Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11 (1992) 1, 57 – 76

# Some Results on the Invertibility of Wiener-Hopf-Hankel Operators

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A study is presented on the invertibility properties of scalar operators defined as the sum of a Wiener-Hopf and a Hankel operator on $L_2(\mathbb{R}^+)$ with symbols in $L_\infty(\mathbb{R})$. This study is based on the properties of a vector Wiener-Hopf operator naturally associated with each of the operators mentioned above. The results obtained are applied to problems in Diffraction Theory.

Key words: *Wiener-Hopf operators, Hankel operators, factorization, diffraction theory*
AMS subject classification: 47 B 35, 47 A 68, 78 A 45

## 1. Introduction

Let $L_2^\pm(\mathbb{R})$ denote the subspaces of $L_2(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}^\pm=\{x\in\mathbb{R}: \pm x>0\}$, such that $L_2(\mathbb{R})=L_2^+(\mathbb{R})\oplus L_2^-(\mathbb{R})$ holds. Further, let $\mathcal{P}^\pm$ be the complementary projection operators associated with this direct sum decomposition and denote by $\mathcal{F}$ the Fourier-Plancherel operator on $L_2(\mathbb{R})$,

$$\mathcal{F}\varphi(\xi)=\int_{-\infty}^{+\infty}\varphi(x)e^{i\xi x}dx \quad , \quad \xi\in\mathbb{R}. \tag{1.1}$$

We will consider Wiener-Hopf-Hankel operators [12], i.e., operators of the form

$$\mathcal{W}(a)+\mathcal{H}(b): L_2^+(\mathbb{R})\to L_2^+(\mathbb{R}) \tag{1.2}$$

where $\mathcal{W}(a)$ is a Wiener-Hopf operator, defined by

$$\mathcal{W}(a)=\mathcal{P}^+\overset{0}{\mathcal{W}}(a)\,|_{L_2^+(\mathbb{R})}: L_2^+(\mathbb{R})\to L_2^+(\mathbb{R}) \quad , \quad \overset{0}{\mathcal{W}}(a)=\mathcal{F}^{-1}a\,\mathcal{F}: L_2(\mathbb{R})\to L_2(\mathbb{R}) \tag{1.3}$$

and $\mathcal{H}(b)$ is the Hankel operator

$$\mathcal{H}(b)=\mathcal{P}^+\overset{0}{\mathcal{W}}(b)\mathcal{J}\,|_{L_2^+(\mathbb{R})}: L_2^+(\mathbb{R})\to L_2^+(\mathbb{R}). \tag{1.4}$$

Here $\mathcal{J}$ stands for the reflection operator, given almost everywhere in $\mathbb{R}$ by

(*)Sponsored by the Deutsche Forschungsgemeinschaft under grant number KO 634/32-1

$$J\varphi(x)=\varphi(-x). \tag{1.5}$$

The symbols $a$ and $b$ of the convolution operators $\overset{0}{\mathcal{W}}(a)$ and $\overset{0}{\mathcal{W}}(b)$ are supposed to be elements of $L_\infty(\mathbb{R})$. For $a$ we impose the additional condition of having a generalized factorization relative to $L_2$, so that the Wiener-Hopf operator $\mathcal{W}(a)$ is Fredholm (cf.[13]).

The Fredholm theory for operators of the form (1.2) with the above assumptions is consequently trivial in the case where $b$ is a continuous function on the one point compactification of $\mathbb{R}$, i.e., $b\in C(\dot{\mathbb{R}})$. Indeed, the last condition is well known to be a necessary and sufficient condition for the compactness of $\mathcal{H}(b)$ (cf.[7]). Nevertheless, even in this rather simple situation, in general the nullity and defect numbers cannot be determined, and in particular no efficient criteria is available for the invertibility of the operator (1.2).

Moreover, if $a,b\in PC(\dot{\mathbb{R}})$ (the algebra of all piecewise continuous functions on $\dot{\mathbb{R}}$, supposed continuous from the left), then the Fredholm theory for the correspondent Wiener-Hopf-Hankel operators is also known. In this case, operators (1.2) are unitary equivalent to singular integral operators on the unit circle $\Gamma$ with Carleman shift (the mapping $z\mapsto z^{-1}$ on $\Gamma$), see [14],[19], and the algebra generated by these operators has been studied by Gohberg and Krupnik [4],[5],[6]. They have obtained necessary and sufficient conditions for the Fredholmness of those operators in terms of a fourth order matrix-valued symbol, which yields as well their total index (see also [3],[9],[21] and the references cited therein).

We further refer to the more recent and general approach of Roch and Silbermann [15],[16], which developed a unified theory for the study of different algebras of convolution type operators generated by several classes of piecewise continuous functions.

All the works cited above, in the context of Banach algebras techniques, yield the images of the different algebras in the correspondent Calkin algebra and therefore give complete descriptions of the Fredholm properties of the operators under consideration, up to the knowledge of the partial indices. Hence, naturally, by these methods no information can be obtained about the invertibility of the operators involved.

The aim of the present work is precisely to provide some possible invertibility criterions for the Wiener-Hopf-Hankel operators (1.2), generalizing the results formerly obtained in [19] for the particular case where $a$ is a complex constant.

Following [19], to each operator (1.2) we associate in a rather natural way a vector Wiener-Hopf operator $\mathcal{W}(G)$, acting on $[L_2^+(\mathbb{R})]^2$, which can be diagonalized by two (at least one-sided) invertible operators $\mathcal{A}$ and $\mathcal{B}$, such that the operator $\mathcal{A}\mathcal{W}(G)\mathcal{B}$ is the direct sum of the identity operator on $L_2^+(\mathbb{R})$ and a scalar operator $\mathcal{S}$, closely related to the original Wiener-Hopf-Hankel operator (see section 2).

In section 3 we relate the Fredholm properties and invertibility of $\mathcal{W}(G)$, known from the general theory of Wiener-Hopf operators [2],[13], with those of $\mathcal{W}(a)\pm\mathcal{H}(b)$, showing in particular that if $a$ has a canonical generalized factorization,then the invertibility of $\mathcal{W}(G)$ is equivalent to the simultaneously invertibility of $\mathcal{W}(a)+\mathcal{H}(b)$ and $\mathcal{W}(a)-\mathcal{H}(b)$.

The results obtained so far are applied, in section 4, to some problems arising in Diffraction Theory [11],[12].

## 2. The Wiener-Hopf operator associated with $\mathcal{W}(a) + \mathcal{H}(b)$.

In this section we associate with a given Wiener-Hopf-Hankel operator (1.2) a vector Wiener-Hopf operator $\mathcal{W}(G)$ acting on $[L_2^+(\mathbb{R})]^2$, with presymbol $G \in [L_\infty(\mathbb{R})]^{2\times 2}$. The connection between the two operators will be established by reducing $\mathcal{W}(G)$ to a diagonal form.

Let $a,b \in L_\infty(\mathbb{R})$ and suppose that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$ (cf.[7]). Consider in $L_2^+(\mathbb{R})$ the equation

$$(\mathcal{W}(a) + \mathcal{H}(b))\varphi^+ = f^+ \tag{2.1}$$

and suppose that $\varphi^+$ is a solution to this equation, which can be written in the equivalent form

$$\overset{0}{\mathcal{W}}(a)\varphi^+ + \overset{0}{\mathcal{W}}(b)\mathcal{J}\varphi^+ = f^+ + \psi^- \tag{2.2}$$

for $\psi^- = \mathcal{P}^-\mathcal{F}^{-1}(a+b\mathcal{J})\mathcal{F}\varphi^+ \in L_2^-(\mathbb{R})$. The use of the Fourier transformation and the relation $\mathcal{F}\mathcal{J} = \mathcal{J}\mathcal{F}$ yields

$$a\,\hat{\varphi}^+ + b\,\mathcal{J}\,\hat{\varphi}^+ = \hat{f}^+ + \hat{\psi}^- \tag{2.3}$$

with $\hat{\varphi}^+ = \mathcal{F}\varphi^+$, $\hat{f}^+ = \mathcal{F}f^+$ and $\hat{\psi}^- = \mathcal{F}\psi^-$. Now, applying the reflection operator $\mathcal{J}$ to both sides of the last equation, we further obtain

$$\tilde{a}\,\mathcal{J}\,\hat{\varphi}^+ + \tilde{b}\,\hat{\varphi}^+ = \mathcal{J}\,\hat{f}^+ + \mathcal{J}\,\hat{\psi}^-. \tag{2.4}$$

Here and in the sequel we use the notation $\tilde{a} = \mathcal{J}a$ *and* $\tilde{b} = \mathcal{J}b$.

The equations (2.3) and (2.4) have the matrix form

$$\begin{bmatrix} a & 0 \\ \tilde{b} & -1 \end{bmatrix}\begin{bmatrix} \hat{\varphi}^+ \\ \mathcal{J}\hat{\psi}^- \end{bmatrix} + \begin{bmatrix} b & -1 \\ \tilde{a} & 0 \end{bmatrix}\begin{bmatrix} \mathcal{J}\hat{\varphi}^+ \\ \hat{\psi}^- \end{bmatrix} = \begin{bmatrix} \hat{f}^+ \\ \mathcal{J}\hat{f}^+ \end{bmatrix}. \tag{2.5}$$

By hypothesis the function $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$. This implies, in particular, that the matrix-valued functions appearing in (2.5) are invertible in $[L_\infty(\mathbb{R})]^{2\times 2}$ (cf. [13]). Let

$$C = \begin{bmatrix} b & -1 \\ \tilde{a} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -\tilde{a} & b \end{bmatrix}\tilde{a}^{-1}. \tag{2.6}$$

Multiplying by $C$ both sides of (2.5), we get the equivalent system of equations

$$G\,\hat{\phi}^+ + \mathcal{J}\,\hat{\phi}^+ = C\,\hat{F} \tag{2.7}$$

where $\hat{\phi}^+ = \mathcal{F}\phi^+$, $\hat{F} = \mathcal{F}F$, with

$$\phi^+ = \begin{bmatrix} \varphi^+ \\ \mathcal{J}\psi^- \end{bmatrix} \in [L_2^+(\mathbb{R})]^2 \quad , \quad F = \begin{bmatrix} f^+ \\ \mathcal{J}f^+ \end{bmatrix} \in [L_2(\mathbb{R})]^2 \tag{2.8}$$

and $G$ is the matrix-valued function

$$G = \begin{bmatrix} \tilde{a}^{-1}\tilde{b} & -\tilde{a}^{-1} \\ -a+b\tilde{a}^{-1}\tilde{b} & -b\tilde{a}^{-1} \end{bmatrix}. \tag{2.9}$$

If we now use the inverse Fourier transformation in equation (2.7), it holds

$$\overset{0}{\mathcal{W}}(G)\,\phi^+ + \mathcal{J}\,\phi^+ = \overset{0}{\mathcal{W}}(C)\,F \tag{2.10}$$

where $\overset{0}{\mathcal{W}}(G)=\mathcal{F}^{-1}G\,\mathcal{F}$ and $\overset{0}{\mathcal{W}}(C)=\mathcal{F}^{-1}C\,\mathcal{F}$ are the convolution operators on $[L_2(\mathbb{R})]^2$ with symbols $G$ and $C$, respectively. Noting that $\mathcal{J}\psi^-=(\mathcal{W}(\tilde{b}) + \mathcal{H}(\tilde{a}))\varphi^+$ due to $\mathcal{J}\mathcal{P}^-=\mathcal{P}^+\mathcal{J}$ (see (2.3)), we have proved the following result.

PROPOSITION 2.1: *Let $a,b\in L_\infty(\mathbb{R})$ and $F$ be given by (2.8). Suppose that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$. Then the equations (2.1) and (2.10) are equivalent in the following sense:*

*(i) If $\varphi^+$ is a solution of (2.1) then*

$$\phi^+=[\varphi^+,\varphi_1{}^+]^T$$

*with $\varphi_1{}^+ = (\mathcal{W}(\tilde{b}) + \mathcal{H}(\tilde{a}))\varphi^+$, is a solution of (2.10).*

*(ii) If $\phi^+=[\varphi^+,\varphi_1{}^+]^T$ is a solution of (2.10) then $\varphi^+$ is a solution of (2.1).*

*Moreover, equation (2.1) is uniquely solvable iff (2.10) is uniquely solvable.*

We immediately recognize that any solution of equation (2.10) is also a solution to the Wiener-Hopf equation

$$\mathcal{W}(G)\,\phi^+ = \mathcal{P}^+\overset{0}{\mathcal{W}}(C)\,F$$

where $\mathcal{W}(G):[L_2^+(\mathbb{R})]^2\to[L_2^+(\mathbb{R})]^2$ is the Wiener-Hopf operator

$$\mathcal{W}(G)=\mathcal{P}^+\overset{0}{\mathcal{W}}(G)\,|_{[L_2^+(\mathbb{R})]^2} \tag{2.11}$$

(here and in the sequel we also denote by $\mathcal{P}^\pm$ the complementary projection operators on $[L_2(\mathbb{R})]^2$ onto $[L_2^\pm(\mathbb{R})]^2$, defined componentwise). The operator $\mathcal{W}(G)$ will be called the Wiener-Hopf operator associated with $\mathcal{W}(a) + \mathcal{H}(b)$ (see [19]). In the remaining part of this section we are going to establish relations between these two operators.

To this end let us introduce some notation and recall basic results. We assumed that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$, which implies that it can be written as (cf. [7],[13])

$$a=a_-\,u_\upsilon\,a_+$$

where, for $r_\pm=(\xi\pm i)^{-1}$, $\xi\in\mathbb{R}$, it holds $\mathcal{F}^{-1}(r_+a_+^{\pm 1})\in L_2^+(\mathbb{R})$ and $\mathcal{F}^{-1}(r_-a_-^{\pm 1})\in L_2^-(\mathbb{R})$. Further $u_\upsilon=[(\xi-i)/(\xi+i)]^\upsilon$, $\xi\in\mathbb{R}$, with $\upsilon\in\mathbf{Z}$. The number $\upsilon$ is called the index of the function $a$ relative to $L_2(\mathbb{R})$

and we write $\upsilon$= ind$a$ (if $a\in C(\dot{\mathbb{R}})$ then $\upsilon$ coincides with the winding number of $a(\xi)$ with respect to the origin). We will use the notation

$$a_0=u_{-\upsilon}a=a_-a_+$$

and we shall refer to this factorization of $a_0$ as a canonical one (cf. [7]). Note that $\mathcal{W}(a_0)$ is an invertible operator, with inverse given by

$$\mathcal{W}^{-1}(a_0)=\mathcal{F}^{-1}a_+^{-1}\mathcal{F}\,\mathcal{P}^+\,\mathcal{F}^{-1}a_-^{-1}\mathcal{F}_{|L_2^+(\mathbb{R})}. \tag{2.12}$$

Moreover $\mathcal{W}(a)$ is left invertible or right invertible according to $\upsilon\geq 0$ or $\upsilon\leq 0$, respectively, with Fredholm index ind $\mathcal{W}(a)=-\upsilon$ (see [7],[13]). The following conventions will also be used:

$$\mathcal{U}_\upsilon=\mathcal{W}(u_\upsilon):L_2^+(\mathbb{R})\to L_2^+(\mathbb{R})$$

and

$$\mathcal{U}_\upsilon^{(2)}=\mathcal{W}(\mathrm{diag}\{u_\upsilon, u_\upsilon\}):[L_2^+(\mathbb{R})]^2\to[L_2^+(\mathbb{R})]^2.$$

Recall that ind $\mathcal{U}_\upsilon=-\upsilon$ and ind $\mathcal{U}_\upsilon^{(2)}=-2\upsilon$ (cf. [7],[13]). Let $\mathcal{I}$, $\mathcal{I}^+$ be the identity operators on $L_2(\mathbb{R})$, $L_2^+(\mathbb{R})$, respectively. A straightforward computation shows that

$$\mathcal{U}_{-|\upsilon|}\mathcal{U}_{|\upsilon|}=\mathcal{I}^+.$$

This relation will be often useful in what follows.

Let $G_0$ denote the matrix-valued function defined by (2.9) with $a$ and $\tilde{a}$ replaced by $a_0$ and $\tilde{a}_0$, respectively. Then it is easily seen that

$$\mathcal{W}(G)=\begin{cases}\mathcal{W}(G_0)\,\mathcal{U}_\upsilon^{(2)} & ,\ if\ \upsilon\geq 0\\ \mathcal{U}_\upsilon^{(2)}\,\mathcal{W}(G_0) & ,\ if\ \upsilon\leq 0\end{cases} \tag{2.13}$$

where $\mathcal{W}(G_0)$ is the Wiener-Hopf operator defined by (2.11), with $G$ replaced by $G_0$.

Now consider the representation of $\mathcal{W}(G_0)$ as a 2×2 matrix of scalar operators (see (2.9))

$$\mathcal{W}(G_0)=\begin{bmatrix}\mathcal{W}(\tilde{a}_0^{-1}\tilde{b}) & -\mathcal{W}(\tilde{a}_0^{-1})\\ -\mathcal{W}(a_0)+\mathcal{W}(b\tilde{a}_0^{-1}\tilde{b}) & -\mathcal{W}(\tilde{a}_0^{-1}\tilde{b})\end{bmatrix}:L_2^+(\mathbb{R})\oplus L_2^+(\mathbb{R})\to L_2^+(\mathbb{R})\oplus L_2^+(\mathbb{R}). \tag{2.14}$$

We are going to prove that $\mathcal{W}(G_0)$ can be diagonalized by means of invertible operators. In the proof of the theorem we shall use the relations

$$\mathcal{W}(ab)=\mathcal{W}(a)\,\mathcal{W}(b)+\mathcal{H}(a)\,\mathcal{H}(\tilde{b}) \tag{2.15}$$

$$\mathcal{H}(ab)=\mathcal{W}(a)\,\mathcal{H}(b)+\mathcal{H}(a)\,\mathcal{W}(\tilde{b}) \tag{2.16}$$

for all $a,b\in L_\infty(\mathbb{R})$, which can be directly obtained from [1, 2.14 Proposition] using the canonical isometry between $L_2(\Gamma)$ ($\Gamma$ being the unit circle) and $L_2(\mathbb{R})$, see [1, Section 9.1].

THEOREM 2.2: *Let $a_0,b\in L_\infty(\mathbb{R})$ and suppose that $a_0$ admits a canonical generalized factorization relative to $L_2(\mathbb{R})$. Further let $\mathcal{A}_0,\mathcal{B}_0: L_2^+(\mathbb{R})\oplus L_2^+(\mathbb{R})\to L_2^+(\mathbb{R})\oplus L_2^+(\mathbb{R})$ be the invertible operators given by*

$$\mathcal{A}_0=\begin{bmatrix} -\mathcal{W}(b\tilde{a}_0^{-1})\mathcal{W}^{-1}(\tilde{a}_0^{-1}) & \mathcal{I}^+ \\ \mathcal{W}^{-1}(\tilde{a}_0^{-1}) & 0 \end{bmatrix}, \quad \mathcal{B}_0=\begin{bmatrix} \mathcal{I}^+ & 0 \\ \mathcal{W}^{-1}(\tilde{a}_0^{-1})\mathcal{W}(\tilde{a}_0^{-1}\tilde{b}) & -\mathcal{I}^+ \end{bmatrix}. \tag{2.17}$$

*Then it holds*

$$\mathcal{A}_0\mathcal{W}(G_0)\,\mathcal{B}_0=\begin{bmatrix} \mathcal{S}_0 & 0 \\ 0 & \mathcal{I}^+ \end{bmatrix} \tag{2.18}$$

*with*

$$\mathcal{S}_0=-(\mathcal{W}(a_0)+\mathcal{H}(b))\,\mathcal{W}^{-1}(a_0)\,(\mathcal{W}(a_0)-\mathcal{H}(b)) \tag{2.19}$$

*where the order of the outer factors can be reversed.*

Proof: First we note that the assumption made on $a_0$ implies that $\mathcal{W}(a_0)$ and $\mathcal{W}^{-1}(\tilde{a}_0^{-1})$ are invertible operators and so the operators $\mathcal{A}_0$ and $\mathcal{B}_0$ are well defined. Further note that these operators are invertible, with inverses given by

$$\mathcal{A}_0^{-1}=\begin{bmatrix} 0 & \mathcal{W}(\tilde{a}_0^{-1}) \\ \mathcal{I}^+ & \mathcal{W}(b\tilde{a}_0^{-1}) \end{bmatrix}, \quad \mathcal{B}_0^{-1}=\begin{bmatrix} \mathcal{I}^+ & 0 \\ -\mathcal{W}^{-1}(\tilde{a}_0^{-1})\mathcal{W}(\tilde{a}_0^{-1}\tilde{b}) & -\mathcal{I}^+ \end{bmatrix}. \tag{2.20}$$

After some direct computations, we get (2.18) with

$$\mathcal{S}_0=-\mathcal{W}(a_0)+\mathcal{W}(b\tilde{a}_0^{-1}\tilde{b})-\mathcal{W}(b\tilde{a}_0^{-1})\mathcal{W}^{-1}(\tilde{a}_0^{-1})\mathcal{W}(\tilde{a}_0^{-1}\tilde{b}). \tag{2.21}$$

Therefore it remains to prove (2.19). To this end let us deduce from (2.16) some useful relations. Substituting in (2.16) $a$ by $\tilde{a}_0^{-1}$ and $b$ by $\tilde{a}_0$, we have

$$0=\mathcal{W}(\tilde{a}_0^{-1})\,\mathcal{H}(\tilde{a}_0)+\mathcal{H}(\tilde{a}_0^{-1})\,\mathcal{W}(a_0).$$

Since $\mathcal{W}(a_0)$ and $\mathcal{W}(\tilde{a}_0^{-1})$ are invertible operators, applying to both sides of this identity $\mathcal{W}^{-1}(\tilde{a}_0^{-1})$ on the left and $\mathcal{W}^{-1}(a_0)$ on the right, we obtain

$$\mathcal{W}^{-1}(\tilde{a}_0^{-1})\,\mathcal{H}(\tilde{a}_0^{-1})=-\mathcal{H}(\tilde{a}_0)\,\mathcal{W}^{-1}(a_0). \tag{2.22}$$

Replacing in (2.16) $a$ by $b\tilde{a}_0^{-1}$ and $b$ by $\tilde{a}_0$, we get

$$\mathcal{H}(b) = \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{H}(\bar{a}_0) + \mathcal{H}(b\bar{a}_0^{-1})\,\mathcal{W}(a_0). \tag{2.23}$$

Consider now the third term in (2.21). Using the identities (2.15),(2.22) and (2.23) we obtain successively:

$$\begin{aligned}
\mathcal{W}(b\bar{a}_0^{-1})\mathcal{W}^{-1}(\bar{a}_0^{-1})\mathcal{W}(\bar{a}_0^{-1}\tilde{b}) &= \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}^{-1}(\bar{a}_0^{-1})\,[\mathcal{W}(\bar{a}_0^{-1})\,\mathcal{W}(\tilde{b}) + \mathcal{H}(\bar{a}_0^{-1})\,\mathcal{H}(b)]\\
&= \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}(\tilde{b}) + \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}^{-1}(\bar{a}_0^{-1})\,\mathcal{H}(\bar{a}_0^{-1})\,\mathcal{H}(b)\\
&= \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}(\tilde{b}) - \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{H}(\bar{a}_0)\,\mathcal{W}^{-1}(a_0)\,\mathcal{H}(b)\\
&= \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}(\tilde{b}) + [\,\mathcal{H}(b\bar{a}_0^{-1})\,\mathcal{W}(a_0) - \mathcal{H}(b)\,]\,\mathcal{W}^{-1}(a_0)\,\mathcal{H}(b)\\
&= \mathcal{W}(b\bar{a}_0^{-1})\,\mathcal{W}(\tilde{b}) + \mathcal{H}(b\bar{a}_0^{-1})\,\mathcal{H}(b) - \mathcal{H}(b)\,\mathcal{W}^{-1}(a_0)\,\mathcal{H}(b)\\
&= \mathcal{W}(b\bar{a}_0^{-1}\tilde{b}) - \mathcal{H}(b)\,\mathcal{W}^{-1}(a_0)\,\mathcal{H}(b).
\end{aligned}$$

Inserting this result in (2.21) we have

$$S_0 = -\,\mathcal{W}(a_0) + \mathcal{H}(b)\,\mathcal{W}^{-1}(a_0)\,\mathcal{H}(b) \tag{2.24}$$

which can also be written as (2.19)

$$S_0 = -\,(\mathcal{W}(a_0) + \mathcal{H}(b))\,\mathcal{W}^{-1}(a_0)\,(\mathcal{W}(a_0) - \mathcal{H}(b))$$

or

$$S_0 = -\,(\mathcal{W}(a_0) - \mathcal{H}(b))\,\mathcal{W}^{-1}(a_0)\,(\mathcal{W}(a_0) + \mathcal{H}(b)) \quad ,$$

i.e., we can commute the outer factors. ∎

REMARK: We like to thank the referee for having suggested the way to prove the above theorem by the use of identities (2.15) and (2.16), making it possible to extend the theorem in a natural way to a larger class of linear operators. Indeed, let $\mathcal{R}$ denote an inverse closed algebra with unit element $e$ (not necessarily commutative) and let $\sim:\mathcal{R}\mapsto\mathcal{R}$ be an automorphism $a\mapsto\tilde{a}$ such that $\tilde{\tilde{a}}=a$ . Further, suppose that we are given a linear space $X$ and that with every element $a\in\mathcal{R}$ two linear operators $\mathcal{W}(a),\mathcal{H}(a)\in\mathcal{L}(X)$ are associated, such that $\mathcal{H}(e)=0$ and relations (2.15), (2.16) are fullfilled for every $a,b\in\mathcal{R}$ . Then Theorem 2.2 remains valid for all $a_0,b\in\mathcal{R}$ such that $\mathcal{W}(a_0)$ and $\mathcal{W}^{-1}(\bar{a}_0^{-1})$ are invertible operators.

There are also a number of different possible generalizations, for instance in Ring Theory or for operators acting between different Banach spaces. Those generalizations can be useful in the setting of General Wiener-Hopf Operator Theory.

We point out that if we take in Theorem 2.2 the invertible operators $\tilde{\mathcal{A}}_0$ and $\tilde{\mathcal{B}}_0$ defined by

$$\tilde{\mathcal{A}}_0=\begin{bmatrix} -\mathcal{W}(b\bar{a}_0^{-1})\mathcal{W}^{-1}(\bar{a}_0^{-1}) & \mathcal{I}^+ \\ \mathcal{I}^+ & 0 \end{bmatrix} \quad , \quad \tilde{\mathcal{B}}_0=\begin{bmatrix} \mathcal{I}^+ & 0 \\ \mathcal{W}^{-1}(\bar{a}_0^{-1})\mathcal{W}(\bar{a}_0^{-1}\tilde{b}) & -\mathcal{W}^{\alpha 1}(\bar{a}_0^{-1}) \end{bmatrix} \tag{2.25}$$

instead of $\mathcal{A}_0$ and $\mathcal{B}_0$, we have yet

$$\tilde{\mathcal{A}}_0\, \mathcal{W}(G_0)\, \tilde{\mathcal{B}}_0=\begin{bmatrix} \mathcal{S}_0 & 0 \\ 0 & \mathcal{I}^+ \end{bmatrix}. \tag{2.26}$$

This remark is convenient for the case $\upsilon \neq 0$, where it seems to be not possible to diagonalize $\mathcal{W}(G)$ by means of triangular (two-sided) invertible operators. In fact, in this case, bearing in mind relation (2.13), the following results can be proved directly, by the use of (2.19), (2.20):

(I) For $\upsilon \geq 0$ and

$$\mathcal{A}=\mathcal{U}_{-\upsilon}^{(2)}\mathcal{A}_0 \quad , \quad \mathcal{B}=\mathcal{U}_{-\upsilon}^{(2)}\,\mathcal{B}_0\,\mathcal{U}_{\upsilon}^{(2)} \tag{2.27}$$

we have

$$\mathcal{A}\,\mathcal{W}(G)\,\mathcal{B}=\begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{I}^+ \end{bmatrix} \tag{2.28}$$

where

$$\mathcal{S}=\mathcal{U}_{-\upsilon}\,\mathcal{S}_0\,\mathcal{U}_{\upsilon}\ , \tag{2.29}$$

with $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{S}_0$ given by (2.17) and (2.19). Note that in this case $\mathcal{A}$ is only a right invertible operator, with ind $\mathcal{A}=2\upsilon$, and that $\mathcal{B}$ is an invertible operator.

(II) For $\upsilon \leq 0$ and

$$\mathcal{A}=\mathcal{U}_{\upsilon}^{(2)}\tilde{\mathcal{A}}_0\mathcal{U}_{-\upsilon}^{(2)} \quad , \quad \mathcal{B}=\tilde{\mathcal{B}}_0\mathcal{U}_{-\upsilon}^{(2)} \tag{2.30}$$

we have

$$\mathcal{A}\,\mathcal{W}(G)\,\mathcal{B}=\begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{I}^+ \end{bmatrix} \tag{2.31}$$

where

$$\mathcal{S}=\mathcal{U}_{\upsilon}\,\mathcal{S}_0\,\mathcal{U}_{-\upsilon}\ , \tag{2.32}$$

with $\tilde{\mathcal{A}}_0$, $\tilde{\mathcal{B}}_0$ and $\mathcal{S}_0$ given by (2.25) and (2.19). Note that now $\mathcal{A}$ is an invertible operator and $\mathcal{B}$ is only a left invertible operator with ind $\mathcal{B}=2\upsilon$.

Furthermore, if $\upsilon \neq 0$ then according to (2.29),(2.32) and using the identity $\mathcal{U}_{-|\upsilon|} \mathcal{W}(a_0) \mathcal{U}_{|\upsilon|} = \mathcal{W}(a_0)$, we have

$$\begin{aligned} \mathcal{S} &= \mathcal{U}_{-|\upsilon|}\, \mathcal{S}_0\, \mathcal{U}_{|\upsilon|} \\ &= \mathcal{U}_{-|\upsilon|}\,(-\mathcal{W}(a_0) + \mathcal{H}(b)\, \mathcal{W}^{-1}(a_0)\, \mathcal{H}(b))\, \mathcal{U}_{|\upsilon|} \\ &= -\mathcal{W}(a_0) + \mathcal{U}_{-|\upsilon|} \mathcal{H}(b)\, \mathcal{W}^{-1}(a_0)\, \mathcal{H}(b)\, \mathcal{U}_{|\upsilon|}\,. \end{aligned}$$

Suppose that $\upsilon \geq 0$. Then using the relation $\mathcal{H}(b)\, \mathcal{U}_\upsilon = \mathcal{U}_{-\upsilon} \mathcal{H}(b)$, we obtain

$$\mathcal{S} = -\mathcal{W}(a_0) + \mathcal{U}_{-\upsilon} \mathcal{H}(b)\, \mathcal{W}^{-1}(a_0)\, \mathcal{U}_{-\upsilon} \mathcal{H}(b)$$

from which we may write

$$\begin{aligned} \mathcal{S} &= -(\mathcal{W}(a_0) + \mathcal{U}_{-\upsilon} \mathcal{H}(b))\, \mathcal{W}^{-1}(a_0)\, (\mathcal{W}(a_0) - \mathcal{U}_{-\upsilon} \mathcal{H}(b)) \\ &= -\mathcal{U}_{-\upsilon}(\mathcal{W}(a) + \mathcal{H}(b))\, \mathcal{W}^{-1}(a_0)\, \mathcal{U}_{-\upsilon}(\mathcal{W}(a) - \mathcal{H}(b)) \end{aligned} \tag{2.33}$$

or equivalently

$$\mathcal{S} = -\mathcal{U}_{-\upsilon}(\mathcal{W}(a) - \mathcal{H}(b))\, \mathcal{W}^{-1}(a_0)\, \mathcal{U}_{-\upsilon}(\mathcal{W}(a) + \mathcal{H}(b)). \tag{2.34}$$

If $\upsilon \leq 0$, the above procedure and the identity $\mathcal{U}_\upsilon\, \mathcal{H}(b) = \mathcal{H}(b)\, \mathcal{U}_{-\upsilon}$ yields

$$\mathcal{S} = -(\mathcal{W}(a) + \mathcal{H}(b))\, \mathcal{U}_{-\upsilon}\, \mathcal{W}^{-1}(a_0)\, (\mathcal{W}(a) - \mathcal{H}(b))\, \mathcal{U}_{-\upsilon} \tag{2.35}$$

$$= -(\mathcal{W}(a) - \mathcal{H}(b))\, \mathcal{U}_{-\upsilon}\, \mathcal{W}^{-1}(a_0)\, (\mathcal{W}(a) + \mathcal{H}(b))\, \mathcal{U}_{-\upsilon}\,. \tag{2.36}$$

The relations (2.33),(2.34) and (2.35),(2.36) establish the connection between the scalar operator $\mathcal{S}$ and the Wiener-Hopf-Hankel operator $\mathcal{W}(a)+\mathcal{H}(b)$, the former obtained through the diagonalization of the vector Wiener-Hopf operator $\mathcal{W}(G)$.

We summarize the results obtained so far in the next theorem.

THEOREM 2.3: *Let $a,b \in L_\infty(\mathbb{R})$ and suppose that a admits a generalized factorization relative to $L_2(\mathbb{R})$ with* ind$a=\upsilon$. *Consider the Wiener-Hopf-Hankel operator $\mathcal{W}(a) + \mathcal{H}(b)$ on $L_2^+(\mathbb{R})$ and let $\mathcal{W}(G)$ be the Wiener-Hopf operator acting on $[L_2^+(\mathbb{R})]^2$ associated with it, defined by (2.11) (see also (2.9)).*

*The operator $\mathcal{W}(G)$ is diagonalized by the operators $\mathcal{A}$ and $\mathcal{B}$, i.e.,*

$$\mathcal{A}\, \mathcal{W}(G)\, \mathcal{B} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{I}^+ \end{bmatrix} : L_2^+(\mathbb{R}) \oplus L_2^+(\mathbb{R}) \to L_2^+(\mathbb{R}) \oplus L_2^+(\mathbb{R}) \tag{2.37}$$

*where:*

*(i) If $\upsilon \geq 0$, $\mathcal{A}$ and $\mathcal{B}$ are given by (2.27) and*

$$\mathcal{S} = -\mathcal{U}_{-\upsilon}(\mathcal{W}(a) + \mathcal{H}(b))\, \mathcal{W}^{-1}(a_0)\, \mathcal{U}_{-\upsilon}(\mathcal{W}(a) - \mathcal{H}(b)). \tag{2.38}$$

*(ii) If $\upsilon \leq 0$, $\mathcal{A}$ and $\mathcal{B}$ are given by (2.30) and*

$$\mathcal{S} = -(\mathcal{W}(a) + \mathcal{H}(b))\, \mathcal{U}_{-\upsilon}\, \mathcal{W}^{-1}(a_0)\, (\mathcal{W}(a) - \mathcal{H}(b))\, \mathcal{U}_{-\upsilon}. \tag{2.39}$$

*Moreover, in these formulas the order of the factors $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ can be reversed.*

### 3. Fredholm properties and Invertibility of $\mathcal{W}(a) + \mathcal{H}(b)$

In this section we are going to exploit the relations between the Wiener-Hopf-Hankel operator (1.2) and its associated Wiener-Hopf operator (2.11) in what the Fredholm properties and invertibility are concerned. Our main interest, however, is focused on the invertibility, since on one-hand it plays a fundamental role in applications (see section 4) and on the other-hand, as mentioned before, the Fredholm property for the Wiener-Hopf-Hankel operators can be determined alternatively through Banach algebras methods [5,[6],[15],[16],[21].

We start with the following auxiliary result.

PROPOSITION 3.1: *Let $a,b \in L_\infty(\mathbb{R})$ and suppose that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$ with* ind$a = \upsilon$. *Further let $\mathcal{W}(G)$ be the Wiener-Hopf operator on $[L_2^+(\mathbb{R})]^2$ defined by (2.11) and $\mathcal{S}$ denote the operator on $L_2^+(\mathbb{R})$ defined by (2.29) if $\upsilon \geq 0$, and by (2.32) if $\upsilon \leq 0$. Then:*

*(i) $\mathcal{W}(G)$ is a Fredholm operator iff $\mathcal{S}$ is a Fredholm operator.*

*(ii) If $\mathcal{W}(G)$ is a Fredholm operator we have*

$$\text{ind}\, \mathcal{W}(G) = \text{ind}\, \mathcal{S} - 2\upsilon. \tag{3.1}$$

Proof: (i) Let us recall that $\mathcal{S}$ is the operator resulting from the diagonalization of $\mathcal{W}(G)$ (see (2.28),(2.31)). The diagonalizing operators $\mathcal{A}$ and $\mathcal{B}$, given by (2.27) if $\upsilon \geq 0$, and by (2.30) if $\upsilon \leq 0$, are Fredholm operators. Then the simultaneous Fredholm property for both operators $\mathcal{W}(G)$ and $\mathcal{S}$ is a direct consequence of the following well known general results: (1) the product of Fredholm operators is a Fredholm operator, (2) if the product of two operators is Fredholm and one of the factors also, then the other has the same property, (3) the direct sum of two operators is Fredholm iff both operators are Fredholm.

(ii) The property (3.1) follows from the relation ind $\mathcal{A}$ + ind $\mathcal{B}$ = $2\upsilon$ (see (2.27),(2.30)) and the general rule for the index of the product of Fredholm operators. ∎

As a consequence of the above proposition and Theorem 2.3, we have

THEOREM 3.2: *Let $a,b \in L_\infty(\mathbb{R})$ and suppose that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$. Consider the Wiener-Hopf-Hankel operator $\mathcal{W}(a)+\mathcal{H}(b)$ on $L_2^+(\mathbb{R})$ and let $\mathcal{W}(G)$ be the Wiener-Hopf operator acting on $[L_2^+(\mathbb{R})]^2$ associated with it (defined by (2.11)). Then:*

*(i) $\mathcal{W}(G)$ is a Fredholm operator iff $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ are Fredholm operators.*

*(ii) If $\mathcal{W}(G)$ is a Fredholm operator we have*

$$\operatorname{ind}\mathcal{W}(G) = \operatorname{ind}(\mathcal{W}(a) + \mathcal{H}(b)) + \operatorname{ind}(\mathcal{W}(a) - \mathcal{H}(b)). \tag{3.2}$$

Proof: (i) From Proposition 3.1 we know that $\mathcal{W}(G)$ is a Fredholm operator iff $\mathbf{S}$, obtained from the diagonalization of $\mathcal{W}(G)$, is a Fredholm operator. Therefore we prove the result for $\mathbf{S}$ instead of $\mathcal{W}(G)$. Theorem 2.3 establish the relation between the operator $\mathbf{S}$ and the operators $\mathcal{W}(a) \pm \mathcal{H}(b)$ (see (2.38),(2.39)). Using the general results from the Theory of Fredholm Operators already mentioned in the proof of Proposition 3.1, we immediately conclude that $\mathbf{S}$ is Fredholm iff both operators $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ are Fredholm.

(ii) By Proposition 3.1, $\mathbf{S}$ is a Fredholm operator if $\mathcal{W}(G)$ is Fredholm. Then it follows from (2.38),(2.39) that

$$\operatorname{ind}\mathbf{S} = \operatorname{ind}(\mathcal{W}(a) + \mathcal{H}(b)) + \operatorname{ind}(\mathcal{W}(a) - \mathcal{H}(b)) + 2\upsilon$$

since $\operatorname{ind}\mathcal{U}_{-\upsilon} = \upsilon$. Combining this result with (3.1) we obtain (3.2). ∎

REMARK: The general assumption in this work is that $a$ admits a generalized factorization relative to $L_2(\mathbb{R})$. In fact, only in this case it was possible to prove the diagonalization of the associated Wiener-Hopf operator stated in Theorem 2.3. However, this condition is not necessary for the Fredholmness of the Wiener-Hopf-Hankel operators $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$, as we shall illustrate in the first example of section 4. Therefore this constitutes a limitation of the present method.

As is known from the general theory of Wiener-Hopf operators, the Fredholm property for the Wiener-Hopf operator $\mathcal{W}(G)$ defined on $[L_2^+(\mathbb{R})]^2$ is equivalent to the existence of a generalized factorization relative to $L_2(\mathbb{R})$ of the matrix-valued function $G$ (cf. [2],[13]).

For arbitrary $a,b \in L_\infty(\mathbb{R})$ (and therefore $G \in L_\infty(\mathbb{R})$) there are no criteria available for the existence of a generalized factorization of $G$ relative to $L_2(\mathbb{R})$. However, if we restrict ourselves to the case where $a,b \in PC(\dot{\mathbb{R}})$, the following necessary and sufficient condition can be stated.

PROPOSITION 3.3: *Let $a,b \in PC(\dot{\mathbb{R}})$. Then $G$ admits a generalized factorization relative to $L_2(\mathbb{R})$ iff*

$$g(\xi,\mu) \neq 0 \qquad \text{for} \qquad (\xi,\mu) \in \dot{\mathbb{R}} \times [0,1] \tag{3.3}$$

*where*

$$g(\xi,\mu) = A(\xi) + [B(\xi) - C(\xi)]\,\mu + [D(\xi) - B(\xi)]\,\mu^2 \quad , \quad (\xi,\mu) \in \dot{\mathbb{R}} \times [0,1], \tag{3.4}$$

with

$$A(\xi)=-a(\xi)/a(-\xi+)\ ,\ B(\xi)=(b(\xi+)-b(\xi))(b(-\xi)-b(-\xi+))/(a(-\xi)a(-\xi+)) \tag{3.5}$$

$$C(\xi)=[a(\xi)(a(-\xi+)-a(-\xi))+a(-\xi)(a(\xi+)-a(\xi))]/(a(-\xi)a(-\xi+)) \tag{3.6}$$

$$D(\xi)=(a(\xi+)-a(\xi))(a(-\xi)-a(-\xi+))/(a(-\xi)a(-\xi+)). \tag{3.7}$$

*Moreover if (3.3) holds then $W(G)$ is a Fredholm operator, whose index is the symmetric of the winding number of $g(\xi,\mu)$ with respect to the origin.*

Proof: We associate with (the left-continuous matrix-valued function) $G$ the matrix-valued function $G': \dot{\mathbb{R}}\times[0,1]\to\mathbb{C}^{2\times2}$, defined by

$$G'(\xi,\mu)=G(\xi)+\mu\,(G(\xi+)-G(\xi))\ ,\quad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1] \tag{3.8}$$

where we use the conventions $G(\infty)=\lim_{\xi\to+\infty}G(\xi)$ and $G(\infty+)=\lim_{\xi\to-\infty}G(\xi)$. Hence, $G$ admits a generalized factorization relative to $L_2(\mathbb{R})$ iff

$$g(\xi,\mu)=\det G'(\xi,\mu)\neq0 \qquad \text{for} \qquad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1].$$

Furthermore, in this condition, the winding number of $g(\xi,\mu)$ coincides with the symmetric of the Fredholm index of $W(G)$ (see [13]).

The rest of the proof is achieved by a straightforward computation of the function $g$ which we omit here. ∎

Now we look for relations between the invertibility of the Wiener-Hopf-Hankel operator and of the associated Wiener-Hopf operator, the latter corresponding to the existence of a canonical generalized factorization relative to $L_2(\mathbb{R})$ for the matrix-valued function $G$ (cf. [2],[13]).

Let us start with the simplest case where $b\in C(\dot{\mathbb{R}})$, i.e., where the Hankel operator $\mathcal{H}(b)$ is compact [7]. In this case we immediately conclude that the operator $W(a)+\mathcal{H}(b)$ is Fredholm if and only if $W(a)$ is Fredholm, which implies that $a\in L_\infty(\mathbb{R})$ admits a generalized factorization relative to $L_2(\mathbb{R})$ (cf. [13]). Also in such case $\operatorname{ind}(W(a)+\mathcal{H}(b))=\operatorname{ind}(W(a))$ and therefore $\upsilon=\operatorname{ind}a=0$ is a necessary condition for the invertibility of the Wiener-Hopf-Hankel operator. This fact motivates the following main result.

THEOREM 3.4: *Let $a,b\in L_\infty(\mathbb{R})$ and suppose that $a$ admits a canonical generalized factorization relative to $L_2(\mathbb{R})$ ($\upsilon=\operatorname{ind}a=0$). Further let $W(G)$ be the Wiener-Hopf operator on $[L_2^+(\mathbb{R})]^2$ associated with $W(a)+\mathcal{H}(b)$ (see (2.11)). Then:*

*(i) The operator $W(G)$ is (left, right) invertible iff $W(a)+\mathcal{H}(b)$ and $W(a)-\mathcal{H}(b)$ are (left, right) invertible operators.*

*(ii) If $W(G)$ is (left, right) invertible then the (one-sided) inverse of $W(a)+\mathcal{H}(b)$ is defined by*

$$(\mathcal{W}(a) + \mathcal{H}(b))^{-1} f^+ = \Pi \ \mathcal{W}^{-1}(G) \ \mathcal{P}^+ \ \overset{0}{\mathcal{W}}(C) \begin{bmatrix} f^+ \\ \mathcal{J}f^+ \end{bmatrix} \quad , \quad f^+ \in L_2^+(\mathbb{R}) \tag{3.9}$$

*where $\mathcal{W}^{-1}(G)$ is the (one-sided) inverse of $\mathcal{W}(G)$, $\overset{0}{\mathcal{W}}(C) = \mathcal{F}^{-1} C \mathcal{F}$ is the convolution operator on $[L_2(\mathbb{R})]^2$ with symbol (2.6), and $\Pi : [L_2(\mathbb{R})]^2 \to L_2(\mathbb{R})$ is the operator given by*

$$\Pi \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \varphi_1 . \tag{3.10}$$

Proof: (i) The assumption on $a$ implies that $\mathrm{ind}\,a = \upsilon = 0$ (so $a_0 = a$, $G_0 = G$, etc.). Recall from section 2 that in this case the operator $\mathcal{W}(G)$ is diagonalized by means of invertible operators $\mathcal{A}_0$ and $\mathcal{B}_0$, see Theorem 2.2. Let $S_0$ be the operator obtained through this diagonalization, i.e., $S_0$ is given by (2.19). From (2.18) we conclude that $\mathcal{W}(G)$ is (left, right) invertible iff $S_0$ is (left, right) invertible, since $\mathcal{A}_0$ and $\mathcal{B}_0$ are invertible operators. Now, from Theorem 2.2, it follows that $S_0$ is (left, right) invertible iff $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ are (left, right) invertible operators. In fact, since the order of the outer factors in (2.18) can be reversed, $S_0$ is injective (respectively surjective) iff $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ are injective (surjective).

(ii) Suppose that $\mathcal{W}(G)$ is (left, right) invertible. Then, by (i), $\mathcal{W}(a) + \mathcal{H}(b)$ is also (left, right) invertible. From Proposition 2.1 it turns out that, for any $f^+$ in the image of $\mathcal{W}(a) + \mathcal{H}(b)$, a solution $\varphi^+$ of equation (2.1) is such that the vector $\phi^+ = [\varphi^+, \varphi_1{}^+]^T$ with $\varphi_1{}^+ = (\mathcal{W}(\tilde{b}) + \mathcal{H}(\tilde{a}))\varphi^+$ is a solution of (2.10). Since $\mathcal{W}(G)$ is (left, right) invertible, this solution is given by

$$\phi^+ = \mathcal{W}^{-1}(G) \, \mathcal{P}^+ \ \overset{0}{\mathcal{W}}(C) \begin{bmatrix} f^+ \\ \mathcal{J}f^+ \end{bmatrix}$$

and, consequently, if $\Pi$ denotes the operator defined by (3.9), we have

$$\varphi^+ = (\mathcal{W}(a) + \mathcal{H}(b))^{-1} f^+ = \Pi \ \mathcal{W}^{-1}(G) \ \mathcal{P}^+ \ \overset{0}{\mathcal{W}}(C) \begin{bmatrix} f^+ \\ \mathcal{J}f^+ \end{bmatrix} \quad , \quad f^+ \in L_2^+(\mathbb{R})$$

which completes the proof of the theorem. ∎

The preceding theorem is restricted to the situation $\upsilon = \mathrm{ind}\,a = 0$. The reason for that restriction lies in the fact that the diagonalization of $\mathcal{W}(G)$ is made by means of invertible operators only when $\upsilon = 0$. Hence, the general case with arbitrary $\upsilon$ is not easy to handle, and it probably needs more sophisticated methods.

However, it can be seen in some particular situations $(a, b \in PC(\dot{\mathbb{R}}))$ that $\upsilon = 0$ is a necessary condition for the simultaneous invertibility of $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$, through a detailed analysis of the symbol $G'(\xi, \mu)$ of the associated Wiener-Hopf operator $\mathcal{W}(G)$ (see (3.8).

For instance, let us take

$$a \in C(\dot{\mathbb{R}}) \quad , \quad b \in PC(\dot{\mathbb{R}}) \tag{3.11}$$

with $a(\xi)\neq 0$, $\xi\in\dot{\mathbb{R}}$, which is a necessary and sufficient condition for $a$ to admit a generalized factorization relative to $L_2(\mathbb{R})$. Suppose that $b$ has $n$ discontinuity points $\xi_j$ $(j=1,...,n)$. From (3.4) we have

$$g(\xi,\mu) = -a(\xi)/a(-\xi) + \mu(1-\mu)B(\xi) \quad , \quad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1] \tag{3.12}$$

with $B$ given by (3.5).

The image of $g(\xi,\mu)$ is formed by the union of the closed curve correspondent to the image of the function $\xi\in\dot{\mathbb{R}} \mapsto a_1(\xi)=-a(\xi)/a(-\xi)$ with $n$ straight-line segments, with end points $a_1(x_j)$ and $a_1(\xi_j) + B(\xi_j)/4$, traversed twice. If $g(\xi,\mu)\neq 0$, this means that the winding numbers of $g(\xi,\mu)$ and $a_1(\xi)$ with respect to the origin coincide. But as $a\in C(\dot{\mathbb{R}})$, the winding number of $a_1(\xi)$ is easily seen to be $2\upsilon$, where $\upsilon$ denotes the winding number of $a(\xi)$ with respect to the origin. Thus ind $\mathcal{W}(G)=-2\upsilon$ (cf. [13]), and from Theorem 3.2 we have

$$\operatorname{ind}(\mathcal{W}(a) + \mathcal{H}(b)) + \operatorname{ind}(\mathcal{W}(a) - \mathcal{H}(b))=-2\upsilon. \tag{3.13}$$

Consequently $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$ can be simultaneously invertible only if $\upsilon=0$. By other words, in condition (3.11) $\upsilon=0$ is a necessary condition for the invertibility of the Wiener-Hopf-Hankel operators $\mathcal{W}(a) \pm \mathcal{H}(b)$.

The same conclusion holds in the following more general situation. Let $a,b \in PC(\dot{\mathbb{R}})$ and denote by $\Omega$ and $\Sigma$ the sets of discontinuity points of $a$ and $b$, respectively. Further assume that

(a) $\Omega \cap \Sigma = \varnothing$
(b) $0,\infty \notin \Omega$
(c) $\xi \in \Omega \Rightarrow -\xi \notin \Omega$.

In this case, from Proposition 3.3 we have

$$g(\xi,\mu)=\begin{cases} -a(\xi)/a(-\xi) & , \ (\xi,\mu)\in(\dot{\mathbb{R}}\setminus\{\Omega\cup\Sigma\})\times[0,1] \\ -a(\xi)/a(-\xi)+\mu(a(\xi)-a(\xi+))/a(-\xi) & , \ (\xi,\mu)\in\Omega\times[0,1] \\ -a(\xi)/a(-\xi)+\mu(1-\mu)B(\xi) & , \ (\xi,\mu)\in\Sigma\times[0,1] \end{cases} \tag{3.14}$$

where $B$ is given by (3.5).

The image of $g(\xi,\mu)$ is the union of the closed curve formed by the image of

$$a'(\xi,\mu)=-a(\xi)/a(-\xi)+\mu(a(\xi)-a(\xi+))/a(-\xi) \quad , \quad (\xi,\mu)\in\dot{\mathbb{R}}\times[0,1] \tag{3.15}$$

with $n$ straight-line segments whose extremes are $a_1(\xi_j)=-a(\xi_j)/a(-\xi_j)$ and $a_1(\xi_j)+B(\xi_j)/4$, where, as before, $\xi_j$ $(j=1,...,n)$ denote the elements of $\Sigma$.

Also in this case the winding number of $g(\xi,\mu)$ is given by $2\upsilon$ as $a(\xi)$ and $a(-\xi)$ do not have common discontinuity points. Therefore we have again ind $\mathcal{W}(G)=-2\upsilon$ and (3.13) still holds. Then once more the Wiener-Hopf-Hankel operators $\mathcal{W}(a) \pm \mathcal{H}(b)$ cannot be simultaneously invertible unless $\upsilon=0$.

In these situations Theorem 3.4 gives a sufficient condition for the invertibility of $\mathcal{W}(a) + \mathcal{H}(b)$ and $\mathcal{W}(a) - \mathcal{H}(b)$, generalizing Theorem 3.2 in [19] which was established for the case where $a$ is a constant.

## 4. Examples and applications to Diffraction Theory

We now apply the results obtained in the previous sections to some examples arising in Diffraction Theory.

Before, however, we give a theoretical example, showing that the condition of $a$ to admit a generalized factorization relative to $L_2(\mathbb{R})$ is not necessary for the Fredholmness or even invertibility of the Wiener-Hopf-Hankel operator $\mathcal{W}(a) + \mathcal{H}(b)$.

EXAMPLE 4.1: Consider the following integral operator on $L_2(\mathbb{R}^+)$:

$$\mathcal{T}\varphi(x) = \lambda\, \varphi(x) - \frac{1}{\pi i}\int_0^{+\infty} \frac{1}{y-x}\,\varphi(y)\,dy - \frac{1}{\pi i}\int_0^{+\infty} \frac{1}{y+x}\,\varphi(y)\,dy\,,\ x>0,\ \ \lambda\in\mathbb{C} \tag{4.1}$$

involving the singular integral operator on $\mathbb{R}^+$ and the Carleman operator. The isomorphism on $L_2(\mathbb{R}^+)$ onto $L_2(\mathbb{R})$ given by $\psi(x)=e^{x/2}\varphi(e^x)$ allows the rewriting of the above operator as a convolution operator on $\mathbb{R}$, whose symbol $\sigma$ is easily seen to be

$$\sigma(\xi)=\lambda - \tanh(\pi\xi) + i\,\mathrm{cosech}(\pi\xi). \tag{4.2}$$

Consequently (4.1) defines an invertible operator for all $\lambda\in\mathbb{C}$ such that $\sigma(\xi)\neq 0,\ \xi\in\dot{\mathbb{R}}$ and $\sigma(\infty+)\neq 0$, and a non-Fredholm operator in the complementary set.

Let us now write (4.1) in the form (identifying $L_2(\mathbb{R}^+)$ with $L_2^+(\mathbb{R})$)

$$\mathcal{T} = \mathcal{W}(a) + \mathcal{H}(b) \tag{4.3}$$

where $\mathcal{W}(a)$ and $\mathcal{H}(b)$ are the Wiener-Hopf and Hankel operators with symbols

$$a(\xi) = \lambda + \mathrm{sign}\xi \quad , \quad b(\xi) = \mathrm{sign}\xi\,. \tag{4.4}$$

The associated Wiener-Hopf operator $\mathcal{W}(G)$ on $[L_2^+(\mathbb{R})]^2$ has the piecewise constant presymbol (see (2.9))

$$G(\xi) = \frac{1}{\lambda - \mathrm{sign}\xi}\begin{bmatrix} -\mathrm{sign}\xi & -1 \\ \lambda^2 & -\mathrm{sign}\xi \end{bmatrix}. \tag{4.5}$$

After some computations, we conclude from Proposition 3.3 that the matrix-valued function $G$ has a generalized factorization relative to $L_2(\mathbb{R})$ if and only if

$$\lambda^2 + \lambda\,(1\mp i)\,\eta\ \mp i \neq 0 \quad \text{for all} \quad \eta\in[-1,1]\,. \tag{4.6}$$

We point out that the set of values of $\lambda$ defined by the above conditions is strictly contained in $\{\lambda\in\mathbb{C}:\ \sigma(\xi)\neq 0\ ,\ \xi\in\dot{\mathbb{R}}\ and\ \sigma(\infty+)\neq 0\}$ (see (4.2)).

Moreover, for all this values of $\lambda$, the generalized factorization of $G$ is a canonical one. Indeed, following the method proposed in [8], such factorization can be worked out explicitly. To this end, let

$$G(\xi) = G_{\ell} \cdot \begin{cases} I & , \quad \xi < 0 \\ G_{\ell}^{-1} G_r & , \quad \xi > 0 \end{cases} \tag{4.7}$$

where $G_{\ell,r} = G(\xi),\ \xi \lessgtr 0$, are constant matrices and $I$ denotes the $2\times 2$ identity matrix. The matrix $G_{\ell}^{-1} G_r$ is diagonalized by the matrix $S$ formed by its eigenvectors, yielding

$$G_{\ell}^{-1} G_r = S\ \mathrm{diag}\{\alpha_1,\alpha_2\}\ S^{-1} \tag{4.8}$$

with eigenvalues $\alpha_{1,2}$ given by

$$\alpha_{1,2} = \frac{\lambda+1}{\lambda-1}\ \frac{\lambda\pm i}{\lambda\mp i}\ . \tag{4.9}$$

Substituting (4.8) in (4.7) we get

$$G(\xi) = G_{\ell}\, S\, D(\xi)\, S^{-1} \tag{4.10}$$

where

$$D(\xi) = \mathrm{diag}\{\beta_1(\xi),\beta_2(\xi)\}\quad ,\quad \beta_{1,2}(\xi) = \frac{(1+\alpha_{1,2})}{2} + \frac{(\alpha_{1,2}-1)}{2}\,\mathrm{sign}\xi\,. \tag{4.11}$$

Now, if conditions (4.6) are satisfied, $\beta_{1,2}$ admit a canonical generalized factorization relative to $L_2(\mathbb{R})$ and consequently the same holds for $D$. As $G_{\ell}$, $S$ are constant matrices, from (4.10) it follows that G has, as claimed, a canonical generalized factorization. Hence the associated Wiener-Hopf operator $\mathcal{W}(G)$ is invertible.

If additionally we impose that $\lambda\notin[-1,1]$, which means that $a$ has a canonical generalized factorization relative to $L_2(\mathbb{R})$ (see (4.4)), then Theorem 3.4 guarantees the invertibility of the Wiener-Hopf-Hankel operator $\mathcal{T}$. For $\lambda\in(-1,1)$, however, $a$ has not a generalized factorization, and Theorem 3.4 cannot be applied. Nevertheless, for such values of $\lambda$, formula (4.2) shows that $\mathcal{T}$ is also an invertible operator.

EXAMPLE 4.2: In [12], the diffraction problem of a time-harmonic electromagnetic wave by a rectangular wedge, one of whose faces is perfectly conducting and the other having a prescribed impedance (finite or infinite), was considered. The problem, initially formulated as an exterior boundary value problem for the two-dimensional Helmholtz equation in the Sobolev space $H_1(\Omega)$, with a Dirichlet condition on one face of the wedge and a third-kind boundary condition on the other, was reduced to an equivalent pseudodifferential equation of Wiener-Hopf-Hankel type in the trace spaces $H^{+}_{\pm 1/2}(\mathbb{R})$ (see [12,Theorem 5.1]).

By the standard lifting procedure, using Bessel potential operators, that equation was seen to be equivalent to a Wiener-Hopf-Hankel equation on $L_2^+(\mathbb{R})$, of the form (see eqs.(5.29)-(5.36) in [12])

$$(\mathcal{W}(a) + \mathcal{H}(b))\varphi^+ = f^+ \tag{4.12}$$

with symbols

$$a(\xi) = \frac{4it^2(\xi)}{t^2(\xi)-\lambda^2} \quad , \quad b(\xi) = \frac{2t(\xi)}{t(\xi)-\lambda}\,\frac{t_-(\xi)}{t_+(\xi)} \tag{4.13}$$

where $t$, $t_\pm$ are the squareroot functions given by

$$t(\xi)=(\xi^2-k_0^2)^{1/2} \quad , \quad t_\pm(\xi)=(\xi \pm k_0)^{1/2} \quad , \quad \xi\in\mathbb{R} \tag{4.14}$$

for suitable branch cuts, such that $t=t_-t_+$ holds (see eqs. (3.3),(4.8) in [12]). The wave number $k_0$ and the impedance parameter $\lambda$ are complex constants with positive imaginary parts, which implies that the symbols $a$ and $b$ are invertible in $L_\infty(\mathbb{R})$. More precisely, $a\in C(\dot{\mathbb{R}})$ with $\operatorname{ind} a=0$, and $b\in PC(\dot{\mathbb{R}})$ has only a discontinuity at infinity with $b(\infty)=-b(\infty+)$, due to the fact that $\lim_{\xi\to\pm\infty} t_-(\xi)/t_+(\xi)=\pm 1$.

The associated Wiener-Hopf operator $\mathcal{W}(G)$ on $[L_2^+(\mathbb{R})]^2$ has the presymbol

$$G = \frac{t^2-\lambda^2}{4it^2}\begin{bmatrix} -\dfrac{2t}{t-\lambda}\,\dfrac{t_+}{t_-} & -1 \\ \dfrac{4t^2}{t^2-\lambda^2}\,\dfrac{3t+\lambda}{t+\lambda} & -\dfrac{2t}{t-\lambda}\,\dfrac{t_-}{t_+} \end{bmatrix}. \tag{4.15}$$

In [12, Lemma 5.2] it was proved that $G$ has a generalized factorization relative to $L_2(\mathbb{R})$ with total index zero. Therefore $\mathcal{W}(G)$ is a Fredholm operator with $\operatorname{ind}\mathcal{W}(G)=0$.

From Theorem 3.2 it follows that the Wiener-Hopf-Hankel operator $\mathcal{W}(a) + \mathcal{H}(b)$ is Fredholm. This yields the Fredholm property for the correspondent boundary value problem referred to above, through the equivalence Theorem 5.1 in [12].

Moreover, for proving the invertibility of $\mathcal{W}(a) + \mathcal{H}(b)$ (i.e., the existence and uniqueness of solution to the boundary value problem) it is now sufficient to prove that $\mathcal{W}(G)$ is invertible (see Theorem 3.4), or equivalently, that $G$ admits a canonical generalized factorization. This remains an open question, and new methods of factorization (and a-priori determination of the partial indices) to deal with matrix-valued functions of the form (4.15) are being investigated. Nevertheless, when $\lambda=0$, which corresponds to the mixed Dirichlet-Neumann boundary value problem, a canonical generalized factorization for $G$ has been explicitly obtained in [10],[18],[20] (see also [12]). This allows to give an explicit representation for the (unique) solution of the boundary value problem through the representation of the inverse of $\mathcal{W}(a) + \mathcal{H}(b)$ given by formula (3.9).

EXAMPLE 4.3: In [11], some mixed boundary-transmission problems for the Helmholtz equation in a half-space were studied, taking different complex wave numbers $k_1$, $k_2$ in each quadrant. The problems with pure Dirichlet and Neumann boundary conditions were solved explicitly. The mixed case, where both a Dirichlet and a Neumann type condition are imposed on each half-axis of the boundary $x_1 \in \mathbb{R}$, $x_2=0$, was proved to be equivalent (in the sense of [11, Theorem 5.1]) to a special Riemann-Hilbert problem in the Sobolev trace spaces $H^+_{1/2}(\mathbb{R}) \times H^+_{-1/2}(\mathbb{R})$, of the form (see (5.23)-(5.25) in [11])

$$\mathcal{F}^{-1}\tilde{G}\,\mathcal{F}\begin{bmatrix}\varphi_1^+\\ \psi_1^+\end{bmatrix} - J\begin{bmatrix}\varphi_1^+\\ \psi_1^+\end{bmatrix} = \begin{bmatrix} f\\ g\end{bmatrix} \tag{4.16}$$

where

$$\tilde{G} = \frac{t_2}{t_1/\rho_1 + t_2/\rho_2}\begin{bmatrix} -\dfrac{t_2/\rho_2 - t_1/\rho_1}{t_2} & \dfrac{2}{\rho_2}\dfrac{1}{t_2} \\ \dfrac{2}{\rho_1}t_1 & \dfrac{t_2/\rho_2 - t_1/\rho_1}{t_2}\end{bmatrix} \tag{4.17}$$

with $t_j(\xi) = (\xi^2 - k_j^2)^{1/2}$, $j=1,2$, (see (4.14)), $\rho_1$, $\rho_2$ complex constants such that $t_1/\rho_1 + t_2/\rho_2 \neq 0$ holds on $\mathbb{R}$ and a given data vector $[f, g]^T \in H_{1/2}(\mathbb{R}) \times H_{-1/2}(\mathbb{R})$.

A similar lifting procedure to that used in [11] to reduce (4.16) to an equivalent Riemann-Hilbert problem on $[L_2^+(\mathbb{R})]^2$ can now be worked out to show that the lifted problem has the form (2.10), therefore corresponding to a Wiener-Hopf-Hankel operator (see Proposition 2.1).

Indeed, if we use the one-to-one mapping from $[L_2^+(\mathbb{R})]^2$ onto $H^+_{1/2}(\mathbb{R}) \times H^+_{-1/2}(\mathbb{R})$ defined by

$$\begin{bmatrix}\varphi_1^+\\ \psi_1^+\end{bmatrix} = \mathcal{F}^{-1}\mathrm{diag}\{\frac{1}{t_{2+}},\ t_{2+}\}\,\mathcal{F}\begin{bmatrix}\varphi^+\\ \psi^+\end{bmatrix} \tag{4.18}$$

for $t_{2\pm} = (\xi \pm k_2)^{1/2}$, $\xi \in \mathbb{R}$, we get, after some calculations, an equivalent system of equations in $[L_2^+(\mathbb{R})]^2$ (see (2.10))

$$\overset{0}{\mathcal{W}}(G)\begin{bmatrix}\varphi^+\\ \psi^+\end{bmatrix} + J\begin{bmatrix}\varphi^+\\ \psi^+\end{bmatrix} = \overset{0}{\mathcal{W}}(C)\,F \tag{4.19}$$

where $C$ and $G$ are the matrix-valued functions given by (2.6) and (2.9), with

$$a(\xi) = -\frac{i}{2}\Big(1 + \frac{\rho_2}{\rho_1}\frac{t_1(\xi)}{t_2(\xi)}\Big) \quad , \quad b(\xi) = -\frac{1}{2}\,\frac{t_{2+}(\xi)}{t_{2-}(\xi)}\Big(1 - \frac{\rho_2}{\rho_1}\frac{t_1(\xi)}{t_2(\xi)}\Big). \tag{4.20}$$

Also $F=[f^{+}, Jf^{+}]^{T}$ for a suitable known function $f^{+} \in L_2^{+}(\mathbb{R})$ (related to the given data functions $f$ and $g$).

Therefore (4.19) corresponds to the Wiener-Hopf-Hankel equation

$$(\mathcal{W}(a) + \mathcal{H}(b))\varphi^{+} = f^{+} \tag{4.21}$$

for the symbols $a \in C(\dot{\mathbb{R}})$, with ind$a=0$, and $b \in PC(\dot{\mathbb{R}})$, with a discontinuity at infinity. The associated Wiener-Hopf operator $\mathcal{W}(G)$ is a Fredholm operator with zero index (see [11, Proposition 5.2]) if and only if

$$\frac{\rho_1}{\rho_2} \in \mathbb{C} \backslash \overline{\mathbb{R}^{-}}. \tag{4.22}$$

From Theorem 3.2 we conclude that $\mathcal{W}(a) + \mathcal{H}(b)$ is also a Fredholm operator if condition (4.22) is satisfied. This means that the correspondent mixed boundary-transmission problem has the Fredholm property whenever (4.22) holds, a result which could not be proved before, as the correspondence between the Riemann-Hilbert problem (4.16) and the Wiener-Hopf-Hankel operator in (4.21) was not known.

Further, by Theorem 3.4, $\mathcal{W}(a) + \mathcal{H}(b)$ is an invertible operator if $\mathcal{W}(G)$ is invertible, i.e., if $G$ has a canonical generalized factorization relative to $L_2(\mathbb{R})$, in which case the mixed boundary-transmission problem is uniquely solvable.

For all $\rho_1, \rho_2$ satisfying (4.22), the existence of a canonical factorization for $G$ was shown in [11, Remark 5.3], for the particular case $k_1=k_2$, i.e., $t_1=t_2$, and such factorization was explicitly given in [10],[18],[20], yielding an analytical representation to the solution of the boundary-transmission problem. However, for different wave numbers $k_1$ and $k_2$, the existence of a canonical factorization could not be established yet. This motivates further efforts in the investigation of factorization methods for matrix-valued functions of the class considered here.

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