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Weighted L^p-Estimates for Pseudo-Differential Operators with Non-Regular Symbols

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Estimates for pseudo-differential operators with non-regular double symbols $a(x,y,\xi)$ are proved in weighted L^p- and Sobolev spaces. The results presented here are generalizations of those by G. Bourdaud [4].

Key words: *Weighted L^p-spaces, pseudo-differential operators*
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1. Let w be a positive locally integrable function defined on the n-dimensional Euclidean number space $\boldsymbol{R}^n$. We say that $w \in A_p$, i.e. w satisfies *Muckenhoupt's A_p* condition, if

$$\sup\left\{(1/|B|)\int_B w(x)\,dx\ \left((1/|B|)\int_B w^{-1/(p-1)}(x)\,dx\right)^{p-1}\right\} < \infty$$

where the supremum is taken over all balls $B \subseteq \boldsymbol{R}^n$. Denote by $L^p(w)$ the weighted L^p-space and let J^s be the Bessel potential of order $s \in \boldsymbol{R}$. The *weighted Sobolev space* $H^{s,p}(w)$ is defined to be the space of all tempered distributions f such that $\|f\|_{H^{s,p}(w)} := \|J^{-s}f\|_{L^p(w)} < \infty$. Our objevtive is to study the action of pseudo-differential operators of the form

$$Op(a)\,f(x) = 1/(2\pi)^n \int\int e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\,dy\,d\xi\ , \tag{1}$$

where $f \in S(\boldsymbol{R}^n)$, the Schwartz space of rapidly decreasing functions, on $L^p(w)$ and $H^{s,p}(w)$.

2. Let $0 \le \delta_1, \delta_2 \le 1$, $r_1, r_2 > 0$ and $N \in \boldsymbol{N}$, the set of natural numbers. Denote by $\boldsymbol{Z}$ the set of integers and by $\boldsymbol{C}$ the set of complex numbers. Define the *symbol class* $S^0_{1,\delta_1,\delta_2}(r_1, r_2, N)$ to be the space of all symbols $a : \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ such that for all multi-indices $\alpha \in \boldsymbol{N}_0^n$

with $|\alpha| \leq N$ it holds that

$$\begin{aligned} |\partial_\xi^\alpha a(x,y,\xi)| &\leq C(1+|\xi|)^{-|\alpha|} \\ \|\partial_\xi^\alpha a(\cdot,y,\xi)\|_{\Lambda^{r_1}} &\leq C(1+|\xi|)^{\delta_1 r_1-|\alpha|} \\ \|\partial_\xi^\alpha a(x,\cdot,\xi)\|_{\Lambda^{r_2}} &\leq C(1+|\xi|)^{\delta_2 r_2-|\alpha|} \\ \|\partial_\xi^\alpha a(\cdot,\cdot,\xi)\|_{\Lambda^{r_1,r_2}} &\leq C(1+|\xi|)^{\delta_1 r_1+\delta_2 r_2-|\alpha|} . \end{aligned} \tag{2}$$

Λ^r denotes the usual homogeneous *Hölder-Zygmund space* (see *Bergh* and *Löfström* [3]) and Λ^{r_1,r_2} the *Hölder-Zygmund space of product type* : $f \in \Lambda^{r_1,r_2}$ if for some constant $C > 0$ and some integers $M_1 > r_1$ and $M_2 > r_2$ it holds that

$$|\Delta_{1,h_1}^{M_1}\, \Delta_{2,h_2}^{M_2}\, f| \leq C|h_1|^{r_1}|h_2|^{r_2} , \tag{3}$$

where $\Delta_{i,h_i}^{M_i}$ denotes the M_i-th order difference operator for the i-th factor. Then we have

Theorem 1. *Let $a \in S^0_{1,\delta_1,\delta_2}(r_1,r_2,N)$, $s \in \mathbf{R}$ and let $N = n+1$ and $-\min\{r_2,(1-\delta_1)r_1\} < s < \min\{r_1,(1-\delta_2)r_2\}$. Then, if $1 < p < \infty$ and $w \in A_p$, $Op(a) : H^{s,p}(w) \to H^{s,p}(w)$ extends as a bounded operator.*

The theorem is proved in Section 4, where we prove an extension of it, Theorem 3. We now turn to the case $s = 0$. Let $\mathbf{R}^+$ be the set of positive real numbers and let $\omega : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ and $\Omega : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be two positive functions, which are slowly varying in the following sense: there exists a constant $C > 0$ such that

$$\omega(t_1,t_2) \leq C\omega(\tau_1,\tau_2)\, ,\ \Omega(t_1,t_2,t_3) \leq C\Omega(\tau_1,\tau_2,\tau_3) \tag{4}$$

whenever $0.5\tau_i \leq t_i \leq 2\tau_i$, $i = 1,2,3$. Suppose, that the symbol a satisfies for all multi-indices α with $|\alpha| \leq n+1$ the estimates

$$\begin{aligned} |\partial_\xi^\alpha a(x,y,\xi)| &\leq C(1+|\xi|)^{-|\alpha|} \\ |\Delta_{1,h}\, \partial_\xi^\alpha a(x,y,\xi)| &\leq C\omega(|h|,1+|\xi|)\,(1+|\xi|)^{-|\alpha|} \\ |\Delta_{2,h}\, \partial_\xi^\alpha a(x,y,\xi)| &\leq C\omega(|h|,1+|\xi|)\,(1+|\xi|)^{-|\alpha|} \\ |\Delta_{1,h_1}\Delta_{2,h_2}\, \partial_\xi^\alpha a(x,y,\xi)| &\leq C\Omega(|h_1|,|h_2|,1+|\xi|)\,(1+|\xi|)^{-|\alpha|} . \end{aligned} \tag{5}$$

Theorem 2. *Let ω and Ω be slowly varying such that $\{\omega(2^{-j},2^j)\} \in l^2(\mathbf{N})$ and $\{\Omega(2^{-j},2^{-j},2^j)\} \in l^1(\mathbf{N})$. Suppose the symbol a satisfies (5). Then, if $1 < p < \infty$ and $w \in A_p$, the operator $Op(a) : L^p(w) \to L^p(w)$ is bounded.*

Theorem 2 is proved in Section 5. Obviously, both theorems extend earlier results by *Coifman* and *Meyer* [6], *Bourdaud* [4], *Alvarez-Alonso* [1], *Wang* and *Li* [10], *Miyachi* and *Yabuta* [9] and others. The regularity in the ξ-variable can be further improved. In fact, if we introduce Hölder regularity in the ξ-variable, both theorems hold in case $N > n$. Actually, we prove the theorems under $B^N_{1,\infty}$-regularity in the ξ-variable, where $B^N_{1,\infty}$ denotes a Besov space.

3. We collect some tools needed for the proofs. First, we need the *Littlewood-Paley decomposition* for the weighted Sobolev spaces $H^{s,p}(w)$. Choose a smooth non-negative function $\varphi : \boldsymbol{R}^n \to \boldsymbol{R}$ supported in the annulus $\{\xi|\ 1/4 \le |\xi| \le 4\}$ such that $\sum_{k\in \boldsymbol{Z}} \varphi(2^{-k}\xi) = 1$, if $\xi \neq 0$. Define φ_k to be $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, if $k \ge 1$ and $\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi_k(\xi)$. If f is a tempered distribution, denote by Ff the Fourier transform of f and let $f_k := F^{-1}(\varphi_k\, Ff)$. Then it holds (see *Bui* [5])

$$\|f\|_{H^{s,p}(w)} \sim \left\| \left(\sum_{k=0}^{\infty} 4^{ks}|f_k|^2 \right)^{1/2} \right\|_{L^p(w)} \tag{6}$$

($a \sim b$ means that a and b are comparable by some fixed constants).

Lemma 1. *Let $1 < p < \infty$ and $w \in A_p$.*

(a) Let $\{f_k\}$ be a sequence of functions such that the spectrum of f_k (i.e. the support of Ff_k) is contained in the annulus $|\xi| \sim 2^k$. Then for each $s \in \boldsymbol{R}$ it holds that

$$\|f\|_{H^{s,p}(w)} \le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks}|f_k|^2 \right)^{1/2} \right\|_{L^p(w)} .$$

(b) Let $\{f_k\}$ be a sequence of functions such that the spectrum of f_k is contained in the ball $|\xi| \le c2^k$. Then for each $s > 0$ it holds that

$$\|f\|_{H^{s,p}(w)} \le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks}|f_k|^2 \right)^{1/2} \right\|_{L^p(w)} .$$

The unweighted case of part (b) of the lemma is due to *Meyer* [8]. His proof extends without any difficulties to the weighted case. Denote by Mf the Hardy-Littlewood maximal operator defined by $Mf(x) = \sup_B (1/|B|) \int_B |f(y)|\,dy$ where the supremum is taken over all balls B with center x. We need the following vector-valued maximal theorem (see *Andersen* and *John* [2]).

Lemma 2. *Let $1 < p < \infty$ and $w \in A_p$. Then for each sequence $\{f_k\}$ of functions it holds that*

$$\left\| \left(\sum_{k=0}^{\infty} |Mf_k|^2 \right)^{1/2} \right\|_{L^p(w)} \le C \left\| \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p(w)} .$$

The following lemma is the basic estimate in our theory. For variants of it see *Marschall* [7].

Lemma 3. *Let the support of the function $\xi \to a(x,y,\xi)$ be contained in a fixed compact set independent of x and y. Then for each $N > n$ there exists a constant $C_N > 0$ such that for each function f and for each $R \geq 1$ it holds*

$$|Op(a)\, f(x)| \leq C_N \sup_y \|a(x,y,R\cdot)\|_{B^N_{1,\infty}} Mf(x) .$$

Proof. Let $K(x,y,x-y) = 1/(2\pi)^n \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi$ be the kernel of $Op(a)$. Let $\psi_k := \varphi(2^{-k}\cdot)$, if $k \in Z$. Then one has

$$\begin{aligned} |Op(a)\, f(x)| &= \left|\int K(x,y,x-y) f(y)\, dy\right| \\ &\leq \sum_{k\in Z} \int |K(x,y,x-y)\psi_k(x-y) f(y)|\, dy \\ &\leq C \left(\sum_{k\in Z} 2^{kn} \sup_y |K(x,y,x-y)\psi_k(x-y)| \right) Mf(x) . \end{aligned}$$

Denote by F_ξ the Fourier transform with respect to the ξ-variable and let $R = 2^l$. Then it follows that

$$\begin{aligned} |K(x,y,z)\psi_k(z)| &\leq \|F_z^{-1}(\psi_k F_\xi a(x,y,\cdot))\|_{L^1} \\ &= 2^{ln} \|F_z^{-1}(\psi_{k+l} F_\xi a(x,y,2^l\cdot))\|_{L^1} . \end{aligned}$$

Now, one has for each $\epsilon > 0$

$$\sum_{j=-l}^{\infty} 2^{(j+l)n} \sup_y \|F_z^{-1}(\psi_{j+l} F_\xi a(x,y,2^l\cdot))\|_{L^1} \leq C_\epsilon \sup_y \sup_{j\geq 0} 2^{j(n+\epsilon)} \|F_z^{-1}(\psi_j F_\xi a(x,y,2^l\cdot))\|_{L^1} ,$$

and since $\|F^{-1}\psi_{j+l}\|_{L^1} = \|F^{-1}\psi_0\|_{L^1}$ we get the estimate

$$\sum_{j=-\infty}^{-l} 2^{(j+l)n} \sup_y \|F_z^{-1}(\psi_{j+l} F_\xi a(x,y,2^l\cdot))\|_{L^1} \leq C \sup_y \|a(x,y,2^l\cdot)\|_{L^1} .$$

The lemma is now an easy consequence of the definition of the Besov space $B^N_{1,\infty}$ (see *Bergh* and *Löfström* [3]) ∎

Denote by F_x (resp. F_y) the Fourier transform with respect to the x-variable (resp. y-variable) and define

$$\begin{aligned} a_k(x,y,\xi) &= a(x,y,\xi)\varphi_k(\xi) \\ a_{i,\cdot,k}(x,y,\xi) &= F_x^{-1}(\varphi_i F_x a(x,y,\xi))\varphi_k(\xi) \\ a_{\cdot,j,k}(x,y,\xi) &= F_y^{-1}(\varphi_j F_y a(x,y,\xi))\varphi_k(\xi) \\ a_{i,j,k}(x,y,\xi) &= F_x^{-1} F_y^{-1}(\varphi_i \otimes \varphi_j F_x F_y a(x,y,\xi))\varphi_k(\xi) . \end{aligned}$$

We say that $a \in \tilde{S}^0_{1,\delta_1,\delta_2}(r_1, r_2, N)$, if for each $i, j, k \in \boldsymbol{N}_0$ one has uniformly in (x, y)

$$\begin{aligned} \|a_k(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\le C \\ \|a_{i,\cdot,k}(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\le C2^{k\delta_1 r_1 - ir_1} \\ \|a_{\cdot,j,k}(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\le C2^{k\delta_2 r_2 - jr_2} \\ \|a_{i,j,k}(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\le C2^{k(\delta_1 r_1+\delta_2 r_2) - ir_1 - jr_2} . \end{aligned} \tag{7}$$

It is an easy conseqence of the Littlewood-Paley decomposition for Hölder-Zygmund spaces of product type that the inclusion $S^0_{1,\delta_1,\delta_2}(r_1, r_2, N) \subseteq \tilde{S}^0_{1,\delta_1,\delta_2}(r_1, r_2, N)$ holds (see *Marschall* [7], Theorem 1.8) . We prove the following theorem, which includes Theorem 1.

Theorem 3. *Let $a \in \tilde{S}^0_{1,\delta_1,\delta_2}(r_1, r_2, N)$, $s \in \boldsymbol{R}$ and let $N > n$ and $-\min\{r_2, (1-\delta_1)r_1\} < s < \min\{r_1, (1-\delta_2)r_2\}$. Then, if $1 < p < \infty$ and $w \in A_p$, $Op(a) : H^{s,p}(w) \to H^{s,p}(w)$ extends as a bounded operator.*

For the proof we need the following lemma, which belongs to the folclore.

Lemma 4. *The following statements are true.*
(a) If $r < 0$, then $(\sum_{k=0}^\infty 4^{kr}(\sum_{j=0}^k |a_j|)^2)^{1/2} \le C(\sum_{k=0}^\infty 4^{kr}|a_k|^2)^{1/2}$,
(b) If $r > 0$, then $(\sum_{k=0}^\infty 4^{kr}(\sum_{j=k}^\infty |a_j|)^2)^{1/2} \le C(\sum_{k=0}^\infty 4^{kr}|a_k|^2)^{1/2}$.

4. Proof of Theorem 3. Step 1. Let $f \in S(\boldsymbol{R}^n)$. We decompose $Op(a)f$ into 9 parts by using the Littlewood-Paley decomposition of the symbol a and of f. Let us denote by $\tilde{f}_k$ the part of f with spectrum contained in the annulus $|\xi| \sim 2^k$. Observe also that

$$F(Op(b)g)(\eta) = \frac{1}{(2\pi)^n} \int\int F_x F_y b(\eta - \xi, \xi - \zeta, \xi)\, Fg(\zeta)\, d\zeta\, d\xi \tag{8}$$

holds for any symbol b with compact support in the ξ-variable and any $g \in S(\boldsymbol{R}^n)$. In this formula, the spectrum of $Op(b)\,g$ is related with the spectrum of b with respect to the x-variable and with ξ whereas the spectrum of g is related with the spectrum of b with respect to the y-variable and with ξ.

Step 2. Let

$$A_1 f = \sum_{k=0}^{\infty} \sum_{i=0}^{k-4} \sum_{j=0}^{k-4} Op(a_{i,j,k})\, \tilde{f}_k .$$

Since $\|F^{-1}(\sum_{i=0}^{k-4} \varphi_i)\|_{L^1} = \|F^{-1}\varphi_0\|_{L^1}$, we obtain from Lemma 3

$$\left|\sum_{i=0}^{k-4} \sum_{j=0}^{k-4} Op(a_{i,j,k})\, \tilde{f}_k(x)\right|$$

$$
\begin{aligned}
&\le C \sup_y \left\| \sum_{i=0}^{k-4} \sum_{j=0}^{k-4} a_{i,j,k}(x,y,2^k \cdot) \right\|_{B^N_{1,\infty}} M\tilde{f}_k(x) \\
&\le C \sup_y \int\int \left| \sum_{i=0}^{k-4} \sum_{j=0}^{k-4} \varphi_i(x-z_1)\varphi_j(y-z_2) \right| \|a_k(z_1,z_2,2^k \cdot)\|_{B^N_{1,\infty}} dz_1 dz_2 \, M\tilde{f}_k(x) \\
&\le C \sup_{x,y} \|a_k(x,y,2^k \cdot)\|_{B^N_{1,\infty}} M\tilde{f}_k(x) \quad \le \quad C\, M\tilde{f}_k(x)\,.
\end{aligned}
$$

The spectrum of $\sum_{i=0}^{k-4}\sum_{j=0}^{k-4} Op(a_{i,j,k})\,\tilde{f}_k$ is contained in the annulus $|\eta| \sim 2^k$, hence by Lemmata 1 and 2

$$
\begin{aligned}
\|A_1 f\|_{H^{s,p}(w)} &\le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |M\tilde{f}_k|^2 \right)^{1/2} \right\|_{L^p(w)} \\
&\le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |\tilde{f}_k|^2 \right)^{1/2} \right\|_{L^p(w)} \le C\|f\|_{H^{s,p}(w)}\,.
\end{aligned}
$$

STEP 3. Since the proofs of the other estimates are similar, we are brief. Let

$$
A_2 f = \sum_{k=0}^{\infty} \sum_{i=0}^{k-4} \sum_{j=k-3}^{k+3} Op(a_{i,j,k}) \left(\sum_{l=0}^{k+6} \tilde{f}_l \right), \quad A_3 f = \sum_{k=0}^{\infty} \sum_{i=0}^{k-4} \sum_{j=k+4}^{\infty} Op(a_{i,j,k})\,\tilde{f}_j\,.
$$

Then we obtain for $s < (1-\delta_2)r_2$, using Lemma 4(a),

$$
\begin{aligned}
\|A_2 f\|_{H^{s,p}(w)} &\le C \left\| \left(\sum_{k=0}^{\infty} 4^{k(s-(1-\delta_2)r_2)} \left(\sum_{l=0}^{k+6} M\tilde{f}_l \right)^2 \right)^{1/2} \right\|_{L^p(w)} \\
&\le C \left\| \left(\sum_{k=0}^{\infty} 4^{k(s-(1-\delta_2)r_2)} |\tilde{f}_k|^2 \right)^{1/2} \right\|_{L^p(w)} \le C\|f\|_{H^{s-(1-\delta_2)r_2,p}(w)}
\end{aligned}
$$

and for $s > -r_2$, using Lemma 4(b)

$$
\begin{aligned}
\|A_3 f\|_{H^{s+(1-\delta_2)r_2,p}(w)} &\le C \left\| \left(\sum_{k=0}^{\infty} 4^{k(s+r_2)} \left(\sum_{j=k}^{\infty} 2^{-jr_2} M\tilde{f}_j \right)^2 \right)^{1/2} \right\|_{L^p(w)} \\
&\le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |\tilde{f}_k|^2 \right)^{1/2} \right\|_{L^p(w)} \le C\|f\|_{H^{s,p}(w)}\,.
\end{aligned}
$$

STEP 4. Let us define

$$
\begin{aligned}
A_4 f &= \sum_{k=0}^{\infty} \sum_{i=k-3}^{k+3} \sum_{j=0}^{k-4} Op(a_{i,j,k})\,\tilde{f}_k\,, \\
A_5 f &= \sum_{k=0}^{\infty} \sum_{i=k-3}^{k+3} \sum_{j=k-3}^{k+3} Op(a_{i,j,k}) \left(\sum_{l=0}^{k+6} \tilde{f}_l \right), \\
A_6 f &= \sum_{k=0}^{\infty} \sum_{i=k-3}^{k+3} \sum_{j=k+4}^{\infty} Op(a_{i,j,k})\,\tilde{f}_j\,.
\end{aligned}
$$

Since the spectrum of $\sum_{i=k-3}^{k+3}\sum_{j=0}^{k-4} Op(a_{i,j,k})\,\tilde{f}_k$ is contained in the ball $|\eta| \le c2^k$, we use Lemma 1(b) and obtain

$$\begin{aligned}
\|A_4 f\|_{H^{s+(1-\delta_1)r_1,p}(w)} &\le C\|f\|_{H^{s,p}(w)}\,,\ if\ s > -(1-\delta_1)r_1\,,\\
\|A_5 f\|_{H^{s+(1-\delta_1)r_1,p}(w)} &\le C\|f\|_{H^{s-(1-\delta_2)r_2,p}(w)}\,,\ if\ -(1-\delta_1)r_1 < s < (1-\delta_2)r_2\,,\\
\|A_6 f\|_{H^{s+(1-\delta_1)r_1+(1-\delta_2)r_2,p}(w)} &\le C\|f\|_{H^{s,p}(w)}\,,\ if\ -\min\{r_2,(1-\delta_1)r_1\} < s\,.
\end{aligned}$$

STEP 5. Finally define

$$\begin{aligned}
A_7 f &= \sum_{k=0}^{\infty}\sum_{i=k+4}^{\infty}\sum_{j=0}^{k-4} Op(a_{i,j,k})\,\tilde{f}_k\,,\\
A_8 f &= \sum_{k=0}^{\infty}\sum_{i=k+4}^{\infty}\sum_{j=k-3}^{k+3} Op(a_{i,j,k})\left(\sum_{l=0}^{k+6}\tilde{f}_l\right),\\
A_9 f &= \sum_{k=0}^{\infty}\sum_{i=k+4}^{\infty}\sum_{j=k+4}^{\infty} Op(a_{i,j,k})\,\tilde{f}_j\,.
\end{aligned}$$

The spectrum of $\sum_{k=0}^{i-4}\sum_j Op(a_{i,j,k})\,g_k$ is contained in the annulus $|\eta| \sim 2^i$. Therefore, we get

$$\begin{aligned}
\|A_7 f\|_{H^{s,p}(w)} &\le C\|f\|_{H^{s-(1-\delta_1)r_1,p}(w)}\,,\ if\ s < r_1\,,\\
\|A_8 f\|_{H^{s,p}(w)} &\le C\|f\|_{H^{s-(1-\delta_1)r_1-(1-\delta_2)r_2,p}(w)}\,,\ if\ s < \min\{r_1,(1-\delta_2)r_2\}\,,\\
\|A_9 f\|_{H^{s,p}(w)} &\le C\|f\|_{H^{s,p}(w)}\,,\ and\ in\ case\ -r_2 < s < r_1\,.
\end{aligned}$$

Since $\sum_{i=1}^{9} A_i f = Op(a)\,f$, the proof is complete ∎

5. Proof of Theorem 2. STEP 1. We decompose $Op(a)\,f$ into 4 parts. Choose a natural number L such that (4) holds with $C = 2^{2L}$ and define

$$\psi(\xi) = \frac{1}{n(2L)!}\sum_{i=1}^{n}\frac{\partial^{2L}}{\partial\xi_i^{2L}}\left(\sum_{j=1}^{n}\xi_j^{2L}\varphi_0(\xi)\right)$$

and $\psi_k(\xi) = \psi(2^{4-k}\xi)$, $k \in N_0$. Note that $\psi_k(0) = 1$ and $\int |z|^{-2L}|F^{-1}\psi(z)|\,dz < \infty$. Then we decompose $Op(a)\,f$ as follows. Let us set

$$\begin{aligned}
a_k^{(1)} &= F_x^{-1}F_y^{-1}(\psi_k\otimes\psi_k F_x F_y a)\varphi_k\,, & a_k^{(2)} &= F_x^{-1}F_y^{-1}(\psi_k\otimes(1-\psi_k)F_x F_y a)\varphi_k\,,\\
a_k^{(3)} &= F_x^{-1}F_y^{-1}((1-\psi_k)\otimes\psi_k F_x F_y a)\varphi_k\,, & a_k^{(4)} &= F_x^{-1}F_y^{-1}((1-\psi_k)\otimes(1-\psi_k)F_x F_y a)\varphi_k\,.
\end{aligned}$$

Then it holds

$$Op(a)\,f = \sum_{k=0}^{\infty}\left(Op(a_k^{(1)})\,\tilde{f}_k + Op(a_k^{(2)})\,f + Op(a_k^{(3)})\,\tilde{f}_k + Op(a_k^{(4)})\,f\right)\,.$$

STEP 2. Similarly to Step 2 in the proof of Theorem 3 one gets

$$\left\|\sum_{k=0}^{\infty} Op(a_k^{(1)})\,\tilde{f}_k\right\|_{L^p(w)} \le C\|f\|_{L^p(w)}\,.$$

STEP3. From $\int F^{-1}\psi_k(z)\,dz = \psi_k(0) = 1$ we get

$$a_k^{(2)}(x,y,\xi) = \int\int F^{-1}\psi_k(x-z_1)F^{-1}\psi_k(y-z_2)(a(z_1,y,\xi)-a(z_1,z_2,\xi))\varphi_k(\xi)\,dz_1dz_2$$

and hence

$$\begin{aligned}\|a_k^{(2)}(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\leq C\int |F^{-1}\psi_k(z)|\omega(|z|,2^k)\,dz\\ &\leq C\int |F^{-1}\psi(z)|\omega(2^{-k}|z|,2^k)\,dz\,.\end{aligned}$$

It follows by induction that $\omega(2^{j-k},2^k) \leq 2^{2L|j|}\omega(2^{-k},2^k)$, $j\in \boldsymbol{Z}$ and therefore

$$\begin{aligned}\|a_k^{(2)}(x,y,2^k\cdot)\|_{B^N_{1,\infty}} &\leq C\sum_{j\in \boldsymbol{Z}} 2^{2L|j|}\int_{|z|\sim 2^j} |F^{-1}\psi(z)|\,dz\,\omega(2^{-k},2^k)\\ &\leq C\int max\{|z|^{-2L},|z|^{2L}\}|F^{-1}\psi(z)|\,dz\,\omega(2^{-k},2^k)\\ &\leq C\omega(2^{-k},2^k)\,.\end{aligned}$$

Since the spectrum of $Op(a_k^{(2)})\,f$ is contained in the annulus $|\eta|\sim 2^k$, we get

$$\left\|\sum_{k=0}^{\infty} Op(a_k^{(2)})\,f\right\|_{L^p(w)} \leq C\left\|\left(\sum_{k=0}^{\infty}\omega(2^{-k},2^k)^2\right)^{1/2} Mf\right\|_{L^p(w)} \leq C\|f\|_{L^p(w)}\,.$$

STEP 4. Analoguosly we get

$$\left\|\sum_{k=0}^{\infty} Op(a_k^{(3)})\,\tilde f_k\right\|_{L^p(w)} \leq C\left(\sum_{k=0}^{\infty}\omega(2^{-k},2^k)^2\right)^{1/2}\left\|\left(\sum_{k=0}^{\infty}(M\tilde f_k)^2\right)^{1/2}\right\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$$

and

$$\left\|\sum_{k=0}^{\infty} Op(a_k^{(3)})\,f\right\|_{L^p(w)} \leq C\sum_{k=0}^{\infty}\Omega(2^{-k},2^{-k},2^k)\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}\,.$$

This completes the proof of the theorem ■

6. Let us add some remarks concerning the sharpness of the theorems. The necessity of $-(1-\delta_1)r_1 < s < r_1$ was shown by *Bourdaud* [4] for symbols independent of y. By duality, it follows that $-r_2 < s < (1-\delta_2)r_2$ is necessary, too. *Wang* and *Li* [10] have constructed a counter-example to the case $N \leq n$. The necessity of $\{\omega(2^{-k},2^k)\}\in l^2$ goes back to *Coifman* and *Meyer* [6]. They used a modulus of continuity $\omega(t)$ independent of t_2, i.e. functions $\omega:\boldsymbol{R}^+\to\boldsymbol{R}^+$ which are monotone, increasing and concave and such that $\omega(0)=0$. Then ω satifies $\omega(\tau)\leq 2\omega(t)$, if $\tau \leq 2t$. In particular, ω is slowly varying and it satisfies $\tau\omega(t)\leq\omega(\tau t)$, if $0<\tau\leq 1$ and $\omega(t)\leq\omega(\tau t)$, if $\tau > 1$.

We prove the necessity of the condition $\{\Omega(2^{-k},2^{-k},2^k)\}\in l^1$. Let Ω be slowly varying and suppose in addition that the functions $t_i \to \Omega(t_1,t_2,t_3)$, i=1,2 are moduli of continuity, an assumption which is reasonable. Let $\Omega_k := \Omega(2^{-k}, 2^{-k}, 2^k)$. We suppose that $n = 1$ and define $a(x,y,\xi) = \sum_{k=1}^{\infty} \Omega_k e^{i2^k(x-y)}\varphi_k(\xi)$. Our remarks on moduli of continuity imply that the symbol a satisfies the inequalities (7) (with $\omega(|h|, 2^k) = \Omega(|h|, 2^{-k}, 2^k)$). We may suppose that $\varphi(\xi) = 1$ on $(1/2, 2)$. If $f \in S(\boldsymbol{R})$ has its spectrum in $(1/2, 2)$, then it follows that $Op(a)\, f = \sum_{k=1}^{\infty} \Omega_k f$. Hence, if $\{\Omega_k\} \notin l^1$, $Op(a)\, f$ does not exist as a distribution.

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