

Semianalytic Discretization of Weakly Nonlinear Boundary Value Problems with Variable Coefficients

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A new technique for the iterative treatment of two-point boundary value problems by means of piecewise simplified operators is investigated. Unlike in previous approaches the main part of the differential operator is also used in the projected right hand side of the recursion of the iteration process. We show the iterates recursively to be piecewise smooth enough to prove the contractivity of the mapping of the iteration. We state the local convergence of the discretization with respect to the step size of the underlying grid.

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1. Introduction

A simple but powerful principle for the numerical solution of boundary value problems consists in an appropriate approximation of the operator involved and an exact solution of the generated approximate problem. This approach is especially advantageous in the case of singularly perturbed problems (see, e.g., [3], [5]) or for generating uniform bounds of the unknown solution (see, e.g., [1], [4], [7]). On the other hand this approach is restricted to simple types of differential operators because the generated approximate problems have to be solved explicitly. In [1] an attempt has been made to solve the subproblems with a certain accuracy only. The aim of the present paper consists in proposing an iterative treatment of the boundary value problem where the differential operator is piecewise approximated and the iteration uses a polynomial approximation of the resulting difference between the exact and the approximated differential operator. We investigate the convergence behaviour of the proposed method and we discuss finite-dimensional approximations.

It should be remarked, that in difference to other iteration methods, here the main part of the differential operator also occurs on the right-hand side of the iteration map. This complicates the convergence analysis significantly because recursive smoothings have to be considered. On the other hand this paves the way for higher order schemes which was restricted as shown in [1]. Moreover, the divergence type of the originally given differential operator can be preserved in the approximation proposed here.

The approximation of the differential operator rests on a piecewise simplification on subintervals of an underlying grid. Finally, we investigate the convergence of the method with respect to the step size of the used grids. Numerical examples are given at the end of the paper.

Denote $\Omega = (0, 1)$ and let $a \in C^1(\Omega)$ satisfy

$$0 < a_0 \leq a(x) \text{ for any } x \in \Omega, \quad (1.1)$$

with some positive constant a_0 . Furthermore, $f : \bar{\Omega} \times R \rightarrow R$ denotes some given sufficiently

smooth function. Especially, we suppose the function f to be locally Lipschitz continuously differentiable with respect to the second argument. With $f = f(y, t)$ this means

$$\left| \frac{\partial f}{\partial t}(x, r) - \frac{\partial f}{\partial t}(x, s) \right| \leq L(\rho)(|x - z| + |r - s|) \quad \forall x, z \in \Omega, |r|, |s| \leq \rho,$$

with some non-decreasing function $L(\cdot)$.

In the present paper we deal with the weakly non-linear two-point boundary value problem

$$-(a(x)y)'(x) = f(x) \quad \text{in } \Omega \quad \text{and} \quad y(0) = y(1) = 0. \tag{1.2}$$

We remark that the class of differential operators under consideration has a variable coefficient $a(\cdot)$ as opposed to the investigations in [2], [4]. Unlike in [1] we take advantage of the Sturmian type of problem (1.2).

The embedding of the originally given problem (1.2) into the framework of weak formulations is a useful tool for the later investigations. Let $V = H_0^1(\Omega)$ denote the adapted Sobolev space and V^* the related dual one. The dual pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. We use the standard Sobolev norm in $H^1(\Omega)$ as norm in V and we denote this norm by $\|\cdot\|$. The closed ball in V with the radius ρ and the center y is denoted by $B_\rho(y)$.

We define operators $L, F : V \rightarrow V^*$ in the usual way by

$$\langle Ly, v \rangle = \int_{\Omega} a(x)y'(x)v'(x) dx \quad \text{and} \quad \langle Fy, v \rangle = \int_{\Omega} f(x, y(x))v(x) dx \quad \forall y, v \in V.$$

Under the assumptions made problem (1.2) is equivalent to the abstract operator equation

$$Ly = Fy. \tag{1.3}$$

We suppose that problem (1.3) possesses some solution $u \in V$ which is stable in the sense

$$\langle (L - F'(u))y, y \rangle \geq \gamma \|u\|^2 \quad \text{for any } y \in V, \tag{1.4}$$

with some $\gamma > 0$.

2. Iterative treatment with a simplified operator

In this chapter we deal with the iterative solution of the operator equation (1.3) by a technique which is close to Newton's method. The standard linearization of the non-linear operator F as well as the linear operator L are locally simplified on a grid $\{x_i\}_{i=0}^N$ on the interval Ω . The grid on Ω has the property

$$0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1.$$

Denote $h_i = x_i - x_{i-1}$, $\Omega_i = (x_{i-1}, x_i)$, $i = 1(1)N$ and $h = \max_i h_i$, which defines the step size of the grid. We assume that the grid is quasi-uniform, i.e. $h_i \geq ch$, $i = 1(1)N$ holds. Here and in the sequel we denote by c, c_1, c_2, \dots some positive constants which can be different at different places of occurrence. We define spaces $C_h^k(\bar{\Omega})$ of piecewise k times continuously differentiable functions by

$$C_h^k(\bar{\Omega}) = \{ v : v|_{\Omega_i} \in C^k(\bar{\Omega}_i), i = 1(1)N \}$$

and we denote

$$\|v\|_{C^k(\bar{\Omega}_i)} = \max_{0 \leq j \leq k} \sup_{x \in \bar{\Omega}_i} |v^{(j)}(x)|, \quad \|v\|_{h,k} = \max_{1 \leq i \leq N} \|v\|_{C^k(\bar{\Omega}_i)} \quad \text{for any } v \in C_h^k(\bar{\Omega}).$$

To simplify the notation we also use $\|\cdot\|_h$ instead of $\|\cdot\|_{h,0}$. The subspace of piecewise polynomials of the maximal degree l is denoted by $P_l \subset C_h(\bar{\Omega})$. Over the grid $\{x_i\}_{i=0}^N$ we substitute the operator L by replacing the function $a(\cdot)$ by some $a_h \in P_0$ with $\|a - a_h\|_h \rightarrow 0$ as $h \rightarrow +0$. Because of (1.1) we select a_h such that $a_h > 0$ holds. The operator L_h related to a_h is defined by

$$\langle L_h y, v \rangle = \int_{\Omega} a_h(x) y'(x) v'(x) dx \quad \text{for any } y, v \in V. \tag{2.1}$$

This leads to the estimation $\|L - L_h\| \leq \|a - a_h\|_h$ in the sense of the operator norm related to $L, L_h : V \rightarrow V^*$.

Additionally we simplify the linearization of the operator F by replacing its derivative $-F'(y)$ by some $D_h(y)$ according to

$$\langle D_h(y)z, v \rangle = \int_{\Omega} d_h[y](x) z(x) v(x) dx \quad \text{for any } z, v \in V,$$

with $d_h[y] \in P_0$ given by

$$d_h[y] = \frac{1}{2} \left[\frac{\partial f}{\partial y}(x_{i-1}, y(x_{i-1})) + \frac{\partial f}{\partial y}(x_i, y(x_i)) \right], \quad x \in \Omega_i, \quad i = 1(1)N. \tag{2.2}$$

Similar to [4] we have

Lemma 1 *There exists some function $s : R^+ \times R^+ \rightarrow R^+$ such that*

$$\|F'(y) + D_h(y)\| \leq s(\rho, h) \quad \text{for any } y \in B_\rho(0)$$

and s satisfies

$$\lim_{h \rightarrow +0} s(\rho, h) = 0 \quad \text{for any } \rho > 0.$$

Proof: Because of $y \in B_\rho(0)$ and the continuous embedding $V \hookrightarrow C(\bar{\Omega})$ we obtain

$$|y(x)| \leq c \|y\| \leq c\rho \quad \text{for any } x \in \bar{\Omega},$$

with some $c > 0$. Using the Lipschitz continuity of $\partial f / \partial t$ this results in

$$\left| \frac{\partial f}{\partial y}(x, y(x)) - \frac{\partial f}{\partial y}(x_i, y(x_i)) \right| \leq L(c\rho) (|x - x_i| + |y(x) - y(x_i)|).$$

With the continuous embedding $V \hookrightarrow C^{0,1/2}(\bar{\Omega})$ we have the estimation $|y(x) - y(x_i)| \leq c h_i^{1/2}$. We remember that c stands for a constant which may be different at different places of occurrence. With the Hölder continuity of y the estimation

$$\left| \frac{\partial f}{\partial y}(x, y(x)) - \frac{\partial f}{\partial y}(x_i, y(x_i)) \right| \leq L(c\rho) (h_i + c\rho h_i^{1/2})$$

is obtained. With a similar estimation concerning the left end point x_{i-1} of the subinterval Ω_i and with the definition (2.2) of $d_h[y] \in P_0$ this results in

$$\left\| \frac{\partial f}{\partial y}(\cdot, y(\cdot)) - d_h[y] \right\|_h \leq L(c\rho)(h + c\rho h^{1/2}).$$

This proves the assertion of the lemma ■

Now we define an iterative technique for solving the operator equation (1.3) by a simplified Newton's method via the recursion

$$(L_h + D_h(y^k))y^{k+1} = (F + D_h(y^k))y^k + (L_h - L)y^k, \quad k = 0, 1, \dots \tag{2.3}$$

with some initial guess $y^0 \in V$. Because of Lemma 1 and of $\lim_{h \rightarrow +0} \|a - a_h\|_h = 0$ indeed (2.3) forms an approximated version of Newton's method. By standard arguments for proving the convergence of perturbed Newton's techniques (compare, e.g., [6]) this results in

Theorem 1 *Let $u \in V$ denote some solution of problem (1.3) that is stable according to (1.4). Then some $\bar{h} > 0$ and some $\delta > 0$ exist such that method (2.3) for any $h \in (0, \bar{h}]$ and $y^0 \in B_\delta(u)$ generates a well-defined sequence $\{y^k\}$ converging to u . More precisely, we have some constants $c_1, c_2 > 0$ such that*

$$\|y^{k+1} - u\| \leq (c_1 \|y^k - u\| + c_2 h) \|y^k - u\|, \quad k = 0, 1, \dots \tag{2.4}$$

Remarks: 1. The inequalities (2.4) show the method (2.3) to possess an asymptotic superlinear convergence for $h \rightarrow +0$.

2. Method (2.3) is not implementable in the strict sense because it cannot be realized using available finite-dimensional spaces with dimensions bounded from above as a rule. In the next chapter we modify method (2.3) to make it implementable. Unlike in earlier investigations (see [1], [4], [5]) in this paper we approximate the main part of the differential operator also. This forms an attempt to extend the technique of simplified operators to more general problems.

3. Finite dimensionally projected iterations

Now, we modify algorithm (2.3) by appropriate projections of the right-hand side of the recursion. First we separate the related operator equation into differential equations on the subintervals Ω_i and into matching conditions for the one-sided derivatives at the inner grid points $x_i, i = 1(1)N - 1$.

Let us restrict our considerations to the subspace $W = V \cap C_h^2(\bar{\Omega})$, which is dense in V , i.e. $cl_V W = V$. We define operators $Q_h, R_h : V \rightarrow V^*$ by

$$\langle Q_h y, v \rangle = - \sum_{i=1}^N \int_{\Omega_i} ((a_h - a)y')'(x)v(x) dx \quad \forall y \in W, v \in V, \tag{3.1}$$

$$\langle R_h y, v \rangle = \sum_{i=1}^{N-1} \{[(a_h - a)y'](x_i - 0) - [(a_h - a)y'](x_i + 0)\} v(x_i) \quad \forall y \in W, v \in V. \tag{3.2}$$

Via integration by parts we obtain the identity in V^*

$$(L - L_h)y = (Q_h + R_h)y \quad \text{for any } y \in W. \tag{3.3}$$

On the other hand we can identify Q_h with a mapping $Q_h : C_h^2(\bar{\Omega}) \rightarrow C_h(\bar{\Omega})$ which is given by

$$[Q_h y](x) = -((a_h - a)y')'(x) \quad \text{for any } x \in \Omega_i, \quad i = 1(1)N. \tag{3.4}$$

Let us introduce a projection $\pi_h : C_h(\bar{\Omega}) \rightarrow P_1$ by taking

$$[\pi_h w](x) = w(x_{i-1} + 0) \frac{x_i - x}{h_i} + w(x_i - 0) \frac{x - x_{i-1}}{h_i}, \quad x \in \Omega_i, \quad i = 1(1)N. \quad (3.5)$$

Using this projection and identity (3.3) we modify recursion (2.3) in the following way:

$$(L_h + D_h(y^k))y^{k+1} = \pi_h(F + D_h(y^k))y^k + \pi_h Q_h y^k + R_h y^k, \quad k = 0, 1, \dots, \quad (3.6)$$

with some initial function $y^0 \in W$. Unlike (2.3) this method can be realized using known finite-dimensional representations by piecewise exponential splines (compare, e.g., [4], [5]). However, we have to suppose the following additional assumption on the variation of the function $a(\cdot)$ to hold:

$$\max_{1 \leq i \leq N} \frac{\|a'\|_{C(\bar{\Omega}_i)}}{a_i} \leq \beta < 1, \quad (3.7)$$

with some $\beta \in (0, 1)$. According to (3.6) we locally define a mapping $T : W \cap B_\rho(u) \rightarrow V$ by means of the linear boundary value problem

$$(L_h + D_h(y))Ty = \pi_h(F + D_h(y))y + \pi_h Q_h y + R_h y \quad (3.8)$$

and we show $Ty \in C_h^2(\bar{\Omega})$ as well as some local contraction property in the norm $\|\cdot\|_{h,2}$. Here $u \in V$ denotes a solution of the operator equation (1.3) and $\rho > 0$ denotes some sufficiently small constant. Owing to Lemma 1 and with the supposed Lipschitz continuity of $\partial f / \partial y$ we can select some $\rho > 0$ and some $\bar{h} > 0$ such that

$$\langle (L_h + D_h(y))v, v \rangle \geq \bar{\gamma} \|v\|^2 \quad \text{for any } v \in B_\rho(u), \quad h \in (0, \bar{h}] \quad (3.9)$$

holds with some $\bar{\gamma} > 0$. Because of (3.8), (3.9) the operator T is well defined on $W \cap B_\rho(u)$. Before analyzing (3.4) we identify $y \rightarrow \pi_h(F + D_h(y))y + \pi_h Q_h y$ with a mapping $S_h : W \rightarrow P_1 \subset C_h(\bar{\Omega})$ defined by

$$[S_h y](x) = [\pi_h(F + D_h(y))y + \pi_h Q_h y](x) \quad \text{for any } x \in \Omega_i, \quad i = 1(1)N. \quad (3.10)$$

This leads to

$$\|S_h y\|_h \leq \|(F + D_h(y))y\|_h + c \|y\|_{h,2}.$$

On the other hand $\|S_h y\|_* \leq c \|S_h y\|_h$ holds with some constant $c > 0$. Taking $y \in B_\rho(u)$ and the supposed properties of the function f into consideration, thus, we obtain an estimation

$$\|S_h y\|_* \leq c \|y\|_{h,2}. \quad (3.11)$$

According to (3.3) we have

$$\langle (Q_h + R_h)y, v \rangle = \int_{\Omega} (a_h - a)(x) y'(x) v'(x) dx.$$

This results in

$$\|(Q_h + R_h)y\|_* \leq \|a_h - a\|_h \|y\|_{h,1}.$$

Furthermore we have

$$\|Q_h y\|_* \leq \|Q_h y\|_h \leq c \|y\|_{h,2}.$$

This leads to

$$\|R_h y\|_* \leq \|Q_h y\|_* + \|(Q_h + R_h)y\|_* \leq c\|y\|_{h,2}.$$

Finally with (3.11) the right-hand side of (3.8) is bounded in the form

$$\|\pi_h(F + D_h(y))y + \pi_h Q_h y + R_h y\|_* \leq c\|y\|_{h,2} \quad \text{for any } y \in W \cap B_\rho(u), \tag{3.12}$$

with some $c > 0$. Combining this with (3.9) we obtain

$$\|Ty\| \leq \frac{c}{\gamma}\|y\|_{h,2} \quad \text{for any } y \in W \cap B_\rho(u).$$

Using the continuous embedding $V \hookrightarrow C^{0,1/2}(\bar{\Omega})$ this leads to

$$\|Ty\| \leq c\|y\|_{h,2} \quad \text{for any } y \in W \cap B_\rho(u). \tag{3.13}$$

Taking the definitions of the operators Q_h, R_h according to (3.1), (3.2) problem (3.8) is equivalent to

$$-a_i z'' + d_i z = r \quad \text{in } \Omega_i, \quad i = 1(1)N \tag{3.14}$$

and

$$a_i z'(x_i - 0) - a_{i+1} z'(x_i + 0) = (a_i - a(x_i))y'(x_i - 0) - (a_{i+1} - a(x_i))y'(x_i + 0), \quad i = 1(1)N - 1. \tag{3.15}$$

Here we denote $z = Ty$ and $r = S_h y$ with the operator S_h defined by (3.10). In the case $d_i > 0$ (see, e.g., [4]) system (3.14), (3.15) can be solved by means of adapted exponential splines for given $y \in W \cap B_\rho(u)$ explicitly with a finite number of arithmetical operations on a computer. Furthermore, $z \in C^2_\lambda(\bar{\Omega})$ holds as a direct consequence of (3.14) and $r \in C(\bar{\Omega})$. Thus the mapping $T : W \cap B_\rho(u) \rightarrow W$ can be implemented in the strict sense unlike the algorithmic mapping related to (2.3).

Now we investigate the local contraction behaviour of recursion (3.6). This has to be done in two different steps. First we derive an estimation of $\|y^{k+2} - y^{k+1}\|_*$ in terms of $\|y^{k+1} - y^k\|_*$ and of the step size h . Then we use embedding theorems and recursively the differential equation (3.14) to obtain an estimation of $\|y^{k+2} - y^{k+1}\|_{h,2}$. This technique is very different from earlier convergence analysis (see [1], [4]) of methods with simplified operators. The previous investigations did not deal with modifications of the main part of the differential operator and the difficulties we meet here just result from this part of the technique.

If no misinterpretation can occur we omit the subscript h in the sequel and we write D^k instead of $D_h(y^k)$. The recursion (3.6) has the form

$$(L_h + D^k)y^{k+1} = \pi(F + D^k)y^k + \pi Q_y^k + Ry^k, \quad k = 0, 1, \dots$$

We abbreviate $w^k = y^{k+1} - y^k$ to shorten the notation. After reformulation (3.6) is equivalent to

$$\begin{aligned} (L_h + D^k)w^k &= \pi(F + D^k)y^k + \pi Q_y^k + Ry^k - L_h y^k - D^k y^k \\ &= (\pi - I)(F + D^k)y^k + (\pi - I)Q_y^k + (R + Q)y^k + Fy^k - L_h y^k. \end{aligned}$$

With (3.14) we obtain

$$(L_h + D^k)w^k = (\pi - I)(F + D^k)y^k + (\pi - I)Q_y^k + Fy^k - Ly^k \tag{3.16}$$

and

$$(L_h + D^{k+1})w^{k+1} = (\pi - I)(F + D^{k+1})y^{k+1} + (\pi - I)Q_y^{k+1} + Fy^{k+1} - Ly^{k+1}.$$

Using the linearity of the operators except F this results in

$$(L_h + D^{k+1})w^{k+1} = (\pi - I)((F + D^{k+1})y^{k+1} - (F + D^k)y^k) + (\pi - I)Qw^k + Fy^{k+1} - Fy^k + D^k w^k + (L_h - L)w^k. \quad (3.17)$$

Now we investigate the behaviour of the summands of the right-hand side of the operator equation (3.17) separately. As a direct consequence of the projection operator defined by (3.5) we have

$$\|(\pi - I)y\|_h \leq ch^m \|y\|_{h,m} \quad \text{for any } y \in C_h^m(\bar{\Omega}), \quad m = 0, 1, 2 \quad (3.18)$$

with some constant $c > 0$. Because of $y^k, y^{k+1} \in C_h^2(\bar{\Omega})$ we can interpret the right-hand side of (3.17) as an element contained in $C_h(\bar{\Omega})$. By definition,

$$\begin{aligned} [Fy^{k+1} - Fy^k + D^k w^k](x) &= f(x, y^{k+1}(x)) - f(x, y^k(x)) - \frac{\partial f}{\partial y}(x, y^k(x)) w^k(x) \\ &+ \left[\frac{\partial f}{\partial y}(x, y^k(x)) - \frac{1}{2} \frac{\partial f}{\partial y}(x_{i-1}, y^k(x_{i-1})) - \frac{1}{2} \frac{\partial f}{\partial y}(x_i, y^k(x_i)) \right] w^k(x), \quad x \neq x_i. \end{aligned}$$

With the Lipschitz continuity of $\partial f / \partial y$ this results in

$$\|Fy^{k+1} - Fy^k + D^k w^k\|_h \leq \frac{L}{2} \|w^k\|_h^2 + L(h + h\|w^k\|_{h,1}) \|w^k\|_h, \quad (3.19)$$

with some $L > 0$ provided $y^k, y^{k+1} \in B_\rho(u)$. Furthermore, we have the estimation

$$\|(D^{k+1} - D^k)y^{k+1}\|_{h,1} \leq L \|w^k\|_h \|y^{k+1}\|_{h,1}. \quad (3.20)$$

Using (3.18) - (3.20) we obtain

$$\begin{aligned} \|(\pi - I)((F + D^{k+1})y^{k+1} - (F + D^k)y^k)\|_h &\leq \|(\pi - I)(Fy^{k+1} - Fy^k + D^k w^k)\|_h + \|(\pi - I)(D^{k+1} - D^k)y^{k+1}\|_h \\ &\leq c_1 \|w^k\|_{h,1}^2 + c_2 h \|w^k\|_{h,1} \quad \text{for any } y^k, y^{k+1} \in W \cap B_\rho(u). \end{aligned} \quad (3.21)$$

With the definition of the operator Q and with the property $a(\cdot) \in C_h^2(\bar{\Omega})$ we obtain

$$[Qy](x) = [a'y'](x) - (a_i - a(x))y''(x) \quad \text{for any } x \in \Omega_i$$

and

$$\|a'y'\|_{h,1} \leq c \|a\|_{h,2}, \quad \|(a_h - a)y''\|_h \leq ch \|y\|_{h,2}.$$

Using (3.18) this results in the estimation

$$\|(\pi - I)Qw^k\|_h \leq ch \|w^k\|_{h,2} \quad \text{for any } y \in W \cap B_\rho(u). \quad (3.22)$$

The last summand in (3.17) can be estimated by

$$\|(L_h - L)w^k\|_* \leq \|a_h - a\|_h \|w^k\|_{h,1}. \quad (3.23)$$

On the basis of (3.19) - (3.21) the right-hand side of equation (3.17) can be bounded in the dual norm $\|\cdot\|_*$ by

$$(c_1 \|w^k\|_{h,2} + c_2 h) \|w^k\|_{h,2},$$

with some constants $c_1, c_2 > 0$ provided $y^k, y^{k+1} \in W \cap B_\rho(u)$. Using the coercivity (3.9) this results in

$$\|w^{k+1}\| \leq \frac{1}{\gamma} \left(c_1 \|w^k\|_{h,2} + c_2 h \right) \|w^k\|_{h,2}.$$

With the continuous embedding $V \hookrightarrow C^{0,1/2}(\bar{\Omega})$ we obtain

$$\|w^{k+1}\|_{C^{0,1/2}(\bar{\Omega})} \leq \left(c_1 \|w^k\|_{h,2} + c_2 h \right) \|w^k\|_{h,2}, \quad (3.24)$$

with some constants $c_1, c_2 > 0$.

In the next part we derive an estimation for the stronger norm $\|w^{k+1}\|_{h,2}$. This can be done by combining the previous results with estimations derived from the differential equations (3.14) on the subintervals Ω_i . Let us consider

$$-a_i (w^{k+1})'' + d_i^{k+1} w^{k+1} = r^k \quad \text{in } \Omega_i \quad (3.25)$$

with

$$d_i^{k+1} = d_i[y^{k+1}]$$

and

$$r^k = (\pi - I) \left((F + D^{k+1})y^{k+1} - (F + D^k)y^k \right) + \pi Q w^k + F y^{k+1} - F y^k + D^k w^k.$$

As shown earlier we obtain the estimation

$$\|r^k\|_{C(\bar{\Omega}_i)} \leq \left(c_1 \|w^k\|_{h,2} + c_2 h \right) \|w^k\|_{h,2} + \|\pi Q w^k\|_{C(\bar{\Omega}_i)}.$$

Now

$$[Q w^k](x) = -((a_i - a)w^k)'(x) \quad \text{in } \Omega_i$$

and (3.5) result in

$$\|\pi Q w^k\|_{C(\bar{\Omega}_i)} \leq (1 + h_i) \|a'\|_{C(\bar{\Omega}_i)} \|w^k\|_{h,2}.$$

Thus, we have

$$\|r^k\|_{C(\bar{\Omega}_i)} \leq \left(c_1 \|w^k\|_{h,2} + c_2 h + \|a'\|_{C(\bar{\Omega}_i)} \right) \|w^k\|_{h,2}. \quad (3.26)$$

With (3.22) - (3.26) this leads to

$$\begin{aligned} \|(w^{k+1})''\|_{C(\bar{\Omega}_i)} &\leq \frac{1}{a_i} \left(|d_i^{k+1}| \|w^{k+1}\|_{C(\bar{\Omega}_i)} + \|r^k\|_{C(\bar{\Omega}_i)} \right) \\ &\leq \left(c_1 \|w^k\|_{h,2} + c_2 h + \frac{\|a'\|_{C(\bar{\Omega}_i)}}{a_i} \right) \|w^k\|_{h,2}. \end{aligned}$$

Owing to assumption (3.7) we have

$$\|(w^{k+1})''\|_{C(\bar{\Omega}_i)} \leq \left(c_1 \|w^k\|_{h,2} + c_2 h + \beta \right) \|w^k\|_{h,2},$$

with some constants $c_1, c_2 > 0, \beta \in (0, 1)$.

Because of the mean value theorem some $\xi_i \in \Omega_i$ exists such that

$$(w^{k+1})'(\xi_i) = (w^{k+1}(x_i) - w^{k+1}(x_{i-1}))/h_i.$$

Using (3.24), now, we obtain

$$|(w^{k+1})'(\xi_i)| \leq \left(c_1 \|w^k\|_{h,2} + c_2 h \right) \|w^k\|_{h,2} h_i^{-1/2}.$$

With the assumed quasi uniformness of the grid this leads to

$$|(w^{k+1})'(\xi_i)| \leq (c_1 h_{-1/2} \|w^k\|_{h,2} + c_2 h^{1/2}) \|w^k\|_{h,2}.$$

Taylor's formula results in

$$\begin{aligned} |(w^{k+1})'(x)| &\leq |(w^{k+1})'(\xi_i)| + \left| \int_{\xi_i}^x (w^{k+1})'(\xi) d\xi \right| \\ &\leq (c_1 h^{-1/2} \|w^k\|_{h,2} + c_2 h^{1/2} + \beta h) \|w^k\|_{h,2} \quad \text{for any } x \in \Omega_i. \end{aligned}$$

Combining all estimations with respect to the function w^{k+1} and its first two derivatives we obtain for sufficiently small $h > 0$

$$\|w^{k+1}\|_{h,2} \leq (c_1 h^{-1/2} \|w^k\|_{h,2} + c_2 h^{1/2} + \beta) \|w^k\|_{h,2}. \tag{3.27}$$

We summarize the conclusions and estimations above in

Theorem 2 *Let $u \in V$ denote some solution of problem (1.3) which satisfies the regularity assumption (1.4). Furthermore, u is supposed to be two times continuously differentiable. Then some $\delta > 0$ and some $\bar{h} > 0$ exist such that method (3.6) is well defined for any $h \in (0, \bar{h}]$ and $y^0 \in V \cap C^2(\bar{\Omega})$ with $\|y_0 - u\|_{h,2} < \delta$. The sequence $\{y^k\}$ generated by the method converges to some y_h approximating the solution u of (1.3) with the accuracy*

$$\|y_h - u\| \leq c h^2$$

with some $c > 0$.

Proof: Owing to (3.16) we have

$$(L_h + D^0)w^0 = (\pi - I)(F + D^0)y^0 + (\pi - I)Qy^0 + Fy^0 - Fu - L(y^0 - u).$$

On the other hand the supposed continuity leads to

$$\begin{aligned} \|(\pi - I)(F + D^0)y^0\|_h &\leq c h \|y^0\|_{h,1}, & \|(\pi - I)Qy^0\|_h &\leq c h \|y^0\|_{h,2}, \\ \|Fy^0 - Fu\|_h &\leq c \|y^0 - u\|_h, & \|L(y^0 - u)\|_h &\leq c \|y^0 - u\|_{C^2(\bar{\Omega}_i)}. \end{aligned}$$

Thus for every $\epsilon > 0$ using Lemma 1 and the contractivity (1.4) we can select $\bar{h} > 0$ and $\delta > 0$ such that $\|w^0\|_{h,2} \leq \epsilon$ for any $y^0 \in V \cap C^2(\bar{\Omega})$ with $\|y^0 - u\|_{h,2} \leq \delta$ and for any $h \in (0, \bar{h}]$. Selecting $\bar{h} > 0$ as well as $\epsilon > 0$ small enough from (3.27) by usual arguments of local versions of Banach's fixed point theorem we obtain recursively $\{y^k\}$ and the sequence is convergent to some $y_h \in W \cap V$ that is close to u .

With the continuity of the related operators from (3.6) we obtain

$$\begin{aligned} (L_h + D_h)y_h &= \pi(F + D_h)y_h + \pi Qy_h + Ry_h \\ &= (\pi - I)(F + D_h)y_h + (\pi - I)Qy_h + (L_h - L)y_h + Fy_h. \end{aligned} \tag{3.28}$$

This is equivalent to

$$(L - F)y_h = (\pi - I)(F + D_h)y_h + (\pi - I)Qy_h. \tag{3.29}$$

The right-hand side of (3.29) can be estimated by

$$\|(\pi - I)(F + D_h)y_h\|_h \leq c h^2 \|y_h\|_{h,2} \quad \text{and} \quad \|(\pi - I)Qy_h\|_h \leq c h^2 \|y_h\|_{h,3},$$

with some $c > 0$. The required smoothness of y_h in (3.29) can be recursively derived from (3.28) if the functions f and a are sufficiently smooth.

Now, (3.29) can be rewritten in the form

$$(L - F'(u))(y_h - u) = (\pi - I)(F + D_h)y_h + (\pi - I)Qy_h + Fy_h - Fu - F'(u)(y_h - u).$$

With the coercivity (1.4) and Taylor's formula we obtain

$$\|y_h - u\|_{c_1 h^2 \|y_h\|_{h,3} + c_2 \|y_h - u\|^2}. \tag{3.30}$$

By selecting $\delta > 0$ and $\bar{h} > 0$ small enough we can guarantee $\|y_h - u\|$ to be sufficiently small. Thus, (3.30) proves the estimation stated in the theorem ■

Remark: Using bounding operators as introduced in [4] instead of the projection defined by (3.3) we can develop numerical methods generating continuous enclosures of the unknown solutions. However, additional analysis is required to cover this situation also. The aim of the present paper consists in paving the way to such analysis as well.

4. Numerical implementations

In this part we will only sketch the finite-dimensional realization of the iteration scheme (3.6). This technique uses the piecewise representation of the solution of the system (3.14), (3.15) of boundary value problems.

The right-hand side of (3.6) consists of piecewise affine functions building the function r in (3.14). Furthermore the patching conditions (3.15) have to be considered. The structure of the differential operators $L_h + D_h^k$ causes the iterates to be formed by piecewise exponential functions z . Taking the values z_i of these functions at the grid points x_i as parameters the system (3.14), (3.15) can be decoupled into boundary value problems

$$\begin{aligned} -a_i z'' + d_i[y]z &= r && \text{in } \Omega_i \\ z(x_{i-1}) = z_{i-1}, z(x_i) &= z_i, && i = 1(1)N. \end{aligned}$$

Condition (3.15) finally results in a tri-diagonal linear system defining the unknown parameters z_i , $i = 1(1)N - 1$. Techniques for solving this type of equation have been studied extensively in [1], [4] (compare also [5]) and the interested reader is referred to these papers for further details.

Finally we conclude that method (3.6) is implementable in finite dimensions and we report some results obtained by (3.6) for the example

$$\begin{aligned} -((2 + \sin(\alpha x))y')'(x) &= -\exp(y(x)) + \exp(x^2(1-x)) - \alpha(2x - 3x^2)\cos(\alpha x) - (2 + \sin(\alpha x))(2 - 6x), \\ y(0) = y(1) &= 0. \end{aligned}$$

Here α denotes some real parameter. This problem has a unique solution $u(x) = x^2(1-x)$. The following table gives the obtained results for various step sizes and different values α . We report the number of iterations needed to reach y_h within a fixed relative accuracy and we give the value

$$\delta_h = \max_{1 \leq i \leq N-1} |y_h(x_i) - u(x_i)|.$$

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Book reviews

M. A. KRASNOSEL'SKIĬ and A. V. POKROVSKIĬ: **Systems with Hysteresis**. Berlin - Heidelberg - New York: Springer-Verlag 1989, XVIII + 410 pp., 77 Fig.

By this voluminous extensive work a careful translation of the russian original from 1983 made by M. Niezgódka is offered. With it at length a many-sided teaching and scientific material, which has - under the collaboration of many soviet mathematicians - its origin in a seminar of the Voronezh university and which was later completed by the authors at the Academic Institute of Control Sciences in Moscow, became accessible to a wide circulation.

The book mainly deals with dynamical systems, in which under the influence of a time dependent input $u(t)$ by means of a "transducer" a unique output $x(t)$ is generated. There are considered especially such appearances, where the output depends very significantly on the monotony behaviour of u . Therefore, finally phenomenons of hysteresis are included as they have been known in electrodynamics and the theory of plasticity for a long time and have been described by many classics. However not these and other physical appearances of that type are

	N=10		N=20		N=40		N=80	
$\alpha=1$	5	3.5E-4	5	8.8E-5	5	2.2E-5	5	5.5E-6
$\alpha=10$	16	4.7E-4	12	1.1E-4	9	2.8E-6	8	7.1E-6
$\alpha=20$	27	5.7E-4	21	1.3E-4	12	3.4E-5	10	8.5E-6

The obtained results show a good coincidence with the predicted convergence behaviour of method (3.6).

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