Some Remarks on Trigonometric Interpolation on the n -Torus

W. **SICKEL**

Let T^n be the *n*-torus and $(I_j f)_{j=0}^{\infty}$,

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\n
$$
\Pi^{n} \text{ be the } n\text{-torus and } (I_{j}f)_{j=0}^{\infty},
$$
\n
$$
I_{j}f(x) = \sum_{k_{1}=j}^{j} \cdots \sum_{k_{n}=j}^{j} f\left(\frac{2\pi k}{2j+1}\right) \prod_{i=1}^{n} \frac{\sin\left(\frac{2j+1}{2}\left(x_{i}-\frac{2\pi k_{i}}{2j+1}\right)\right)}{\left(2j+1\right)\sin\frac{1}{2}\left(x_{i}-\frac{2\pi k_{i}}{2j+1}\right)} \left(\frac{x-(x_{1},...,x_{n}) \in \mathbb{R}^{n}}{k-(k_{1},...,k_{n}) \in \mathbb{Z}^{n}}\right)
$$
\nsequence of Lagrange interpolating polynomials. Then we give a complete chara-
\nion of the set of functions f with\n
$$
\left(\sum_{j=1}^{\infty} [j^{s} \|f-j_{j}f| L_{p}(\mathbf{T}^{n})\|]^{q}\right)^{1/q} < \infty \qquad \text{if } 1 \leq p \leq \infty, 0 \leq q \leq \infty, s \geq n/p
$$

the sequence of Lagrange interpolating polynomials. Then **we give a complete characterization of the set of functions f** with

$$
\left(\sum_{j=1}^{\infty} [j^{s}||f - i_{j}f|L_{p}(\mathbf{T}^{n})||]^{q}\right)^{1/q} < \infty
$$
 if $1 < p < \infty, 0 < q \le \infty, s > n/p$
\n
$$
\left\|\sum_{j=1}^{\infty} [j^{s}|f(x) - i_{j}f(x)]^{q}\right)^{1/q} |L_{p}(\mathbf{T}^{n})|| < \infty \quad \text{if } 1 < p < \infty, 0 < q \le \infty, s < n/p
$$

and

$$
\left\|\sum_{j=1}^{\infty} [j^{s}|f(x)-j_{j}f(x)]^{q}\right|^{1/q}\left|L_{p}(\mathbf{T}^{n})\right\|<\infty \quad \text{if } 1
$$

in terms of Besov-Triebel-Lizorkin spaces on T'3.

Key words: *Periodic spaces of Besov-Triebel-Lizorkin type, Lagrange interpolating polynomials, approximation of functions*

AMS **subject classification: 41A25, 42A15, 46E15**

0. Introduction

As usual, \mathbb{R}^n denotes the Euclidean n-space, \mathbb{Z}^n the set of all lattice points having integer components, N the set of all natural numbers and N_0 the set of all non-negative integers. The aim of the paper is to show that the Besov-Triebel-Lizorkin spaces on the n -torus T^n can be completely characterized by the sequence $(I_j f)_{j=0}^{\infty}$ of Lagrange interpolating polynomials. Here *1,f* is given by al, \mathbb{R}^n denotes the Euclidean n -space, \mathbb{Z}^n the set of all lattice points having
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re $I_j f$ is given by
 $\sum_{i=1}^{j} f(\frac{2\pi k}{2j+1}) \prod_{i=1}^{n} \frac{\$ i.e., Z^n the set of all lattice points having inte

bers and N_o the set of all non-negative integ.

Besov-Triebel-Lizorkin spaces on the *n*-to

the sequence $(I_j f)_{j=0}^{\infty}$ of Lagrange interpolat
 $\frac{i+1}{2}x_i - k_i \pi$
 the Euclidean *n*
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is to show that
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is given by
 $\left(\frac{2\pi k}{2j+1}\right)\prod_{i=1}^{n}\frac{s}{(2j+1-i)^{i}}$ uction

Its, Nⁿ denotes the Euclidean *n*-space, Zⁿ the set of all lattice points havin

Ints, N the set of all natural numbers and N_o the set of all non-negative

of the paper is to show that the Besov-Triebel-Lizo

$$
\sum_{k_1=-j}^j \dots \sum_{k_n=-j}^j f\left(\frac{2\pi k}{2j+1}\right) \prod_{i=1}^n \frac{\sin\left(\frac{2j+1}{2}x_i - k_i\pi\right)}{(2j+1)\sin\frac{1}{2}\left(x_i - \frac{2\pi k_i}{2j+1}\right)}, \quad x = (x_1, \dots, x_n) \in \mathbb{T}^n \tag{0.1}
$$

It turns out that for periodic continuous functions *f* the following equivalences are true $(1 < p < \infty, s > n/p)$:

aim of the paper is to show that the Besov-Triebel-Lizorkin spaces on the
\ncan be completely characterized by the sequence
$$
(I_j f)_{j=0}^{\infty}
$$
 of Lagrange into
\nnomials. Here $I_j f$ is given by
\n
$$
\sum_{k_1=-j}^{j} \cdots \sum_{k_n=-j}^{j} f(\frac{2\pi k}{2j+1}) \prod_{i=1}^{n} \frac{\sin(\frac{2j+1}{2}x_i - k_i \pi)}{(2j+1)\sin(\frac{1}{2}(x_i - \frac{2\pi k_i}{2j+1}))}, \quad x = (x_1, ..., x_n) \in \mathbb{T}^n
$$
\n
$$
k = (k_1, ..., k_n) \in \mathbb{Z}^n
$$
\n
$$
p < \infty, s > n/p
$$
\n
$$
f \in B_{p,q}^s(\mathbb{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} ||f - j_f|| \lfloor \frac{1}{p} \rfloor \right)^{1/q} < \infty \qquad (0 < q \leq \infty),
$$
\n
$$
f \in F_{p,q}^s(\mathbb{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} ||f - j_f|| \lfloor \frac{1}{p} \rfloor \right)^{1/q} \left| L_p(\mathbb{T}^n) \right| < \infty \qquad (1 < q < \infty).
$$
\n
$$
f \in F_{p,q}^s(\mathbb{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} ||f - j_f(x)||]^q \right)^{1/q} \left| L_p(\mathbb{T}^n) \right| < \infty \qquad (1 < q < \infty).
$$
\n
$$
f \in F_{p,q}^s(\mathbb{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} ||f - j_f(x)||]^q \right)^{1/q} \left| L_p(\mathbb{T}^n) \right| < \infty \qquad (1 < q < \infty).
$$
\n
$$
f \in F_{p,q}^s(\mathbb{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} ||f - j_f(x)||]^q \right)^{1/q} \left| L_p(\mathbb{T}^n) \right| < \infty \qquad (1 < q < \infty
$$

The main tools of proof used here are the characterization of the underlying function spa-

ces via approximation and the $L_{\mathbf{o}}$ -stability of trigonometric polynomials *t* of degree $t \leq j$ expressed by the following inequalities:

\n
$$
C_1 \left(\frac{1}{(2j+1)^n} \sum_{k_1 = -j}^{j} \dots \sum_{k_p = -j}^{j} \left| t \left(\frac{2\pi k}{2j+1} \right)^p \right|^{1/p} \leq \| t \| L_p(\mathbf{T}^n) \| \leq c_2 \sum_{k_1 = -j}^{j} \dots \sum_{k_p = -j}^{j} \left| t \left(\frac{2\pi k}{2j+1} \right)^p \right|^{1/p}
$$
\n

\n\n (15.8.98) for some constants $C_1 \subset C_2 \geq 0$, independent of t and i (cf. A. Zvemund [19]. P.1).\n

 $(1 < p < \infty)$ for some constants $c_1, c_2 > 0$, independent of t and *j* (cf. A. Zygmund [19], P.I. Lizorkin and D.G. Orlovskij [4]).

The paper is organized as follows:

After collecting some necessary informations about Besov-Triebel-Lizorkin spaces on $Tⁿ$ in the first section, Section 2 deals with our main result concerning the characterization of the function spaces. Therefore we investigate the uniform boudedness of I_j in $\|\cdot\|_{L_p}(\mathbf{T}^n)\|$, $1 \leq p \leq \infty$. As a complement and more or less to show the great similarity between approximation via partial sums and approximation via Lagrange interpolating on **T**ⁿ in the first section, Section 2 deals with our main result concerning the characterization of the function spaces. Therefore we investigate the uniform boudedness of I_j in $\|\cdot |L_p(\mathbf{T}^n)\|$, $1 \le p \le \infty$. As a

1. Besov-Triebel-Lizorkln spaces

1.1 Notations and definitions. The n -torus \mathbf{T}^n may be represented by the set

{ *x E* R': — it x it *(j* I.....n)},

where opposite sides are identified. D_{π} and D_{π} denote the set of all complex-valued infinitely differentiable functions on T^n and its dual space, respectively. Furthermore we put $\{x \in \mathbb{R}^n : -\pi \le x_j \le \pi \ (j = 1,...,n)\},\$

re opposite sides are identified. D_{π} and *i*

y differentiable functions on T^n and its
 $\hat{f}(k) = (2\pi)^{-n} f(e^{-ikx}) \ (k \in \mathbb{Z}^n, f \in D_{\pi}^{\prime}).$

n any $f \in D_{\pi}^{\prime}$ can be repre

$$
\hat{f}(k) = (2\pi)^{-n} f(e^{-ikx}) \ (k \in \mathbb{Z}^n, \ f \in D_{\pi}').
$$

Then any $f \in D'_\pi$ can be represented by its Fourier series

$$
f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx} \text{ (convergence in } D'_\pi)
$$

(cf. H.-J. Schmeißer and H. Triebel [14]). The space of continuous functions on T^n is denoted by $C(T^n)$, the space of p-th power integrable functions by $L_p(T^n)$. If there is no confusion possible we drop T^n in notations. Let $\{x \in \mathbb{R}^n : \neg x \le x_j \le \pi \ (j = 1,...,n) \}$,
 $\{x \in \mathbb{R}^n : \neg x \le x_j \le \pi \ (j = 1,...,n) \}$,
 $\{x \in \mathbb{R}^n : \neg x \le x_j \le \pi \ (j = 1,...,n) \}$,
 $\{x \in \mathbb{R}^n : \neg x \in \mathbb{R}^n : \neg x \in \mathbb{R}^n : \neg x \in \mathbb{R}^n \}$,
 $\{x \in \mathbb{R}^n : \neg x \in \mathbb{R}^n :$ ace of continuous functions on T^n is de-
able functions by $L_p(T^n)$. If there is no
with the properties (1.1)
 $\varphi_1(2^{-1+1}x)$ $(l = 2,3,...)$. (1.2) $\hat{f}(k) e^{ikx}$ (convergence in D'_n)

meißer and H. Triebel [14]). The space of continuous functions on \mathbf{T}^n is de-
 \hat{f} , the space of p -th power integrable functions by $L_p(\mathbf{T}^n)$. If there is no

sible we

Let ψ be an infinitely differentiable function with the properties

$$
\psi(x) = 1 \text{ if } |x| \le 1 \text{ and } \psi(x) = 0 \text{ if } |x| \ge 3/2. \tag{1.1}
$$

Further we put

$$
\psi(x) = 1 \text{ if } |x| \le 1 \text{ and } \psi(x) = 0 \text{ if } |x| \ge 3/2. \tag{1.1}
$$
\n
$$
\text{her we put}
$$
\n
$$
\varphi_0(x) = \psi(x), \varphi_1(x) = \psi(x/2) - \psi(x), \varphi_1(x) = \varphi_1(2^{-1+1}x) \quad (I = 2, 3, \dots). \tag{1.2}
$$

Hence, we have

$$
\sum_{i=0}^{\infty} \varphi_i(x) = 1 \quad (x \in \mathbb{R}^n)
$$

Definition: Let $0 < q \le \infty$ and $-\infty < s < \infty$. (i) Let $1 \le p \le \infty$. Then

$$
\varphi_{0}(x) = \psi(x), \varphi_{1}(x) = \psi(x/2) - \psi(x), \varphi_{1}(x) = \varphi_{1}(2^{-1+x}) \quad (1 = 2, 3, ...).
$$

\nwe, we have
\n
$$
\sum_{l=0}^{\infty} \varphi_{l}(x) = 1 \quad (x \in \mathbb{R}^{n})
$$

\n**Definition:** Let $0 < q \le \infty$ and $-\infty < s < \infty$.
\n(i) Let $1 \le p \le \infty$. Then
\n
$$
B_{p,q}^{s}(\mathbf{T}^{n}) = \left\{ f \in D_{\pi} : ||f||B_{p,q}^{s}|| = \left(\sum_{l=0}^{\infty} 2^{lsq} || \sum_{k \in \mathbb{Z}^{n}} \varphi_{l}(k) \hat{f}(k) e^{ikx} || L_{p} ||^{q} \right)^{1/q} < \infty \right\}.
$$

(ii) Let $1 \leq p \leq \infty$. Then

Trigonometric Interpolation on the *n*-Torus
\n(ii) Let
$$
1 < p < \infty
$$
. Then
\n
$$
F_{p,q}^s(\mathbf{T}^n) = \left\{ f \in D_{\pi} : ||f||F_{p,q}^s|| = ||\left(\sum_{l=0}^{\infty} 2^{lsq} \left|\sum_{k \in \mathbb{Z}} p_l(k) \hat{f}(k) e^{ikx} \right|^q \right)^{1/q} |L_p|| < \infty \right\}.
$$
\nRemark 1: All spaces defined above are quasi-Banach spaces (Banach spaces if

Remark 1: All spaces defined above are quasi-Banach spaces (Banach spaces if $q \ge 1$). They are independent of the special choice of ψ in (1.1) (equivalent quasi-norms). These periodic spaces of Besov-Triebel-Lizorkin type are extensively investigated in the book by H.-J. Schmeißer and H. Triebel [14].

Remark 2: The above definition can be understand as a uniform approach to different types of classical function spaces. In particular, we have (i) $F_{p,2}^o = L_p$,

(ii) $F_{p,2}^o = L_p$,

(ii) $F_{p,2}^m = W_p^m$ (Sobolev spaces) if $m \in \mathbb{N}$, **Remark** 2: The above definition can be understand as a uniform approach to different

s of classical function spaces. In particular, we have

(*i*) $F_{p,2}^{0} = L_p$,

(*ii*) $F_{p,2}^{m} = W_p^{m}$ (Sobolev spaces) if $m \in \mathbb{N}$

$$
(i) F_{p,2}^o = L_p,
$$

-
- (iii) $B_{p,q}^s = A_{p,q}^s$ (Besov-Lipschitz classes) if $s > 0$, and
- (iv) B_{∞}^s = C^s (Hoelder-Zygmund classes) if $s > 0$

(cf. H.-J. SchmeiBer and H. Triebel [14]).

Remark 3: Of some importance are the embedding relations

$$
(B_{p,q}^s \circ F_{p,q}^s) \hookrightarrow L_p \text{ if } s > 0 \tag{1.4}
$$

and

(i)
$$
F_{p,2}^o = L_p
$$
,
\n(ii) $F_{p,2}^m = W_p^m$ (Sobolev spaces) if $m \in \mathbb{N}$,
\n(iii) $B_{p,q}^s = \Lambda_{p,q}^s$ (Besov-Lipschitz classes) if $s > 0$, and
\n(iv) $B_{\infty,\infty}^s = C^s$ (Hoelder-Zygmund classes) if $s > 0$
\n \therefore H.-J. Schmeißer and H. Triebel [14]).
\n**Remark 3:** Of some importance are the embedding relations
\n $(B_{p,q}^s \circ F_{p,q}^s) \hookrightarrow L_p$ if $s > 0$
\n(d
\n $(B_{p,q}^s \circ F_{p,q}^s) \hookrightarrow C$ if $s > n/p$
\n \therefore H.-J. Schmeißer and H. Triebel [14]).

(cf. H.-J. Schmeißer and H. Triebel [14]).

1.2 Characterization via approximation. The spaces defined above are well-adapted to problems in approximation theory. To show this we recall the following facts. Let

$$
T_j = \left\{ t \in D'_\pi \colon \hat{t}(k) = \text{ of } t \text{ at } k \in \mathbb{Z}^n, \, |k| > j \right\} \, (j \in \mathbb{N}_0).
$$

Let X be an appropriate quasi-Banach space. Then we put

$$
E_j(f, X) = \inf_{g \in T_j} \|f - g\|X\| \ (j \in N_o).
$$

Proposition 1: Let $0 < q \leq \infty$.

Proposition 1: Let
$$
0 < q \le \infty
$$
.
\n(i) Let $1 \le p \le \infty$ and $s > 0$. Then
\n
$$
B_{p,q}^s = \left\{ f \in L_p: ||f| L_p|| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q} E_j(f, L_p)]^q \right)^{1/q} < \infty \right\}
$$

in the sense of equivalent quasi-norms.

(ii) Let

$$
E_j(f, X) = \inf_{g \in \mathcal{F}_j} ||f \cdot g|X|| \quad (j \in \mathbb{N}_0).
$$

\nProposition 1: Let $0 < q \le \infty$.
\n(i) Let $1 \le p \le \infty$ and $s > 0$. Then
\n
$$
B_{p,q}^s = \left\{ f \in L_p: ||f|L_p|| + \left(\sum_{f=0}^{\infty} [(1+j)^{s-1/q} E_j(f, L_p)]^q \right)^{1/q} < \infty \right\}
$$

\n*e* sense of equivalent quasi-norms.
\n(ii) Let
\n
$$
S_j f(x) = \sum_{|k_1| \le j} ... \sum_{|k_n| \le j} \hat{f}(k) e^{ikx} \quad (k = (k_1, ..., k_n), j \in \mathbb{N}_0).
$$
\n(1.6)
\ncan replace $E_j(f, L_n)$ by $||f - S_j f|L_n||$ in (i) if $1 \le p \le \infty$.

We can replace $E_j(f, L_p)$ *by* $||f - S_j f| L_p ||$ *in (i) if* $1 < p < \infty$.

 $\label{eq:2} \frac{1}{\sqrt{2}}\int_{0}^{\infty}\frac{1}{\sqrt{2}}\,d\mu_{\mu}$

(iii) *Let*

554 W. SICKEL
\n(iii) Let
\n
$$
V_j f(x) = \sum_{k \in \mathbb{Z}^n} \psi(j^{-1}k) \hat{f}(k) e^{ikx} \quad (j \in \mathbb{N}),
$$
\n(1.7)
\nwhere ψ is the function from (1.1). Then $E_j(f, L_p)$ can be replaced by $||f - V_j f| L_p ||$ in (i).

Proposition 2: Let $1 < p, q < \infty$ and $s > 0$. Then

$$
V_j f(x) = \sum_{k \in \mathbb{Z}^n} \psi(j^{-1}k) f(k) e^{ikx} \quad (j \in \mathbb{N}),
$$

\n
$$
F_{\text{proposition}}(1, 1) \quad \text{Then } E_j(f, L_p) \text{ can be replaced by } \|f - V_j f\|
$$

\nProposition 2: Let $1 < p, q < \infty$ and $s > 0$. Then
\n
$$
F_{p,q}^s = \begin{cases} \n\frac{3}{2} \left(g_j \right)_{j=1}^{\infty}, g_j \in T_j \text{ for } j \in \mathbb{N}, \text{ such that } g_j \to f \text{ in } L_p \text{ and} \\
\frac{3}{2} \left(g_j \right)_{j=1}^{\infty}, g_j \in T_j \text{ for } j \in \mathbb{N}, \text{ such that } g_j \to f \text{ in } L_p \text{ and} \\
\frac{3}{2} \left[g_j \right]_{j=1}^{\infty}, g_j \in T_j \text{ for } j \in \mathbb{N}, \text{ such that } g_j \to f \text{ in } L_p \text{ and} \\
\frac{3}{2} \left[g_j \right]_{j=1}^{\infty} \left(g_j \right) \\
\text{where } g_j = S_j f \left(j \in \mathbb{N} \right). \text{ Remark 4: Proofs of Propositions 1 and 2 may be found in H.-J. Schm\n
$$
F_{\text{per}}(1, 1) \text{ and } W. \text{ Sickel } [15] \text{ (cf. also H.-J. Schmeißer and W. Sickel } [12],
$$
\n
$$
\text{Remark 5: For later use we mention also that } (0 < q < \infty, s \ge 0)
$$
\n
$$
\|f - S_j f| B_{p,q}^s \|_{\frac{1}{j} \to \infty} \to 0 \text{ if } 1 < p < \infty \text{ and } \|f - V_j f| B_{p,q}^s \|_{\frac{1}{j} \to \infty} \to 0 \text{ if } 1 \text{ is an } \infty.
$$
$$

in the sense of equivalent norms. Moreover, we can choose $g_j = S_j f$ *(* $j \in \mathbb{N}$ *).*

Remark *4:* Proofs of Propositions I and 2 may be found in H.-J. SchmeiBer and H. Triebel [14] and W. Sickel [15] (cf. also H.-J. SchmeiBer and W. Sickel [12, 13]).

Remark 5: For later use we mention also that $(0 < q < \infty, s \ge 0)$

$$
\|f - S_j f\|_{p,q}^s \|\xrightarrow[j \to \infty]{} 0 \text{ if } 1 < p < \infty \text{ and } \|f - V_j f\|_{p,q}^s \|\xrightarrow[j \to \infty]{} 0 \text{ if } 1 \leq p \leq \infty.
$$

These are consequences of

$$
\hat{f}(k) - S_j \hat{f}(k) = 0
$$
 if $|k_j| < j$ $(i = 1,...,n)$ and $\hat{f}(k) - \hat{V_j} f(k) = 0$ if $|k| < j$

and of

$$
\sup_j \|S_j f(L_p\| \le c \|f|L_p\| \ (1 < p < \infty) \quad \text{and} \quad \sup_j \|V_j f(L_p\| \le c \|f|L_p\| \ (1 \le p \le \infty).
$$

2. Trigonometric interpolation

We start with a uniform lattice on \mathbf{T}^n , characterized by the nodes

$$
\sup_{j} ||S_{j}f||_{L_{p}} || \leq c ||f||_{L_{p}} || (1 \leq p \leq \infty) \text{ and } \sup_{j} ||V_{j}f||_{L_{p}} || \leq c ||f||_{L_{p}} || (1 \leq p \leq \infty)
$$

riigonometric interpolation
start with a uniform lattice on Tⁿ, characterized by the nodes

$$
x^{r} = (x_{1}^{r},...,x_{n}^{r}) = \left(\frac{2\pi r_{1}}{2j+1},...,\frac{2\pi r_{n}}{2j+1}\right) (-j \leq r_{i} \leq j (i = 1,...,n), j \in \mathbb{N}_{0}), r \in \mathbb{Z}^{n}.
$$

$$
Q_{m}^{j} = \left\{ k \in \mathbb{Z}^{n}: -j-1/2 \leq k_{i} - m_{i}(2j+1) \leq j+1/2 (i = 1,...,n) \right\} (m \in \mathbb{Z}^{n}, j \in \mathbb{N})
$$

$$
f \in C. \text{ Then the function } I_{j}f \text{ defined by (0.1) is the unique solution of}
$$

$$
g(x^{r}) = f(x^{r}), r \in Q_{0}^{j} \text{ and } \hat{g}(k) = 0 \text{ if } k \in Q_{0}^{j}.
$$

$$
\text{pose additionally}
$$

$$
\sum_{k \in \mathbb{Z}^{n}} |\hat{f}(k)| < \infty.
$$

$$
\text{we can rewrite}
$$

Let

$$
x^{r} = (x_{i}^{r},...,x_{n}^{r}) = \left(\frac{2x_{i}}{2j+1},..., \frac{2x_{i}}{2j+1}\right) (-j \le r_{i} \le j (i = 1,...,n), j \in \mathbb{N}_{0}), r \in \mathbb{Z}^{n}.
$$

$$
Q_{m}^{j} = \left\{k \in \mathbb{Z}^{n}: -j-1/2 \le k_{i} - m_{i}(2j+1) \le j+1/2 (i = 1,...,n)\right\} (m \in \mathbb{Z}^{n}, j \in \mathbb{N}_{0})
$$

and $f \in C$. Then the function $I_j f$ defined by (0.1) is the unique solution of

$$
g(x^r) = f(x^r), r \in Q_0^j \text{ and } \hat{g}(k) = 0 \text{ if } k \in Q_0^j.
$$

Suppose additionally

$$
\sum_{k \in \mathbb{Z}^n} |\hat{f}(k)| < \infty. \tag{2.1}
$$

Then we can rewrite

$$
Q_m^j = \{k \in \mathbb{Z}^n : -j - 1/2 \le k_i - m_i(2j + 1) \le j + 1/2 \ (i = 1, ..., n)\} \ (m \in \mathbb{Z}^n, j \in \mathbb{N}_0)
$$

$$
f \in C. \text{ Then the function } I_j f \text{ defined by (0.1) is the unique solution of}
$$

$$
g(x^r) = f(x^r), r \in Q_0^j \text{ and } \hat{g}(k) = 0 \text{ if } k \in Q_0^j.
$$

pose additionally

$$
\sum_{k \in \mathbb{Z}^n} |\hat{f}(k)| < \infty.
$$

(2.1)

$$
\sum_{m \in \mathbb{Z}^n} \left| \sum_{k \in Q_m^j} \hat{f}(k) e^{ikx} \right| e^{-im(2j+1)x} \ (j \in \mathbb{N}_0)
$$

(2.2)
A. Zygmund [19]).

(cf. A. Zygmund [19]).

Remark 6: Formula (2.2) shows the great similarity between Lagrange polanomials and Whittaker's cardinal series. The latter one is defined as

Trigonometric Interpolati
\n**Remark 6:** Formula (2.2) shows the great similarity between
\nWhittaker's cardinal series. The latter one is defined as
\n
$$
I_j^* f(x) = \sum_{k \in \mathbb{Z}^n} f\left(\frac{2\pi k}{2j+1}\right) \prod_{i=1}^n \frac{\sin\left(\frac{2j+1}{2}x_i - k_i\pi\right)}{\left(\frac{2j+1}{2}x_i - k_i\pi\right)} \qquad (x \in \mathbb{R}^n).
$$
\nhave the identity

We have the identity

$$
I_j^*f(x) = \sum_{k \in \mathbb{Z}^n} \left(F^{-1} \left[\chi_m^j F f \right] \right)(x) e^{-im(2j+1)x}
$$

where F, F^{-1} are the Fourier transform and its inverse, respectively, and χ_m^j denotes the characteristic function of Q_m^j (cf. P.L. Butzer [3], W. Sickel [16]).

Remark 7: If we put $A_{\pi} = \{f \in D_{\pi}: \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)| < \infty\}$, then $B_{2,1}^{n/2} \hookrightarrow A_{\pi} \hookrightarrow B_{\infty,1}^0(\text{cf.})$ H. Triebel [17]). $\int f(x) \, dx = \int f(x) \, dx$
 ourier transform and its inverse, respectively, and $\chi \frac{d}{dt}$ denotes the
 i of $Q^{\frac{1}{2}}_m$ (cf. P.L. Butzer [3], W. Sickel [16]).
 ut $A_{\pi} = \{f \in D_{\pi}: \sum_{k \in \mathbb{Z}} n | \hat{f}(k) | < \infty \}$, then B **Remark** 7: If we put $A_{\pi} = \{f \in D_{\pi} : \sum_{k \in \mathbb{Z}^n} |f(k)| < \infty\}$, then $B_{2,1}^{n/2} \hookrightarrow A_{\pi} \hookrightarrow B_{\infty,1}^0$ (cf.

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In our investigations a crucial role is played by the following
 Lemma 1: Let $1 \le p \le \infty$.

In our investigations a crucial role is played by the following

Lemma 1: $Let 1 < p < \infty$. *(i) There exists a constant c such that*

$$
||I_j f| L_p || \le c(1+j)^{-n/p} ||f| |B_{p,1}^{n/p} || \quad (j \in \mathbb{N}_0)
$$
\n(2.3)

holds for all f $\in B_{p,1}^{n/p}$ with $\hat{f}(k) = 0$, $k \in Q_0^j$.

(ii) There exists a constant c such that

$$
||f - I_j f| L_p || \le c(1+j)^{-n/p} ||f| B_{p,1}^{n/p} || (j \in N_0)
$$
 (2.4)

holds for all $f \in B_{p,1}^{n/p}$ *.*

Proof: First, note that I_i is a projection, that means $I_i f = f$ for all f with $\hat{f}(k) = 0$, $k \in Q_0^{\cdot j}$. Now we split

$$
f - I_i f = f - S_i f + I_i (S_i f - f). \tag{2.5}
$$

 $|l_j f|L_p|| \le c(1+j)^{-n/p} ||f|B_{p_1}^{n/p}|| (j \in N_0)$ (2.3)
 for all f $\in B_{p_1}^{n/p}$ *with* $\hat{f}(k) = 0$, $k \in Q_0^j$.
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 $|f - l_j f|L_p|| \le c(1+j)^{-n/p} ||f|B_{p_1}^{n/p}|| (j \in N_0)$ (2.4)
 for all f To prove (2.4) we can use Proposition 1 and (2.3). So, it remains to prove (2.3). Let $\{\varphi_i\}$ (ii) There exists a constant c such that
 $||f - I_j f| L_p|| \le c(1 + j)^{-n/p} ||f| B_{p,1}^{n/p} || (j \in N_0)$

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 $k \in Q_0^j$. Now we split
 $f - I_j f = f - S_j f + I_j$ *p₁(k)f²(k)e^{ikx} (1∈ N₀). Let 2^{<i>t*} ≤ *j* ≤ 2^{t+1} . The properties of ψ guarantee $I_j(S_j f - f) = \sum_{i=t-1}^{\infty} I_j(S_j f_i - f_i)$ in D'_π . Applying (0.2), we find *c*
ise Proposition
in (1.2). We pu
ψ guarantee
property of I_j
 $c(1+j)^{-n/p}$ *I_j*(*S_jf -f*).

1 use Proposition 1 and (2

1 in (1.2). We put $f_i(x) = \sum$

5 ψ guarantee $I_j(S_jf - f)$

1 property of I_jf , $I_jf(k) =$
 $\leq c(1+j)^{-n/p} \left(\sum_{k \in Q_0} |I_j(S_j) \right)$
 $\leq c(1+j)^{-n/p} \left(\sum_{k \in Q_0} |C_j f_k(z)| \right)$
 $\leq c$

$$
2^{t+1}
$$
. The properties of ψ guarantee $I_j(S_j f - f) = \sum_{i=t-1}^{\infty} I_j(S_j f_i - f_i)$ in D_{π} . Applying (0.2),
using the interpolation property of $I_j f$, $I_j f(k) = 0$ if $k \in Q_0^j$ and $\hat{f}_j(k) = 0$ if $|k| > (3/2)2^{1-1}$
we find

$$
||I_j(S_j f_1 - f_1)||_{L_p}|| \le c(1+j)^{-n/p} \Big(\sum_{k \in Q_0^j} |I_j(S_j f_1 - f_1)(\frac{2\pi k}{2j+1})|^{p}\Big)^{1/p}
$$
(2.6)

$$
\le c(1+j)^{-n/p} \Big(\sum_{k \in Q_0^j} |(S_j f_1 - f_1)(\frac{2\pi k}{2j+1})|^{p}\Big)^{1/p},
$$

where c is independent of j, l , and f . Next we pick out a sequence of meshes $\{M_l\}$ such that

$$
\Big\{ \Big(\frac{2\pi k}{2j+1}\Big): k \in \mathbb{Z}^n \Big\} \subset M_l = \Big\{ \Big(\frac{2\pi k}{2M_{l+1}}\Big): k \in \mathbb{Z}^n \Big\} \quad (l = t-1, ...),
$$

$$
\left\{\left(\frac{2\pi k}{2j+1}\right): k\in\mathbb{Z}^n\right\} \subset M_I = \left\{\left(\frac{2\pi k}{2M_{I+1}}\right): k\in\mathbb{Z}^n\right\} \ (I = t-1,\dots),
$$

where $(3/2)2^{1-\frac{1}{2}} \le M_1 \le c2^1$ (c independent of *I* and *t*) holds. According to M_1 we apply

again (0.2). This leads to

w.
$$
SICKEL
$$

\nn (0.2). This leads to
\n $\left(\sum_{k \in Q_0^j} |(S_j f_1 - f_1)(\frac{2\pi k}{2j+1})|^p\right)^{1/p} \le c2^{ln/p} \|S_j f_1 - f_1|L_p\| \le c2^{ln/p} \|f_1|L_p\|.$ (2.7)
\ning (2.7) into (2.6), summing up from $t - 1$ to ∞ the desired inequality (2.3) follows **Example**
\n**Remark 8:** Using Remark 5 we can sharper (2.4) a little bit. We have
\n $j^{n/p} \|f - I_j f|L_p\| \to 0$ if $j \to \infty$ (2.8)
\nany $f \in B_{p,1}^{n/p}$. In case $n = 1$ this was observed first by K.I. Oskolkov [5].
\n**Remark 9:** In the one-dimensional case J. **Prestin** [7 - 10] has proved a result similar

Putting (2.7) into (2.6), summing up from $t - 1$ to ∞ the desired inequality (2.3) follows **E**

Remark 8: Using Remark 5 we can sharpen (2.4) a little bit. We have

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Remark 9: In the one-dimensional case J. Prestin [7 - 10] has proved a result similar to (2.8), but with $B_{p,1}^{1/p}(\mathbf{T}^1)$ replaced by the set of functions with bounded variation.

As a consequence of Lemma I one obtains some estimates of the approximation error in $\|\cdot|C\|$.

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\nRemark 9: In the one-dimensional case J. Prestin [7 - 10] has proved a result similar
\n2.8), but with $B_{p,1}^{1/p}(T^4)$ replaced by the set of functions with bounded variation.
\nAs a consequence of Lemma 1 one obtains some estimates of the approximation error
\n $|\mathcal{C}||$.
\nLemma 2: Let $p < \infty$ and $s > 0$.
\n(i) For any $f \in B_{p,1}^{n/p} \cup A_{\pi}$, there holds
\n $||f - I_j f | \mathcal{C}|| \rightarrow 0$ as $j \rightarrow \infty$.
\n(ii) There exists a constant c such that
\n $\sup_{j \in N_0} (1 + j)^s ||f - I_j f | \mathcal{C}|| \le c ||f |B_{p,\infty}^{s+n/p} ||$ for all $f \in B_{p,\infty}^{s+n/p}$.
\n(iii) There exists a constant c such that
\n $\sup_{j \in N_0} (1 + j)^s |[0, (1 + j))^n ||f - I_j f | \mathcal{C}|| \le c ||f |C^s||$ for all $f \in C^s = B_{\infty,\infty}^s$.
\n(2.10)
\nProof: (i) Let $f \in A_{\pi}$. Then (2.9) follows from (2.2) since

(ii) There exists a constant c such that

$$
\sup_{j\in\mathbf{N}_0} (1+j)^s \|f - I_j f\| C \| \le c \|f\|_{\mathcal{P},\infty}^{s+n/\mathcal{P}} \|\text{ for all } f \in B_{\mathcal{P},\infty}^{s+n/\mathcal{P}}.
$$
 (2.10)

(iii) There exists a constant c such that

$$
\sup_{j \in \mathbb{N}_0} (1+j)^s (\log(1+j))^{-n} \|f - I_j f\| C \| \le c \|f\| C^s \| \text{ for all } f \in C^s = B^s_{\infty, \infty}.
$$
 (2.11)

Proof: (i) Let $f \in A_{\pi}$. Then (2.9) follows from (2.2) since

 $|f(x) - I_j f(x)| \le \sum_{|m|>0} \sum_{k \in Q} \frac{j}{k} |\hat{f}(k)|$.

Let $f \in B_{\rho,1}^{\,n \neq \rho}$. Then we use the decomposition

$$
f - I_j f = f - V_{j/2} f + I_j (V_{j/2} f - f),
$$
\n(2.12)

It is a constant c such that
 $\int f^2 |C|| \le c ||f| |B_{p,\infty}^{s+n/p}||$ for a

It is a constant c such that
 $\log(1+j))^{-n} ||f - I_j f |C|| \le c ||f|$
 $\int f \in A_{\pi}$. Then (2.9) follows from
 $\int \sum_{|m|>0} \sum_{k \in Q_m} f(k) |C_{m}$.

we use the decompositio with $V_{j/2}$ defined in (1.7). From the embeddings $B_{p,1}^{n/p} \hookrightarrow B_{\infty,1}^{\circ} \hookrightarrow C$ (cf. H.-J. Schmeißer and H. Triebel [14]) and Remark 5 we know that (iii) There exists a constant c such that

sup $(1+j)^s(\log(1+j))^{-n}||f - I_jf||C|| \le c||$
 $j \in N_o$
 Proof: (i) Let $f \in A_\pi$. Then (2.9) follows $f(x) - I_jf(x)|| \le \sum_{|m|>o} \sum_{k \in Q_m} \left| \hat{f}(k) \right|$.
 $f \in B_{p,1}^{n/p}$. Then we use the decompos Then we use the decomposition
 $f - V_{j/2}f + I_j(V_{j/2}f - f)$, (2.12)

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el [14]) and Remark 5 we know that
 $f|C|| \rightarrow 0$ as $j \rightarrow \infty$. (2.13)

$$
\|f - V_{i/2}f\|C\| \to 0 \text{ as } j \to \infty. \tag{2.13}
$$

Next we apply the Nikol'skij inequality (cf. H.-J. Schmeißer and H. Triebel [14]) and (2.3). This yields

$$
\big\| I_j \big(V_{j'2} f - f \big) \big| C \big\| \le c \big(1+j \big)^{n/p} \big\| I_j \big(V_{j'2} f - f \big) \big| L_p \big\| \le c \, \big\| f - V_{j'2} f \, \big| B_{p,1}^{n/p} \big\|.
$$

Using again Remark S we find

$$
||I_j(V_{j\prime 2}f - f)|C|| \to \text{as } j \to \infty.
$$
 (2.14)

Now, (2.13) and (2.14) complete the proof of (2.9).

(ii) We use the splitting stated in (2.12), Proposition 1, and (2.3). This yield:

$$
V_r
$$
, (2.13) and (2.14) complete the proof of (2.9).
\n(ii) We use the splitting stated in (2.12), Proposition 1, and (2.3). This $|f - I_j f| \subset ||s||f - V_{j/2}f| \subset ||t - V_{j/2}f| \cdot ||f - V_{j/2}f| \cdot ||g_{p,1}^{n/p}|| \leq c(1+j)^s ||f| |g_{p,\infty}^{n/p+s}||$.

Frigonometric interpolation on the *n*-Torus 557

For the last step we have used on the one hand the embedding $B_{\rho,\infty}^{n/p+s} \subset B_{\infty,\infty}^s = C^s$ and

on the other hand Proposition 3 (see Section 3). This proves (2.10).

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we obtain

the last step we have used on the one hand the embedding
$$
B_{p,\infty}^{n/p+s} \hookrightarrow B_{\infty,\infty}^s = C^s
$$
 and
\ne other hand Proposition 3 (see Section 3). This proves (2.10).
\n(iii) Since the first part of inequality (0.2) remains true if $p = 1$ (cf. A. Zygmund [19])
\nbtain
\n
$$
I_j f(x) = \left(\frac{1}{2j+1}\right)^n \sum_{r \in Q_j} \sum_{k \in Q_j} f(x^r) e^{ik(x-x)^r} \Big|
$$
\n
$$
\leq \sup_{r \in Q_0} |f(x^r)| \left(\frac{1}{2j+1}\right)^n \sum_{r \in Q_j} \sum_{k \in Q_j} e^{ik(x-x)^r} \Big|
$$
\n
$$
\leq c ||f| C || \sum_{k \in Q_j} e^{ikx} |L_1|| \leq c (\log(1+j))^n ||f| C ||.
$$
\ng this with $f - V_{j/2}f$ instead of f , the desired inequality follows from (2.12) as in (ii) **Example**
\nThe main result of this paper is formulated in the next
\n**Theorem 1:** Let $1 \leq p \leq \infty$ and $s > n/p$.
\n(i) Let $0 \leq q \leq \infty$. Then
\n
$$
B_{p,q}^s = \left\{ f \in C : |f(0)| + \left(\sum_{j=0}^{\infty} [1+j)^{s-j/q} ||f - I_jf| L_p ||]^q \right)^{1/q} \leq \infty \right\}
$$
\nwe sense of equivalent quasi-norms.

Using this with $f - V_{j/2}f$ instead of f, the desired inequality follows from (2.12) as in (ii) \blacksquare

The main result of this paper is formulated in the next

Theorem 1: Let $1 < p < \infty$ and $s > n/p$. *(i)* $Let \ 0 \leq q \leq \infty$. Then

(i) Let
$$
0 < q \le \infty
$$
. Then
\n
$$
B_{p,q}^s = \left\{ f \in C: |f(0)| + \left(\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} ||f - I_j f| L_p || \right]^q \right)^{1/q} < \infty \right\}
$$

in the sense of equivalent quasi-norms.

(ii) Let $1 < q < \infty$ *. Then*

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$$
\ne sense of equivalent quasi-norms.
\n(ii) Let $1 < q < \infty$. Then
\n
$$
F_{p,q}^s = \left\{ f \in C: |f(0)| + \left\| \sum_{j=0}^{\infty} \left[1 + j \right)^{s-1/q} |f(x)|^q \right\}^{1/q} |L_p| < \infty \right\}
$$
\ne sense of equivalent norms.
\n**Proof:** (i) Comparing the above characterization of $B_{p,q}^s$ with
\nove that
\n $f(0)| + \left(\sum_{j=0}^{\infty} \left[1 + j \right)^{s-1/q} ||f - I_jf|L_p|| \right]^q \right\}^{1/q} \le c ||f||B_{p,q}^s||$
\nc independent of f . Let $0 < q < \infty$. Again we use (2.3) and Pro

in the sense of equivalent norms.

to prove that

$$
F_{p,q}^{s} = \left\{ f \in C: |f(0)| + \left\| \sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} |f(x)|^{q} \right]^{1/q} \middle| L_{p} \right\| < \infty \right\}
$$

we sense of equivalent norms.
Proof: (i) Comparing the above characterization of $B_{p,q}^{s}$ with Proposition 1 it remains
rove that

$$
|f(0)| + \left(\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \left\| f - I_{j} f \right| L_{p} \right\| \right]^{q} \right)^{1/q} \le c \left\| f \left| B_{p,q}^{s} \right\|
$$
(2.15)

with *c* independent of f. Let $0 < q < \infty$. Again we use (2.3) and Proposition 1. This leads to

(ii) Let
$$
1 < q < \infty
$$
. Then
\n
$$
F_{p,q}^s = \left\{ f \in C : |f(0)| + \left\| \sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} |f(x)|^q \right)^{1/q} |L_p \right\| < \infty \right\}
$$
\n
$$
\text{ence, for } q \text{ equivalent norms.}
$$
\n**Proof:** (i) Comparing the above characterization of $B_{p,q}^s$ with Proposition 1 it remains
\nover that\n
$$
|f(0)| + \left(\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \|f - I_j f| L_p \right] \right)^q \right)^{1/q} \le c \|f| B_{p,q}^s \| \qquad (2.15)
$$
\nc independent of f . Let $0 < q < \infty$. Again we use (2.3) and Proposition 1. This leads to
\n
$$
\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \|f - I_j f| L_p \right] \Big|^q
$$
\n
$$
\le \sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \left(\|f - S_j f| L_p \right\| + \|I_j (f - S_j f)| L_p \right] \Big|^q \qquad (2.16)
$$
\n
$$
\le c \|f| B_{p,q}^s \| \le \sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \left(\|f - S_j f| L_p \right\| + \|I_j (f - S_j f)| L_p \right] \Big|^q.
$$
\n
$$
\text{proced with an estimate of the second term on the right-hand side of (2.16) Using}
$$

We proceed with an estimate of the second term on the right-hand side of (2.16). Using $f_l(x) = \sum_{k \in \mathbb{Z}^D} \varphi_l(k) \hat{f}(k) e^{ikx}$ ($l \in \mathbb{N}_0$) we find

$$
\leq c \|f| B_{p,q}^s\|^q + \sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} (1+j)^{-n/p} \|f - S_j f| B_{p,1}^{n/p} \| \right]^q.
$$

proceed with an estimate of the second term on the right-hand side

$$
= \sum_{k \in \mathbb{Z}^n} \varphi_l(k) \hat{f}(k) e^{ikx} (I \in \mathbb{N}_0) \text{ we find}
$$

$$
\left\{ \sum_{t=0}^{\infty} \sum_{j=2^{t}-1}^{2^{t+n}-2} 2^{t(s-n/p)q} 2^{-t} \left(\sum_{l=t-1}^{\infty} 2^{ln/p} \| (S_j f - f)_l |L_p \| \right)^q \right\}^{\min(1, q)/q}
$$

$$
\leq c \sum_{l=0}^{\infty} \left(\sum_{t=0}^{\infty} 2^{t(s-n/p)q} 2^{(1+t)qn/p} \left\| f_{1+t-1} |L_p \right\|^q \right)^{\min(1, q)/q}
$$
\n
$$
\leq c \sum_{l=0}^{\infty} 2^{-l(s-n/p)\min(1, q)} \left(\sum_{t=0}^{\infty} 2^{t sq} \left\| f_t |L_p \right\|^q \right)^{\min(1, q)/q}
$$
\n
$$
\leq c \left\| f \left| B_{p,q}^s \right\| \right)^{\min(1, q)},
$$
\n(2.17)

since *s> n/p* and

$$
\sup_{j} \|(S_{j}f)_{l}|L_{p}\| = \sup_{j} \|S_{j}(f_{l})|L_{p}\| \leq c \|f_{l}|L_{p}\|
$$

(put $\varphi_{-1} = 0$). Note that $I_0f = f(0)$. In view of this fact, Lemma 1 and (2.16), (2.17) the desired inequality (2.15) follows if $q < \infty$. In case $q = \infty$ one has to modify the above considerations in an obvious way. s c $||f||B_{p,q}^{s}||^{min(1, q)}$,

e s > n/p and

sup $||(S_j f)_l|L_p|| = \sup_j ||S_j(f_j)|L_p|| \le c ||f_j||$,

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red inequality (2.15) follows if $q < \infty$. In contains in an obvious way.

(ii) Using Prop *I I* and (2.16), (2.17) the

one has to modify the above con-
 *I*blish the inequality
 $\leq c \left\| f \left| F_{p,q}^s \right| \right\|$. (2.18)

the case $s > n$. Because of $F_{p,q}^s \right\|$

(ii) Using Proposition 2 the proof is reduced to establish the inequality

$$
|f(0)| + \left\| \left(\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} |f(x) - I_j f(x)| \right]^q \right)^{1/q} L_p \right\| \leq c \|f| F_{p,q}^s \|.
$$
 (2.18)

Step 1: In order to prove (2.18) we consider at first the case $s \ge n$. Because of $F_{\alpha, \alpha}^s$ A_{π} (cf. Remark 7) we can apply (2.2). This yields

Step 1: In order to prove (2.18) we consider at first the case
$$
s > n
$$
. Because
\n
$$
[f(x) - f(f(x))] = f(x) - f(x) - \sum_{|m|>0} \left(\sum_{k \in \mathbb{Z}^n} \chi_m^j(k) \hat{f}(k) e^{ikx} \right) e^{-ikx}.
$$

where χ^{j}_{m} is the characteristic function of $Q^{j}_{m}.$ With the help of Proposition 1 a proof of

Step 1: In order to prove (2.18) we consider at first the case
$$
s > n
$$
. Because of $F_{p,q}^s \hookrightarrow A_{\pi}$ (cf. Remark 7) we can apply (2.2). This yields
\n $f(x) - I_j f(x) = f(x) - S_j f(x) - \sum_{|m|>0} \left(\sum_{k \in \mathbb{Z}^n} \chi_m^j(k) \hat{f}(k) e^{ikx} \right) e^{-i x m (2j + 1)},$
\nwhere χ_m^j is the characteristic function of Q_m^j . With the help of Proposition 1 a proof of
\n(2.18) is now reduced to a proof of
\n
$$
\left\| \left(\sum_{j=0}^{\infty} \left[(1+j)^{s-1/q} \Big| \sum_{|m|>0} \left(\sum_{k \in \mathbb{Z}^n} \chi_m^j(k) \hat{f}(k) e^{ikx} \right) e^{-i x m (2j + 1)} \right] \right]^q \right\}^{1/q} |L_p| \le c \|f| |F_{p,q}^s|.
$$
\n(2.19)
\nIn order to obtain (2.19) we make use of Lizorkin's vector-valued Fourier-multiplier theo-
\nrem for cubes with sides parallel to the axis (cf. H - 1. Schmeilfer and H. Triabel [14]) and

In order to obtain (2.19) we make use of Lizorkin's vector-valued Fourier-multiplier theorem for cubes with sides parallel to the axis (cf. H.-J. SchmeiBer and H. Triebel [14]) and of w reduced to a proof of
 $\left(1+j\right)^{s-1/q}\Big|\sum_{|m|>0}\Big(\sum_{k\in\mathbb{Z}^n}\chi_m^j(k)\Big)$

obtain (2.19) we make use obes with sides parallel to th
 $K_{t+1}^{N_0, N_1}$ if $2^t - 1 \le j \le 2^{t+1}$

$$
Q_m^j \in K^{N_0, N_1}_{t+1} \text{ if } 2^t - 1 \leq j \leq 2^{t+1} - 2, 2^l \leq |m| < 2^{l+1},
$$

where

In order to obtain (2.19) we make use of Lizorkin's vector-valued Fourier-multiplier the
rem for cubes with sides parallel to the axis (cf. H.-J. Schmeißer and H. Triebel [14]) an
of

$$
Q_m^j \in K_{t+1}^{N_0,N_1}
$$
 if $2^t - 1 \le j \le 2^{t+1} - 2$, $2^l \le |m| < 2^{l+1}$,
where
 $K_0^{N_0,N_1} = \{x : |x_j| \le 2^{-N_1} (j = 1,...,n) \}$,
 $K_t^{N_0,N_1} = \{x : |x_j| \le 2^{t+N_0} (j = 1,...,n) \} \setminus \{x : |x_j| \le 2^{t-1-N_1} (j = 1,...,n) \}$ (t \in N)
for appropriate $N_0, N_1 \in N_0$. Let $\chi(K_t^{N_0,N_1}, \cdot)$ be the characteristic function of $K_t^{N_0,N}$.
These yields

$$
\iint_{\infty}^{\infty} 2^{t+1-2} 2^{-t} \cdot \iint_{\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y^{j} (k) \hat{f}(k) e^{ikx} |e^{-ixm(2j+1)}|^q V^{[q]} I}{\prod_{i=1}^{n} (k - 1) \cdot \iint_{\infty}^{\infty} |f^{(i)}(k)|^q} d|I_{ij}^{\frac{1}{2}}
$$

These yields

$$
Q_{m}^{j} \in K_{t+1}^{N_{0},N_{1}} \text{ if } 2^{t} - 1 \leq j \leq 2^{t+1} - 2, 2^{l} \leq |m| < 2^{l+1},
$$
\n
$$
K_{0}^{N_{0},N_{1}} = \left\{ x : |x_{j}| \leq 2^{-N_{1}} \ (j = 1,...,n) \right\},
$$
\n
$$
K_{t}^{N_{0},N_{1}} = \left\{ x : |x_{j}| \leq 2^{t+N_{0}} \ (j = 1,...,n) \right\} \setminus \left\{ x : |x_{j}| \leq 2^{t-1-N_{1}} \ (j = 1,...,n) \right\} \ (t \in \mathbb{N})
$$
\n
$$
\text{appropriate } N_{0}, N_{1} \in \mathbb{N}_{0}. \text{ Let } \chi(K_{t}^{N_{0},N_{1}}, \cdot) \text{ be the characteristic function of } K_{t}^{N_{0},N_{1}}.
$$
\n
$$
\left\| \left(\sum_{i=0}^{\infty} 2^{t s q} \sum_{j=2^{l-1}}^{2^{t+1}-2} 2^{-t} \right| \sum_{|m|>0} \left(\sum_{k \in \mathbb{Z}^{n}} \chi_{m}^{j}(k) \hat{f}(k) e^{ikx} \right) e^{-i x m (2j+1)} \right|^{q} \right\}^{N/q} \left| L_{p} \right|\right\}
$$
\n
$$
\leq \sum_{l=0}^{\infty} \sum_{2^{j} \leq |m| \leq 2^{j+1}} \left\| \left(\sum_{i=0}^{\infty} 2^{t s q} \sum_{j=2^{l-1}}^{2^{t+1}-2} 2^{-t} \right| \sum_{k \in \mathbb{Z}^{n}} \chi_{m}^{j}(k) \chi(K_{t+1}^{N_{0},N_{1}},k) \hat{f}(k) e^{ikx} \right|^{q} \right\}^{N/q} \left| L_{p} \right|\right\}
$$
\n(2.20)

Trigonometric Interpolatic
\n
$$
s \ c \sum_{j=0}^{\infty} 2^{jn} \left\| \left(\sum_{r=0}^{\infty} 2^{r} sq \right| \sum_{k \in \mathbb{Z}^n} \chi(K_{r+1}^{N_0, N_1}, k) \hat{f}(k) e^{ikx} \right|^q \right\}^{1/q} |L_p|
$$
\n
$$
s \ c \sum_{j=0}^{\infty} 2^{l(n-s)} \|f| F_{p,q}^s \| \le c \|f| F_{p,q}^s \|,
$$
\n
$$
V = \text{trivial to a so-called Lizorkin-type representation of } F_{p,q}^s \text{ (cf. 14]).}
$$
\n
$$
Step 2: \text{We remove the restriction } s > n. \text{ Note, that } B_{p,p}^s = F_{p}^s
$$
\n
$$
F_{p_0,q_0}^s, F_{p_1,q_1}^s \bigg|_{\Theta} = F_{p,q}^s \qquad s = (1 - \Theta)
$$
\n
$$
[L_{\Omega}(A), L_p(B)]_{\Theta} = L_p([A, B]_{\Theta}) \qquad \text{with} \qquad \frac{1}{p} = \frac{1 - \Theta}{p_0}
$$

according to a so-called Lizorkin-type representation of $F_{p,q}^s$ (cf. H.-J. Schmeißer and H. Triebel [141). *Step 2* :We remove the restriction S **>** *n.* Note, that *^B ⁵*= Furthermore, we have

$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|f| F_{p,q}^{s} \| \le c \|f| F_{p,q}^{s} \|,
$$

\n
$$
= 1 \ \text{or } s \text{ or called } \text{Lizorkin-type representation of } F_{p,q}^{s} \ (cf. H.-J. SchmeiBer and H.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|f| F_{p,q}^{s} \| \le c \|f| F_{p,q}^{s} \|,
$$

\n
$$
s \ t \text{ for all } s \geq n.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|f| F_{p,q}^{s} \| \le c \|f| F_{p,q}^{s} \|,
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|f| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|f| F_{p,q}^{s} \| \le c \|f| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{i=0}^{\infty} 2^{i(n-s)} \|F| F_{p,q}^{s} \|.
$$

\n
$$
s \ c \ \sum_{
$$

(cf. Triebel [18]). We shall use (2.21) with $A_j = j^{s_0} \mathbb{C}$, $B_j = j^{s_1} \mathbb{C}$, and $A = I_{q_0}(A_j)$, $B = I_{q_1}(B_j)$. $\left[l_{q_0}(A_j), l_{q_1}(B_j)\right]_{\mathfrak{B}} = l_q([A_j, B_j]_{\mathfrak{B}})$ $\frac{1}{q} = \frac{1-\mathfrak{B}}{q_0} + \frac{\mathfrak{B}}{q_1}$
(cf. Triebel [18]). We shall use (2.21) with $A_j = j^{s_0} \mathbb{C}$, $B_j = j^{s_1} \mathbb{C}$, and $A = i$
Here \mathbb{C} is the complex plane Here C is the complex plane. Considering the linear operator $R: F_{p,q}^s \to L_p(I_q(j^{s-1/q})\mathbb{C})$, $Rf = \{f - I_j f\}_{j=0}^\infty$ we know from the proof of (i) and from Step 2 that *R* is bounded if *s* > *n/p* and $p = q$ or $s > n$ and $1 < p, q < \infty$. Hence, *R* is bounded as a mapping with respect to the intermediate spaces $R: F_{p,q}^s \to L_p(I_q(j^{s-1/q})\mathbb{C}))$ (1 < $p,q < \infty; s > n/p$). That means, (2.18) is true also under these restrictions \blacksquare

Remark 10: The restriction $s > n/p$ in Theorem 1 seems to be natural. If $s < n/p$, then unbounded functions are contained in $B_{\rho,q}^s$ and hence, I_jf makes no sense in general.

Remark 11: Parts of the assertions of Theorem 1 and of the Lemmas 1 and 2 are known = 1. We refer to J. Prestin [7, 10] and K.I. Oskolkov [5]. Corresponding results in case thit taker's cardinal series are obtained in S if $n = 1$. We refer to J. Prestin [7,10] and K.I. Oskolkov [5]. Corresponding results in case of Whittaker's cardinal series are obtained in Sickel [16]. ediate spaces $R: F_{p,q}^S \to L_p(I_q(j^{s-1/q})\mathbb{C}))$ $(1 < p,q < \infty; s > n/p)$. That means, (2.18)
ie also under these restrictions \blacksquare
Remark 10: The restriction $s > n/p$ in Theorem 1 seems to be natural. If $s < n/p$, then
unded functi

we can employ an inequality due to Leindler [3]. Let $0 < \mu < \infty$. Then \mathcal{N} e a
an en

$$
\left\| \left(2^{-1} \sum_{j=2}^{2^{I+1}-1} |f(x) - S_j f(x)|^{\mu} \right)^{1/\mu} \right| L_{\infty}(\mathbf{T}^1) \right\| \le c \, E_{2^j}(f, C(\mathbf{T}^1)), \tag{2.22}
$$

where c is independent of f and l ϵ $\mathsf{N_{o}}.$ This implies

We are also interested in a characterization of function spaces if
$$
p = \infty
$$
. To this end
\ncan employ an inequality due to Leindler [3]. Let $0 < \mu < \infty$. Then
\n
$$
\left\| \left(2^{-1} \sum_{j=2}^{2^{n}-1} |f(x) - S_j f(x)|^{\mu} \right)^{j} / \mu \right| L_{\infty}(\mathbf{T}^1) \leq c E_{2j}(f, C(\mathbf{T}^1)),
$$
\n(2.22)
\n
$$
\text{For } c \text{ is independent of } f \text{ and } l \in \mathbb{N}_0. \text{ This implies}
$$
\n
$$
\sup_{l \in \mathbb{N}_0} 2^{(1+1/\mu)l} \left\| \left(2^{-1} \sum_{j=2}^{2^{l}+1} |f(x) - l_j f(x)|^{\mu} \right)^{j} / \mu \right| C(\mathbf{T}^1) \leq c \| f |B_{\infty,1}^{1+1/2}(\mathbf{T}^1) \| \tag{2.23}
$$

and

re c is independent of f and
$$
I \in \mathbb{N}_{0}
$$
. This implies\n\n
$$
\sup_{I \in \mathbb{N}_{0}} 2^{(1+1/\mu)I} \left\| \left(2^{-I} \sum_{j=2}^{2^{I+1}-1} |f(x) - I_{j}f(x)|^{\mu} \right)^{i/\mu} |C(\mathbf{T}^{i})| \right\| \leq c \left\| f \left| B_{\infty,1}^{i+1/\mu}(\mathbf{T}^{i}) \right\| \tag{2.23}
$$
\n
$$
\sup_{I \in \mathbb{N}_{0}} 2^{sI} \left\| \left(2^{-I} \sum_{j=2^{I}}^{2^{I+1}-1} |f(x) - I_{j}f(x)|^{\mu} \right)^{i/\mu} |C(\mathbf{T}^{i})| \right\| \leq c \left\| f \left| C^{s}(\mathbf{T}^{i}) \right\| \right\} \tag{2.24}
$$
\n
$$
\sup_{i \in \mathbb{N}} 2^{sI} \left\| \left(2^{-I} \sum_{j=2^{I}}^{2^{I+1}-1} |f(x) - I_{j}f(x)|^{\mu} \right)^{i/\mu} |C(\mathbf{T}^{i})| \right\| \leq c \left\| f \left| C^{s}(\mathbf{T}^{i}) \right\| \right\} \tag{2.24}
$$
\n
$$
\sup_{i \in \mathbb{N}} 2^{sI} \left\| \left(2^{-I} \sum_{j=2^{I}}^{2^{I+1}-1} |F(x) - I_{j}f(x)|^{\mu} \right)^{i/\mu} |C(\mathbf{T}^{i})| \right\| \leq c \left\| f \left| C^{s}(\mathbf{T}^{i}) \right\| \right\} \tag{2.25}
$$

if $1 \le \mu \le \infty$ and $s > 1 + 1/\mu$. Extending (2.24) to I_{α} -norms one obtains a characterization of $B_{\infty, q}^s(\mathbf{T}^1)$.

Theorem 2: Let $1 \le \mu \le \infty$, $0 \le q \le \infty$ and $s > 1/\min(1,q) + 1/\mu$. Then

$$
B_{\infty,\,q}^{s}(\mathbf{T}^{\mathbf{i}})=\bigg\{f\in C(\mathbf{T}^{\mathbf{i}})\colon\bigg\|f\,\big|C(\mathbf{T}^{\mathbf{i}})\big\|
$$

Let

\n
$$
+ \left(\sum_{i=0}^{\infty} 2^{isq} \left\| \left(2^{-i \sum_{j=2}^{i} |f(x) - f_j f(x)|^{\mu} \right)^{i/\mu}} \right| C(T^i) \right\|^{q} \right)^{1/q} < \infty
$$
\nequivalent quasi-norms.

\nAssetions of this type with $I_i f$ replaced by $S_i f$ may be the

in the sense of equivalent quasi-norms.

Remark 12: Assertions of this type with $I_j f$ replaced by $S_j f$ may be found in H.-J. Schmeißer and W. Sickel [13].

3. Approximation in Besov and Sobolev norms

In several papers the approximation order of $f-I_jf$ is studied in stronger norms than $\|\cdot L_p\|$ (cf. R. Haverkamp [2], J. Prestin [7 - 10], S. ProBdorf and B. Silbermann [ii]). The results derived in the preceding section can be generalized in a convenient way. The first step in doing this is the following characterization of Besov spaces (cf. A. Pietsch [61). papers the approxinute exproxing [2], J. Pre
the preceding sect
is the following charge is the following charge of
sition 3: Let $1 \le p \le$
 ℓe have
 $= \left\{ f \in B_{p, q_1}^t : ||f||B \right\}$
see of equivalent que

Proposition 3: Let $1 \le p \le \infty$, $0 \le q_0, q_1 \le \infty$, and $t, s \ge 0$.

(i) We have

\n We'd in the preceding section can be generalized in a convenient way. If this is the following characterization of Besov spaces (cf. A. Pietsch **Proposition 3:** Let
$$
1 \leq p \leq \infty
$$
, $0 < q_0$, $q_1 \leq \infty$, and $t, s > 0$.\n

\n\n (i) We have\n \[\n \left\|\n \int_{p_1, q_0}^{s+t} \mathbf{f} \left[f \left[B_{p_1, q_1}^t \right] \right] \mathbf{f} \left[B_{p_1, q_1}^t \right] + \left(\sum_{t=1}^{\infty} \left[f^{s-1/q_0} E_j(f, B_{p_1, q_1}^t) \right]^{q_0} \right)^{1/q_0} < \infty \right\}\n \]\n

in the sense of equivalent quasi-norms.

(ii) If
$$
1 \le p \le \infty
$$
, then $E_j(f, B_{p,q_1}^t)$ can be replaced by $||f - S_j f||B_{p,q_1}^t||$ in (i).

As a consequence of this proposition and Theorem I we obtain the following

Then we have

Theorem 3: Let
$$
1 < p < \infty
$$
, $0 < q_0$, $q_1 \le \infty$, $t \ge 0$ and $s > 0$. Let additionally $s + t > n/p$.
\n*n* we have
\n
$$
B_{p, q_0}^{s+t} = \left\{ f \in B_{p, q_1}^t: \|f\| B_{p, q_1}^t\| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-t/q_0}] \|f - I_j f\| B_{p, q_1}^t\|]^{q_0} \right)^{1/q_0} < \infty \right\}
$$

in the sense of equivalent quasi-norms.

Proof: By Proposition 3 it is sufficient to prove
\n
$$
||f||B_{p,q_1}^t|| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q_0} ||f - I_j f| B_{p,q_1}^t ||]^{q_0}\right)^{1/q_0} \le c ||f|| B_{p,q_0}^{s+t}||
$$

for some constant *c,* independent of *f.* Therefore, we use the splitting from (2.5). Again by applying Proposition 3 it suffices to consider the term $I_j(S_jf-f)$. Let 2^v \leq j < 2^{v+1}. Then (2.3) implies

Proof: By Proposition 3 it is sufficient to prove
\n
$$
||f|B_{p,q_1}^t|| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q_0} ||f - I_j f|B_{p,q_1}^t||]^{q_0}\right)^{1/q_0} \le c ||f|B_{p,q_0}^{s+t}||
$$
\nsome constant c, independent of f. Therefore, we use the splitting from (lying Proposition 3 it suffices to consider the term $I_j(S_j f - f)$. Let $2^V \le$.)
\nimplies
\n
$$
||I_j(S_j f - f)||B_{p,q_1}^t|| \le \left(\sum_{j=0}^{V+1} 2^{Itq_1} || \sum_{k \in \mathbb{Z}^n} \varphi_j(k) I_j(S_j f - f)(k) e^{ikx} ||_{L_p} ||q_1|^{1/q_1}
$$
\n
$$
\le c(1+j)^t ||I_j(S_j f - f)||_{L_p} || \le c(1+j)^t (||f - S_j f||_{L_p} || + ||f - I_j f||_{L_p} ||).
$$

This leads to

$$
\leq c(1+j)^t \|I_j(S_jf - f)\|L_p\| \leq c(1+j)^t
$$

s leads to

$$
\left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q_0}] \|f - I_jf\|B_{p,q_1}^t\|]^{q_0}\right)^{1/q_0}
$$

ζV

Trigonometric Interpolation on the *n*-Torus
\n
$$
\leq c \left(\left\| f \left| B_{p,\,q_0}^{s+t} \right\| + \left(\sum_{j=0}^{\infty} [(1+j)^{s+t} (||f - S_j f|L_p|| + ||f - I_j f|L_p||) \right]^{q_0} \right)^{1/q_0} \right) \leq c \left\| f \left| B_{p,\,q_0}^{s+t} \right\|
$$
\n*e* $s + t > n/p$ ensures that Theorem 1 can be applied **B**

since $s + t > n/p$ ensures that Theorem 1 can be applied **B**

Remark 13: As a consequence of embeddings for Besov-Triebel-Lizorkin spaces on the *n*-torus one obtains characterizations of $B_{\rho,q}^s$ via approximation by Lagrange inters $c \left(||I||B_{p,q_0}||^* \left(\sum_{j=0}^{\lfloor (1+j)/ \rfloor} (||I - 5_jI||L_p|| + ||I - 1_jI||L_p||) \right) \right)$ s $c \left(||I||B_{p,q_0}|| \right)$
since $s + t > n/p$ ensures that Theorem 1 can be applied **if**
Remark 13: As a consequence of embeddings for Besov-Triebel-Liz Theorem 1 as an application of Theorem 3. Furthermore, by $B_{p,1}^t \nightharpoonup W_p^t \nightharpoonup B_{p,\infty}^t$ ($t \in \mathbb{N}$) one can replace B_{p,q_1}^t in (3.1) by the Sobolev spaces W_p^t . This improves some results of S. Prößdorf and B. Silbermann [11] and J. Prestin [7,10].

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562 W. SICKEL

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