathbf{R}^2).$$

Then we have

$$u \in H^{s+2}(B^-(p_0; R')) \quad \forall R' < R.$$

For the proof cf. G. Folland [5], Theorem 7.29.

Lemma 2.4. *Let G' be a bounded domain of class C^2 in $\mathbf{R}^2_-$ such that $\Gamma(p_0; R) \subset \partial G'$ for a fixed $R > 0$ (cf. Fig. 3). Let $u \in H^1(G')$ and $\psi \in H^s(\Gamma(p_0; R))$ with $s = k + 1/2$, $k \in \mathbf{N}_0$, such that*

$$\int_{G'} \nabla u \cdot \nabla \bar{v} dp = \int_{\partial G'} \psi \bar{v} ds \quad \forall v \in X_R(p_0) \cap C^\infty(\mathbf{R}^2). \tag{2.1}$$

Then we have $u \in H^{k+2}(B^-(p_0; R'))$ and therefore $T_0 u \in H^{k+3/2}(\Gamma(p_0; R'))$ for all $0 < R' < R$. Especially, if $\psi \in C^\infty(\overline{\Gamma(p_0; R)})$, then $u \in C^\infty(B^-(p_0; R))$.

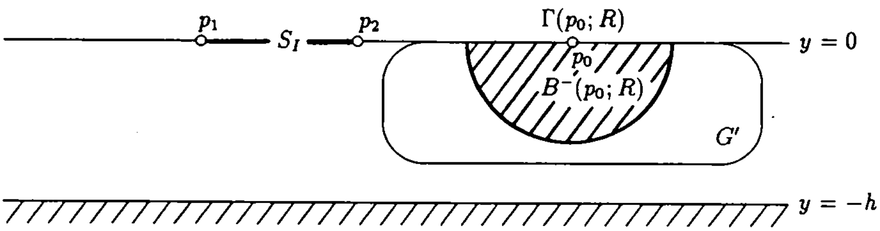

Fig. 3

Proof. Fix R' with $0 < R' < R$ and numbers R_1, R_2 such that $R > R_1 > R_2 > R'$. Next choose a cut-off function $\zeta \in C^\infty(\partial G')$ such that $0 \le \zeta \le 1$ and

$$\zeta = \begin{cases} 1 & \text{on } \Gamma(p_0; R_1) \\ 0 & \text{on } \partial G' \setminus \Gamma(p_0; R). \end{cases}$$

Then $\zeta\psi$ lies in $H^s(\partial G')$ and by Lemma 2.2 we can find a function $\Psi \in H^{k+2}(G')$ with $T_1(\Psi) = \zeta\psi$. For $\tilde{u} := u - \Psi$ we obtain by partial integration from (2.1)

$$\begin{aligned} \int_{G'} \nabla \tilde{u} \cdot \nabla \bar{v} dp &= \int_{\partial G'} \psi \bar{v} ds - \int_{G'} \nabla \Psi \cdot \nabla \bar{v} dp \\ &= \int_{\partial G'} \psi \bar{v} ds + \int_{G'} \Delta \Psi \bar{v} dp - \int_{\partial G'} T_1(\Psi) \bar{v} ds \\ &= \int_{\partial G'} (1 - \zeta) \psi \bar{v} ds + \int_{G'} \Delta \Psi \bar{v} dp \\ &= \int_{G'} \Delta \Psi \bar{v} dp \end{aligned}$$

for all $v \in X_{R_2}(p_0) \cap C^\infty(\mathbf{R}^2)$. Since $\Delta\Psi \in H^k(G')$ we obtain by Lemma 2.3 $\tilde{u} \in H^{k+2}(B^-(p_0; R'))$ and therefore $u = \tilde{u} + \Psi \in H^{k+2}(B^-(p_0; R'))$ ∎

As an immediate consequence we obtain

Lemma 2.5. *If $u \in H^{k+2}(B^-(p_0; R'))$ for all $R' < R$, then $T_0 u \in H^{k+3/2}(\Gamma(p_0; R'))$ holds for all $R' < R$.*

Proof. Fix R', $0 < R' < R$, choose $R_1 \in]R', R[$ and a domain G of class C^2 such that $B^-(p_0; R') \subset G \subset B^-(p_0; R_1)$ and $\Gamma(p_0; R') \subset \partial G$. Then $u \in H^{k+2}(G)$ and by Lemma 2.2 we get $T_0 u \in H^{k+3/2}(\partial G) \subset H^{k+3/2}(\Gamma(p_0; R'))$ ∎

2.4. A direct consequence of Lemma 2.4 is the regularity of the weak solutions on the bottom surface S_B and on the floating beam S_I.

Lemma 2.6. *For each solution u of Problem C we have $u \in C^\infty(\Omega \cup S_B)$.*

Proof. Take $p_0 \in S_B$ and choose $R \in]0, h[$. Using the change of coordinates $p \mapsto -p - (0, h)$ we can apply Lemma 2.4 with $\psi = 0$ and obtain $u \in C^\infty(\overline{B(p_0; R) \cap \Omega})$. Since this is valid for all $p_0 \in S_B$ we obtain with Lemma 2.1 $u \in C^\infty(\Omega \cup S_B)$ ∎

On the other hand, we have

Lemma 2.7. *For $f \in H^{3/2}(S_I)$ take a solution u of Problem C and a point $p_0 \in S_I$. Then $u \in H^3(B^-(p_0; R))$ for all $R > 0$ with $\operatorname{dist}(B(p_0; R), \partial\Omega \setminus S_I) > 0$.*

Proof. Take $p_0 \in S_I$ and $R > 0$ as above. Choose $\epsilon > 0$ such that $\operatorname{dist}(B(p_0; R+\epsilon), \partial\Omega \setminus S_I) > 0$, and obtain the assertion by applying Lemma 2.4 to $\psi := f \in H^{3/2}(\Gamma(p_0; R + \epsilon))$ ∎

2.5. As a consequence of Lemma 2.4 and Lemma 2.5 we obtain the regularity of the weak solutions of Problem C on the free surface S_F:

Lemma 2.8. *Let u be a solution of Problem C. Then $u \in C^\infty(\Omega \cup S_F)$.*

Proof. Fix $p_0 \in S_F$ and take $R > 0$ such that $\Gamma(p_0; R) \subset S_F$. Now choose a domain $G \subset \Omega$ (cf. Fig. 3) with

$$(i)\ \ \Gamma(p_0; R) \subset \partial G, \qquad (ii)\ \ \operatorname{dist}(\partial G, S_I) > 0.$$

Then (1.4) reduces to

$$\int_G \nabla u \cdot \nabla \bar{v}\, dp = \lambda \int_{\partial G} u\bar{v}\, ds \quad \forall v \in X_R(p_0) \cap C^\infty(\mathbf{R}^2) (\subset H^1(\Omega)). \tag{2.2}$$

First note that Lemma 2.2 implies $\lambda T_0 u \in H^{1/2}(\Gamma(p_0; R))$, and in view of (2.2) Lemma 2.4 gives $u \in H^2(B^-(p_0; R'))$. Suppose now that we have $u \in H^{k'+2}(B^-(p_0; R'))$ for a $k' \in \mathbf{N}_0$ and all $R' < R$. Then Lemma 2.5 shows $\lambda T_0 u \in H^{k'+3/2}(\Gamma(p_0; R'))$ for all $R' < R$. Applying Lemma 2.5 again we obtain $u \in$

$H^{k'+3}(B^-(p_0;R'))$ for all $R' < R$. Therefore we obtain by induction on k

$$u \in H^{k+2}(B^-(p_0;R')) \quad \forall k \in \mathbf{N}, \forall R' < R.$$

Sobolev's embedding theorem (cf. J. Wloka [21], Theorem 6.2) now shows $u \in C^\infty(\overline{B^-(p_0;R')})$ for all $R' < R$. Since p_0 was arbitrary the assertion follows ∎

2.6. As a consequence of Lemma 2.1, Lemma 2.6 and Lemma 2.8 we have regularity away from the beam:

Theorem 2.9. *Each weak solution of Problem C lies in* $C^\infty(\Omega \cup S_F \cup S_B)$.

2.7. For our main regularity result for the solution of Problem C we need the local H^1-regularity on $\partial\Omega$, i.e. $\phi u \in H^1(\partial\Omega)$ for all $\phi \in C_0^\infty(\partial\Omega)$. In view of Theorem 2.9 it is sufficient to prove $u \in H^1(\partial G)$, where $G \subset \Omega$ is an arbitrary but fixed bounded domain with C^2-boundary such that

$$S_I \subset [p_1 - (\epsilon, 0), p_2 + (\epsilon, 0)] \subset \partial G$$

for a suitable $\epsilon > 0$ (see Fig. 4) and to show $u \in H^1(G)$.

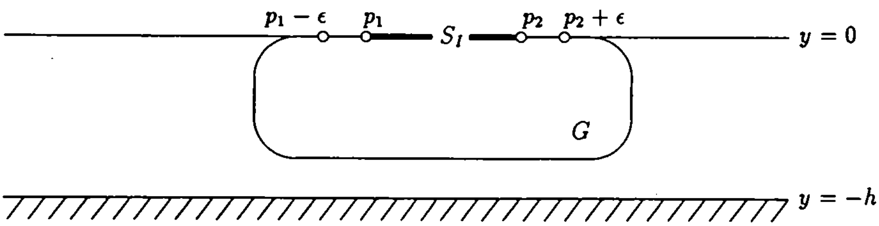

Fig. 4

To this end we consider the Dirichlet form $D : H^1(G) \times H^1(G) \to \mathbf{C}$, defined by

$$D(u,v) = \int_G \nabla u \cdot \nabla \bar{v} dp + \int_{\partial G} u\bar{v} ds$$

and denote by $< \cdot, \cdot >$ the duality bracket between $H^{-1/2}(\partial G)$ and $H^{1/2}(\partial G)$. Because of the fact that D is $H^1(G)$-elliptic by the Friedrichs-Poincaré inequality (cf. J. Wloka [21], Theorem 2.7) and the Lax-Milgram theorem we have :

Lemma 2.10. *(i) For each* $\psi \in H^{-1/2}(\partial G)$ *there exists exactly one* $u \in H^1(G)$ *such that*

$$D(u,v) = < \psi, v >$$

holds for all $v \in H^1(G)$.

(ii) There exists a linear bounded (solution) operator $\mathcal{L} : H^{-1/2}(\partial G) \to H^1(G)$ *such that*

$$D(\mathcal{L}(\psi), v) = < \psi, v >$$

holds for all $\psi \in H^{-1/2}(\partial G)$ *and for all* $v \in H^1(G)$.

Furthermore we have (cf. K. Doppel/B. Schomburg [4], Theorem 11)

Lemma 2.11. *Let* $\psi \in H^{1/2}(\partial G)$, $u \in H^1(G)$ *such that*

$$\int_G \nabla u \cdot \nabla \bar{v} dp = \int_{\partial G} \psi \bar{v} ds \quad \forall v \in C^\infty(\overline{G}). \tag{2.3}$$

Then we have $u \in H^2(G)$.

Proof. For the given $\psi \in H^{1/2}(\partial G)$ there exists a $\Psi \in H^2(G)$ with $T_1\Psi = \psi$ by Lemma 2.2. For $F := u - \Psi \in H^1(G)$ we obtain by (2.3)

$$\int_G \nabla F \cdot \nabla \bar{v} dp = \int_{\partial G} \psi \bar{v} ds - \int_G \nabla \psi \cdot \nabla \bar{v} dp \quad \forall v \in C^\infty(\overline{G}).$$

Partial integration of the last integral gives

$$\int_G \nabla F \cdot \nabla \bar{v} dp = \int_G \Delta \Psi \bar{v} dp \quad \forall v \in C^\infty(\overline{G}).$$

Since $\Delta\Psi \in L^2(G)$ the classical regularity theory (cf. J. L. Lions/E. Magenes [15],Ch.2) shows that $F \in H^2(G)$ and therefore $u = F + \Psi \in H^2(G)$ ∎

Lemma 2.12. *The solution operator* $\mathcal{L}$ *maps* $H^{1/2}(\partial G)$ *continuously into* $H^2(G)$.

Proof. By the closed graph theorem it suffices to show $\mathcal{L}(H^{1/2}(\partial G)) \subset H^2(G)$. Let $\psi \in H^{1/2}(\partial G)$. By Lemma 2.10 $u := \mathcal{L}(\psi)$ fulfils

$$\int_G \nabla u \cdot \nabla \bar{v} dp = \int_{\partial G} \tilde{\psi} \bar{v} ds \quad \forall v \in H^1(G)$$

where $\tilde{\psi} := \psi - T_0 u$ lies in $H^{1/2}(\partial G)$ by the Lemma 2.2. Lemma 2.11 gives the assertion ∎

Lemma 2.13. The operator $\mathcal{L}$ maps $L^2(\partial G)$ into $H^{3/2}(G)$, i.e. $\mathcal{L}(L^2(\partial G)) \subset H^{3/2}(G)$.

Proof. By interpolation theory (cf. J. L. Lions/E. Magenes [15], Theorem 7.7 (p.36) and Theorem 9.6 (p.43)) we have for the Sobolev spaces $H^s(\partial G), s \in \mathbf{R}$, and $H^s(G), s > 0$,

$$[H^{s_1}(\partial G), H^{s_2}(\partial G)]_{1/2} = H^{s_1+s_2/2}(\partial G), \quad s_1 > s_2, \tag{2.4}$$

$$[H^{s_1}(G), H^{s_2}(G)]_{1/2} = H^{s_1+s_2/2}(G), \quad s_1 > s_2 > 0. \tag{2.5}$$

On the other hand Lemma 2.11 and Lemma 2.12 imply

$$\mathcal{L}([H^{1/2}(\partial G), H^{-1/2}(\partial G)]_{1/2}) \subset [H^2(G), H^1(G)]_{1/2},$$

so the assertion follows from (2.4) and (2.5), respectively ∎

Lemma 2.14. *Let $\psi \in H^{1/2}(\partial G)$, $u \in H^1(G)$ be such that*

$$\int_G \nabla u \cdot \nabla \bar{v} dp = \int_{\partial G} \psi \bar{v} ds \quad \forall v \in C^\infty(\overline{G}). \tag{2.6}$$

Then we have $u \in H^{3/2}(G)$ and furthermore $T_0 u \in H^1(\partial G)$.

Proof. For the given $\psi \in L^2(\partial G)$ and a solution u of (2.6) we get

$$D(u,v) = \int_{\partial G} (\psi + T_0 u)\bar{v} ds \quad \forall v \in H^1(G).$$

Since by Lemma 2.2 $\psi + T_0 u \in L^2(\partial G)$ we get by Lemma 2.10 $\mathcal{L}(\psi + T_0 u) = u$, so Lemma 2.13 implies $u \in H^{3/2}(G)$. To continue we get by the properties of the trace operator T_0 (cf. Lemma 2.2)

$$T_0([H^2(G), H^1(G)]_{1/2}) \subset [H^{3/2}(\partial G), H^{1/2}(\partial G)]_{1/2}$$

and by (2.4) and (2.5) the second part of the assertion ∎

We are able to prove the announced $H^1_{\text{loc}}(\partial\Omega)$-regularity of weak solutions of Problem C.

Theorem 2.15. For each solution $u \in H^1(\Omega)$ of Problem C we have $u|_{\partial\Omega} \in H^1_{\text{loc}}(\partial\Omega)$.

Proof. Take the domain G as described at the beginning of Section 2.7. Let u be a solution of Problem C. Because of Theorem 2.9 it is sufficient to prove $u|_{\partial G} \in H^1(\partial G)$. Set $\tilde{u} = u|_G (\in H^1(G))$ and define a function $\psi : \partial G \to \mathbf{C}$ by

$$\psi = \begin{cases} f & \text{on } S_I \\ \lambda u & \text{on } S_F \cap \partial G \\ u_n|_{\partial G} & \text{on } \partial G \setminus (S_F \cup S_I). \end{cases} \tag{2.7}$$

It is clear by Lemma 2.2 and Theorem 2.9 that $\psi \in L^2(\partial G)$. Now take a partition of unity $\{\phi_1, \phi_2\} \subset C^\infty(\overline{G})$ with $\phi_1(p) = 1$ for all $p \in O$, where O is an open neighbourhood of $\partial G \setminus (S_F \cup S_I)$, and $\operatorname{supp} \phi_1 \cap S_I = \emptyset$. Let $v \in C^\infty(\overline{G})$, define $v_j = \phi_j v$, $j = 1, 2$ and get by Theorem 2.9

$$\begin{aligned} \int_G \nabla \tilde{u} \cdot \nabla \overline{v_1} dp &= \int_{\operatorname{supp} \phi_1} (-\Delta \tilde{u}) \overline{v_1} dp + \int_{\operatorname{supp} \phi_1 \cap \partial G} \tilde{u}_n \overline{v_1} ds \\ &= \int_{\partial G} \psi \overline{v_1} ds, \end{aligned} \tag{2.8}$$

where the last equality follows from (1.3) and (2.7). On the other hand

$$\int_G \nabla \tilde{u} \cdot \nabla \overline{v_2} dp = \int_G \nabla u \cdot \nabla \overline{(\chi_{\overline{G}} v_2)} dp,$$

where $\chi_{\overline{G}}$ denotes the characteristic function of $\overline{G}$. For the last integral we have by (1.1), (1.2) and (1.4)

$$\begin{aligned} \int_G \nabla u \cdot \nabla \overline{(\chi_{\overline{G}} v_2)} dp &= \lambda \int_{S_F} u \, \overline{v_2 \chi_{\overline{G}}} ds + \int_{S_I} f \, \overline{v_2 \chi_{\overline{G}}} ds \\ &= \int_{\partial G} \psi \overline{v_2} ds, \end{aligned} \tag{2.9}$$

where the last identity follows again by (2.7). Adding (2.8) and (2.9) we obtain

$$\int_G \nabla\tilde{u}\cdot\nabla\overline{v}dp = \int_{\partial G}\psi\overline{v}ds \quad \forall v \in C^\infty(\overline{G}).$$

Since $\psi \in L^2(\partial G)$ we can apply Lemma 2.14 and conclude that $u|_{\partial G} = \tilde{u}|_{\partial G} \in H^1(\partial G)$ ∎

3. Weighted regularity in the corners

3.1. In this Section we study the regularity in the points p_1, p_2. We restrict our attention to p_1 and assume without loss of generality p_1 to be the origin of the $\mathbf{R}^2$. Define in the lower halfspace $\mathbf{R}^2_-$ the bounded domain

$$G_{a,b} = \{p \in \mathbf{R}^2_- \mid a < |p| < b\},$$

where $0 < a < b$. If we take a function $\phi \in C^\infty(\overline{G_{a,b}})$ with $\operatorname{supp}\phi \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ we obtain by partial integration

$$2\int_{G_{a,b}} |\phi_{xy}|^2dp = \int_{G_{a,b}}(\phi_{xx}\overline{\phi_{yy}} + \phi_{yy}\overline{\phi_{xx}})dp + \int_{\partial G_{a,b}}(\phi_t\overline{\phi_{nt}} - \phi_n\overline{\phi_{tt}})ds,$$

where ϕ_t denotes the tangential derivative along $\partial G_{a,b}$ of ϕ and ϕ_n the outer normal derivative of ϕ on $\partial G_{a,b}$. Another partial integration leads to

Lemma 3.1. *For all $\phi \in C^\infty(\overline{G_{a,b}})$ with* supp $\phi \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ *we have*

$$2\int_{G_{a,b}} |\phi_{xy}|^2dp = \int_{G_{a,b}}(\phi_{xx}\overline{\phi_{yy}} + \phi_{yy}\overline{\phi_{xx}})dp + 2\mathrm{Re}\int_{\partial G_{a,b}}\phi_t\overline{\phi_{nt}}ds.$$

If we use the Sobolev seminorms given by

$$|\phi|^2_{j,G_{a,b}} := \sum_{|\alpha|=j}\int_{G_{a,b}}|\partial^\alpha\phi|^2dp, \quad j = 1,2,$$

we can rewrite Lemma 3.1 in the following form.

Lemma 3.2. *For all $\phi \in C^\infty(\overline{G_{a,b}})$ with* supp $\phi \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ *we have*

$$|\phi|^2_{2,G_{a,b}} \le \|\Delta\phi\|^2_{0,G_{a,b}} + |\phi|^2_{1,\partial G_{a,b}} + |\phi_n|^2_{1,\partial G_{a,b}}.$$

Proof. Since by definition $|\phi|^2_{2,G_{a,b}} = |\phi_{xx}|^2_{0,G_{a,b}} + |\phi_{xy}|^2_{0,G_{a,b}} + |\phi_{yy}|^2_{0,G_{a,b}}$ Lemma 3.1 implies

$$\begin{aligned}|\phi|^2_{2,G_{a,b}} &\le \int_{G_{a,b}}(|\phi_{xx}|^2 + |\phi_{yy}|^2 + \phi_{xx}\overline{\phi_{yy}} + \phi_{yy}\overline{\phi_{xx}})dp + 2\mathrm{Re}\int_{\partial G_{a,b}}\phi_t\overline{\phi_{nt}}ds \\ &= \int_{G_{a,b}}|\Delta\phi|^2dp + 2\mathrm{Re}\int_{\partial G_{a,b}}\phi_t\overline{\phi_{nt}}ds.\end{aligned}$$

The Cauchy–Schwarz inequality gives

$$|\phi|^2_{2,G_{a,b}} \le \|\Delta\phi\|^2_{0,G_{a,b}} + 2\|\phi_t\|_{0,\partial G_{a,b}}\|\phi_{nt}\|_{0,\partial G_{a,b}}$$

and the assertion follows ■

We transfer the situation of Lemma 3.2 to Sobolev space functions.

Lemma 3.3. *For all* $v \in H^3(G_{a,b})$ *with* supp $v \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ *we have*

$$|v|^2_{2,G_{a,b}} \le \|\Delta v\|^2_{0,G_{a,b}} + |v|^2_{1,\partial G_{a,b}} + |v_n|^2_{1,\partial G_{a,b}}.$$

Proof. Fix $v \in H^3(G_{a,b})$ with supp $v \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$. Then there exists a sequence $(\phi_k) \subset C^\infty(\overline{G_{a,b}})$ such that $\|v - \phi_k\|_3 \to 0 \quad (k \to \infty)$. Now take $\eta \in C_0^\infty(\{p \in \mathbf{R}^2 \mid a < |p| < b\})$ with $\eta|_{\text{supp } v} = 1$ and define $\tilde{\phi}_k = \eta\phi_k,\ k \in \mathbf{N}$. For these functions we have $\tilde{\phi}_k \in C^\infty(\overline{G_{a,b}})$, supp $\tilde{\phi}_k \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ and

$$\|v - \tilde{\phi}_k\|_3 = \|\eta v - \eta\phi_k\|_3 = \|\eta(v - \phi_k)\|_3 \le c(\eta)\|v - \phi_k\|_3 \to 0.$$

But this implies

$$\begin{array}{llll} (i) & |v - \tilde{\phi}_k|_2 \le \|v - \tilde{\phi}_k\|_3 & \to 0 & (k \to \infty) \\ (ii) & \|\Delta v - \Delta\tilde{\phi}_k\|_0 \le 2\|v - \tilde{\phi}_k\|_3 & \to 0 & (k \to \infty) \end{array}$$

and by the trace theorem (using the support properties of v and $\tilde{\phi}_k$ in $\{p \in \mathbf{R}^2 \mid a < |p| < b\}$)

$$\begin{array}{llll} (iii) & |v - \tilde{\phi}_k|_{1,\partial G_{a,b}} & \to 0 & (k \to \infty) \\ (iv) & |v_n - (\tilde{\phi}_k)_n|_{1,\partial G_{a,b}} & \to 0 & (k \to \infty). \end{array}$$

Lemma 3.2 and a density argument now give the assertion ■

As a consequence of the preceeding lemmas we obtain the following a priori estimate.

Theorem 3.4. *Let* $\rho > 0$. *Then there exists a constant* $c = c(\rho) > 0$ *such that*

$$|v|^2_{2,G_{\rho,2\rho}} \le c\left(\|\Delta v\|^2_{0,G_{\rho/2,4\rho}} + \|v\|^2_{1,G_{\rho/2,4\rho}} + \|v\|^2_{1,\partial G_{\rho/2,4\rho}} + \|v_n\|^2_{1,\partial G_{\rho/2,4\rho}}\right)$$

holds for all $v \in H^3(G_{\rho/2,4\rho})$.

Proof. Fix $v \in H^3(G_{\rho/2,4\rho})$. Let $\xi \in C_0^\infty(]\rho/2, 4\rho[)$ such that $\xi = 1$ on $[\rho, 2\rho]$. We define $\eta = \xi(|\cdot|)$ and $\tilde{v} = \eta v$. Obviously, $\tilde{v} \in H^3(G_{a,b})$ and supp $\tilde{v} \subset \{p \in \mathbf{R}^2 \mid a < |p| < b\}$ with $a = \rho/2, b = 4\rho$. Thus we can apply Lemma 3.3 and obtain

$$|\tilde{v}|^2_{2,G_{\rho/2,4\rho}} \le \|\Delta\tilde{v}\|^2_{0,G_{\rho/2,4\rho}} + |\tilde{v}|^2_{1,\partial G_{\rho/2,4\rho}} + |\tilde{v}_n|^2_{1,\partial G_{\rho/2,4\rho}}. \tag{3.1}$$

Because of $|v|^2_{2,G_{\rho,2\rho}} \le |\eta v|^2_{2,G_{\rho/2,4\rho}} = |\tilde{v}|^2_{2,G_{\rho/2,4\rho}}$ we get

$$|v|^2_{2,G_{\rho,2\rho}} \le \|\Delta\tilde{v}\|^2_{0,G_{\rho/2,4\rho}} + |\tilde{v}|^2_{1,\partial G_{\rho/2,4\rho}} + |\tilde{v}_n|^2_{1,\partial G_{\rho/2,4\rho}}.$$

On the other hand we have $\Delta\tilde{v} = \eta\Delta v + 2\nabla\eta \cdot \nabla v + \Delta\eta v$, and therefore for the right-hand side of (3.1)

$$\begin{aligned} \|\Delta\tilde{v}\|_{0,G_{\rho/2,4\rho}} &\le \|\eta\Delta v\|_{0,G_{\rho/2,4\rho}} + 2\|\nabla\eta \cdot \nabla v\|_{0,G_{\rho/2,4\rho}} + \|\Delta\eta v\|_{0,G_{\rho/2,4\rho}} \\ &\le c(\eta)(\|\Delta v\|_{0,G_{\rho/2,4\rho}} + \|v\|_{1,G_{\rho/2,4\rho}}). \end{aligned}$$

Furthermore,

$$\begin{aligned}|\tilde{v}|_{1,\partial G_{\rho/2,4\rho}} &= \|(\eta v)_t\|_{0,\partial G_{\rho/2,4\rho}} = \|\eta v_t + \eta v_t\|_{0,\partial G_{\rho/2,4\rho}} \\ &\leq \|\eta_t v\|_{0,\partial G_{\rho/2,4\rho}} + \|\eta v_t\|_{0,\partial G_{\rho/2,4\rho}} \leq c(\eta)\|v\|_{1,\partial G_{\rho/2,4\rho}}\end{aligned}$$

and analogously

$$|\tilde{v}_n|_{1,\partial G_{\rho/2,4\rho}} \leq c(\eta)\|v_n\|_{1,\partial G_{\rho/2,4\rho}}.$$

Putting altogether the assertion follows ■

3.2. We now apply Theorem 3.4 for a special domain. To this end we set for $k \in \mathbf{Z}$ (cf. Fig. 5)

$$G_k = G_{2^{-k},2^{-k+1}}, \quad \hat{G}_k = G_{2^{-k-1},2^{-k+2}} = G_{k-1} \cup \overline{G_k} \cup G_{k+1}$$

and finally

$$I_k = [2^{-k}, 2^{-k+1}] \cup [-2^{-k+1}, -2^{-k}], \quad \hat{I}_k = [2^{-k-1}, 2^{-k+2}] \cup [-2^{-k+2}, -2^{-k-1}].$$

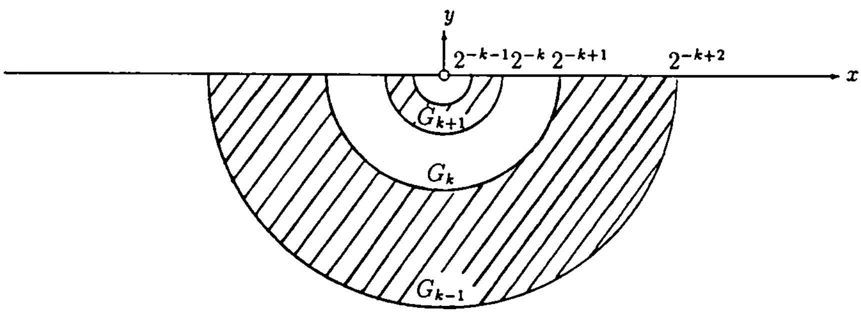

Fig. 5

According to Theorem 3.4 we have for $\rho := 1$ the following

Corollary 3.5. *There exists a constant $c > 0$ such that*

$$|v|^2_{2,G_0} \leq c\left(\|\Delta v\|^2_{0,\hat{G}_0} + \|v\|^2_{1,\hat{G}_0} + \|v\|^2_{1,\hat{I}_0} + \|v_n\|^2_{1,\hat{I}_0}\right)$$

holds for all $v \in H^3(\hat{G}_0)$.

3.3. We are going to apply Corollary 3.5 to a solution u of Problem C for sufficiently smooth f. To be precise, we will assume throughout this Section that $f \in H^{3/2}(S_I)$. Remember that $h > 0$ is the finite depth of the fluid domain. Let be $\xi \in C^\infty(]0,\infty[)$ a function such that

$$\begin{aligned}&(i) && 0 \leq \xi(r) \leq 1 && \forall r \in\,]0,\infty[,\\ &(ii) && \xi(r) = 0 && \forall r > r_1 := \max(|p_2|, h)/2,\\ &(iii) && \xi(r) = 1 && \forall r \in\,]0, r_0[\end{aligned}$$

for a certain $r_0, 0 < r_0 < r_1$. Now take a solution u ($\in H^1(\Omega)$) of Problem C and define

$$U = \phi u \in H^1(\mathbf{R}^2_-), \tag{3.2}$$

where $\phi := \xi(|\cdot|) \in C_0^\infty(\mathbf{R}^2)$. Then U is a weak solution of the system

$$\Delta U = u\Delta\phi + 2\nabla u \cdot \nabla\phi =: \Phi \quad \text{in } \mathbf{R}^2_- \tag{3.3}$$

$$U_n = \begin{cases} \phi f =: F & \text{on } \mathbf{R}^+ \\ \lambda U & \text{on } \mathbf{R}^-. \end{cases} \tag{3.4}$$

Note that $\Phi \in L^2(\mathbf{R}^2_-)$ with $\Phi(p) = 0$ for $|p| > r_0$, $F \in H^1(\mathbf{R}^+)$ with $F(x) = 0$ for $x > r_1$ (cf. Fig. 6).

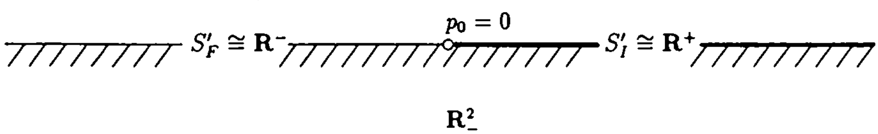

Fig. 6

Furthermore, note that by Lemma 2.7 and Theorem 2.9

$$U \in H^3(G_k) \quad \forall k \in \mathbf{Z}, \quad U \in H^3(\hat{G}_k) \quad \forall k \in \mathbf{Z}.$$

For the function U, introduced above, we now show the following

Lemma 3.6. *There is a constant $c' > 0$ such that*

$$\begin{aligned}(3.5)\quad \sum_{|\alpha|=2} \int_{\mathbf{R}^2_-} |p|^{2+\epsilon}|\partial^\alpha U(p)|^2 dp \;\le\; c' \Big(& \int_{\mathbf{R}^2_-} |p|^{2+\epsilon}|\Delta U(p)|^2 dp \\
+ & \int_{\mathbf{R}^2_-} |p|^{\epsilon}|\nabla U(p)|^2 dp + \int_{\mathbf{R}^2_-} |p|^{\epsilon-2}|U(p)|^2 dp \\
+ & \int_{\mathbf{R}^-} |p|^{\epsilon+1}|U_t(p)|^2 dx + \int_{\mathbf{R}^+} |p|^{\epsilon+1}|U_t(p)|^2 dx \\
+ & \int_{\mathbf{R}^-} |p|^{\epsilon-1}|U(p)|^2 dx + \int_{\mathbf{R}^+} |p|^{\epsilon-1}|U(p)|^2 dx \\
+ & \int_{\mathbf{R}^-} |p|^{\epsilon+3}|U_{nt}(p)|^2 dx + \int_{\mathbf{R}^+} |p|^{\epsilon+3}|U_{nt}(p)|^2 dx \\
+ & \int_{\mathbf{R}^-} |p|^{\epsilon+1}|U_n(p)|^2 dx + \int_{\mathbf{R}^+} |p|^{\epsilon+1}|U_n(p)|^2 dx \Big)\end{aligned}$$

holds for all $\epsilon \in\,]0, 1[$.

Proof. Fix $k \in \mathbf{Z}$, set $\sigma = 2^{-k}$ and consider $v : \hat{G}_0 \to \mathbf{C}, p' \mapsto U(\sigma p')$. Then $v \in H^3(\hat{G}_0)$ and by Corollary 3.5 we have

$$|v|^2_{2,G_0} \le c\left(\|\Delta v\|^2_{0,\hat{G}_0} + \|v\|^2_{1,\hat{G}_0} + \|v\|^2_{1,I_0} + \|v_n\|^2_{1,I_0}\right). \tag{3.6}$$

Applying the identity $\partial^\alpha v(p') = \sigma^{|\alpha|}\partial^\alpha U(\sigma p')$ for all $\alpha \in \mathbf{N}_0^2$ to (3.6) we obtain

$$\begin{aligned}
&\sum_{|\alpha|=2}\int_{G_0}\sigma^4|\partial^\alpha U(\sigma p')|^2dp' \\
&\leq c \Big(\int_{\hat G_0}\sigma^4|\Delta U(\sigma p')|^2dp' + \int_{\hat G_0}\sigma^2|\nabla U(\sigma p')|^2dp' \\
&\quad + \int_{\hat G_0}|U(\sigma p')|^2dp' + \int_{\hat I_0}\sigma^2|U_t(\sigma p')|^2dx' + \int_{\hat I_0}|U(\sigma p')|^2dx' \\
&\quad + \int_{\hat I_0}\sigma^4|U_{nt}(\sigma p')|^2dx' + \int_{\hat I_0}\sigma^2|U_n(\sigma p')|^2dx'\Big).
\end{aligned}$$

Using the dilation $p' \mapsto \sigma p' = p$ we transform the integrals over $G_0, \hat G_0, \hat I_0$ to integrals over $G_k, \hat G_k, \hat I_k$ and obtain

$$\begin{aligned}
&\sum_{|\alpha|=2}\int_{G_k}\sigma^2|\partial^\alpha U(p)|^2dp \\
&\leq c \Big(\int_{\hat G_k}\sigma^2|\Delta U(p)|^2dp + \int_{\hat G_k}|\nabla U(p)|^2dp \\
&\quad + \int_{\hat G_k}\sigma^{-2}|U(p)|^2dp + \int_{\hat I_k}\sigma|U_t(p)|^2dx + \int_{\hat I_k}\sigma^{-1}|U(p)|^2dx \\
&\quad + \int_{\hat I_k}\sigma^3|U_{nt}(p)|^2dx + \int_{\hat I_k}\sigma|U_n(p)|^2dx\Big).
\end{aligned}$$

Multiplying the above inequality with σ^ϵ, $\epsilon \in]0,1[$ and using

$$0 \leq |p| \leq 2\sigma \quad \forall p \in G_k, \qquad \sigma/2 \leq |p| \leq 4\sigma \quad \forall p \in \hat G_k$$

we conclude

$$\begin{aligned}
(3.7)\quad &\sum_{|\alpha|=2}\int_{G_k}|p|^{2+\epsilon}|\partial^\alpha U(p)|^2dp \\
&\leq c \Big(\int_{\hat G_k}|p|^{2+\epsilon}|\Delta U(p)|^2dp + \int_{\hat G_k}|p|^\epsilon|\nabla U(p)|^2dp \\
&\quad + \int_{\hat G_k}|p|^{\epsilon-2}|U(p)|^2dp + \int_{\hat I_k}|p|^{\epsilon+1}|U_t(p)|^2dx + \int_{\hat I_k}|p|^{\epsilon-1}|U(p)|^2dx \\
&\quad + \int_{\hat I_k}|p|^{\epsilon+3}|U_{nt}(p)|^2dx + \int_{\hat I_k}|p|^{\epsilon+1}|U_n(p)|^2dx\Big).
\end{aligned}$$

with a generic constant c independent of ϵ. Since each point $p \in \mathbf{R}^2_-$ lies in at most three ring sectors $\hat G_k$ we have

$$\sum_{k\in\mathbf{Z}}\int_{\hat G_k}\tau(p)dp = 3\sum_{k\in\mathbf{Z}}\int_{G_k}\tau(p)dp = 3\int_{\mathbf{R}^2_-}\tau(p)dp$$

and

$$\sum_{k\in\mathbf{Z}}\int_{\hat I_k}\tau(p)dx = 3\sum_{k\in\mathbf{Z}}\int_{I_k}\tau(p)dx = 3\Big(\int_{\mathbf{R}^+}\tau(p)dx + \int_{\mathbf{R}^-}\tau(p)dx\Big)$$

for all $\tau \in L^1(\mathbf{R}^2_-) \cap L^1(\mathbf{R}^-) \cap L^1(\mathbf{R}^+)$. Especially, summation over $k \in \mathbf{Z}$ in (3.7) gives the assertion ∎

Next we show

Theorem 3.7. For all $\epsilon \in]0,1[$ we have

$$\sum_{|\alpha|=2} \int_{\mathbf{R}^2_-} |p|^{2+\epsilon} |\partial^\alpha U(p)|^2 dp < +\infty.$$

Proof. First note that by (3.2) $U(p) = 0$ for all $p, |p| \geq h$ and by (3.3) $\Phi = \Delta U(p) = 0$ for all $p, |p| \geq h$. Since $U \in H^1(\mathbf{R}^2_-)$ and $\Phi \in L^2(\mathbf{R}^2_-)$ we conclude that the volume integrals on the right-hand side of (3.5) with positive exponent of $|p|$ are bounded. Analogously we see by (3.4)

$$U_n(p) = 0, \quad p = (x,0), |x| \geq h.$$

Since $F \in H^1(\mathbf{R}^+)$ by assumption and $U \in H^1(\mathbf{R}^-)$ by Theorem 2.15 we see again that all line integrals on the right-hand side of (3.5) with positive exponent of $|p|$ are bounded.

So it remains to show that integrals in (3.5) where the exponents of $|p|$ are negative are finite. The boundary integrals can be estimated with the help of the famous Hardy inequality (cf. G. H. Hardy et al. [8], A. Kufner [13])

$$\int_{\mathbf{R}^+} |p|^{\epsilon-1} |U(p)|^2 dx \leq (2/\epsilon)^2 \int_{\mathbf{R}^+} |p|^{\epsilon+1} |U_t(p)|^2 dx < \infty,$$

where the last conclusion again follows from Theorem 2.15. Analogously,

$$\int_{\mathbf{R}^-} |p|^{\epsilon-1} |U(p)|^2 dx < \infty.$$

Finally, the remaining integral can be estimated by a generalization of Hardy's inequality for domains of arbitrary dimension by V. A. Kondrat'ev (cf. [12], Lemma 4.9). Since $U \in H^1(\mathbf{R}^2_-)$ vanishes for all $p, |p| > h$ there exists for each $\epsilon \in]0,1[$ a constant c such that

$$\int_{\mathbf{R}^2_-} |p|^{\epsilon-2} |U(p)|^2 dp \leq c \|U\|_{1,\mathbf{R}^2_-} < \infty$$

and the assertion is completely proved ∎

Corollary 3.8. *Take a test function $\phi \in C_0^\infty(\mathbf{R}^2)$ such that $p_2 \notin \operatorname{supp} \phi$. Then for each solution u of Problem C*

$$\sum_{|\alpha|=2} \int_\Omega |p|^{2+\epsilon} |\phi(p)|^2 |\partial^\alpha u(p)|^2 dp < +\infty$$

holds for all $\epsilon \in]0,1[$.

Proof. Without loss of generality we can assume that supp ϕ is contained in a ball $B(0;M)$ such that $M < \min(h, |p_2|)/4$. Then choose a radial symmetric test function $\xi(|\cdot|)$ such that $\xi(|p|) = 1$ for all $p \in B(0;M)$ and supp $\xi(|\cdot|) \subset B(0;2M)$. By Theorem 3.7 we know that $|\cdot|^{1+\epsilon/2} \partial^\alpha(\xi(|\cdot|)u) \in L^2(\mathbf{R}^2_-)$ for all $\alpha, |\alpha| = 2$. In view of the fact that $u \in H^1(\Omega)$ the last statement is equivalent to

$$|\cdot|^{1+\epsilon/2} \xi(|\cdot|) \partial^\alpha u \in L^2(\mathbf{R}^2_-) \quad \forall \alpha \in \mathbf{N}_0^2, |\alpha| = 2.$$

Now define the function $\tilde{\phi}$ by

$$\tilde{\phi} = \begin{cases} \phi/\xi(|\cdot|) & \text{on } B(0;M) \\ 0 & \text{elsewhere} \end{cases}$$

and note that $\tilde{\phi} \in C_0^\infty(\mathbf{R}^2)$. Thus

$$|\cdot|^{1+\epsilon/2}\phi(|\cdot|)\partial^\alpha u = |\cdot|^{1+\epsilon/2}\tilde{\phi}\xi(|\cdot|)\partial^\alpha u \in L^2(\mathbf{R}^2_-) \quad \forall \alpha \in \mathbf{N}_0^2, |\alpha| = 2$$

and the assertion follows ∎

3.4. Now we are able to prove the following

Theorem 3.9. *If $f \in H^{3/2}(S_I)$, then each solution of Problem C lies in $H^2(\Omega;\rho_\epsilon)$ for all $\epsilon > 0$, where ρ_ϵ is defined as in (1.5).*

Proof. Without loss of generality we can assume that $\epsilon \in]0,1[$. Now let $\phi \in C_0^\infty(\mathbf{R}^2)$ be an arbitrary test function. If $\operatorname{supp}\phi \cap \{p_1,p_2\} = \emptyset$, then $u \in H^2(\Omega;\rho_\epsilon)$ by Theorem 2.9 and Lemma 2.7. Assume $\operatorname{supp}\phi \cap \{p_1,p_2\} \neq \emptyset$. Let $\{\psi_1,\psi_2\}$ be a partition of unity in $C^\infty(\mathbf{R}^2)$ with $p_1 \notin \operatorname{supp}\psi_2$ and $p_2 \notin \operatorname{supp}\psi_1$. Then there exist constants c_1, c_2 such that

$$\begin{aligned}(3.8)\qquad \int_{\mathbf{R}^2_-} |\phi(p)|^2 \operatorname{dist}(p,\{p_1,p_2\})^{2+\epsilon}|\partial^\alpha u(p)|^2 dp \\ \leq c_1 \int_{\mathbf{R}^2_-} \psi_1|\phi(p)|^2|p-p_1|^{2+\epsilon}|\partial^\alpha u(p)|^2 dp \\ + c_2 \int_{\mathbf{R}^2_-} \psi_2|\phi(p)|^2|p-p_2|^{2+\epsilon}|\partial^\alpha u(p)|^2 dp\end{aligned}$$

for all $\epsilon \in]0,1[$ and $\alpha \in \mathbf{N}_0^2, |\alpha| = 2$. Furthermore, since $\sqrt{\psi_j}\phi \in C_0^\infty(\mathbf{R}^2)$ for $j = 1,2$, Corollary 3.8 gives

$$(3.9)\qquad \sum_{|\alpha|=2} \int_\Omega |p-p_1|^{2+\epsilon}\psi_1(p)|\phi(p)|^2|\partial^\alpha u(p)|^2 dp < +\infty$$

$$(3.10)\qquad \sum_{|\alpha|=2} \int_\Omega |p-p_2|^{2+\epsilon}\psi_2(p)|\phi(p)|^2|\partial^\alpha u(p)|^2 dp < +\infty$$

for all $\epsilon \in]0,1[$. Combining (3.8),(3.9) and (3.10) we obtain the assertion ∎

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