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On a Generalization of the Spaces of Quasi-Constant Curvature

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Some properties and metrics are obtained for the Riemannian manifold satisfying the conditions

$$\begin{aligned} R(X,Y)Z &= k(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + m(\langle Y,Z\rangle\langle X,Q\rangle - \langle X,Z\rangle\langle Y,Q\rangle)U \\ R(X,U)U &= hX \end{aligned} \qquad (\forall\ X,Y,Z \in U^{\perp}).$$

These manifolds are a natural generalization of the spaces of quasi-constant curvature.

Key words: *Sectional curvature, metric, Euler-Poincaré characteristic*
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1. Introduction

S.S.Chern [3] has studied some properties of the curvature and characteristic classes of a Riemannian manifold (M,g) whose Riemann curvature tensor R satisfies the relation

$$\langle R(Z,W)Y,X\rangle = S(X,Z)S(Y,W) - S(X,W)S(Y,Z), \tag{1}$$

where S is a symmetric tensor of type (0,2), and $\langle\,,\rangle$ the inner product with respect to g. It is known that the spaces of quasi–constant curvature [1] are of form (1). In physical literature, the 4–dimensional Lorentzian manifolds of quasi–constant curvature are called infinitesimally spatially isotropic, they are conformally flat solutions of Einstein's gravitational field equations for the perfect fluid matter [5].

In this note we shall consider a class of Riemannian manifolds which is a natural generalization of the spaces of quasi–constant curvature. These manifolds do not satisfy the relation of form (1), but still have some remarkable properties in their curvature and characteristic classes. Their physical background is the cosmological models with heat flow [6], with an assumption that they are conformally flat.

2. Conditions on curvature

The *space of quasi–constant curvature*, due to Vranceanu [7], is an n–dimensional Riemannian manifold which locally admits an orthonormal frame field $\{e_A\}$ such that

$$\gamma_{ABAB} = k, \quad \gamma_{nBnB} = h \qquad (A,B < n;\ A \neq B) \tag{2a}$$

$$\gamma_{ABCD} = O \qquad (A,B,C,D < n;\ C \neq A,B) \tag{2b}$$

$$\gamma_{AnBn} = \gamma_{ABnB} = O \qquad (A \neq n,B) \tag{2c}$$

where k and h are two functions, and

$$\gamma_{ABCD} = \langle R(e_C, e_D)e_B, e_A \rangle.$$

We consider now a Riemannian manifold (M, g) of dimension $n \geqslant 4$, where g is positively definite. Suppose there exist on M two unit vector fields U and Q which are orthogonal each other, and three real-valued functions k, m and h, such that

$$R(X,Y)Z = k(\langle Y,Z \rangle X - \langle X,Z \rangle Y) + m(\langle Y,Z \rangle \langle X,Q \rangle - \langle X,Z \rangle \langle Y,Q \rangle)U, \tag{3}$$

for all $X,Y,Z \in U^{\perp}$, and

$$R(X,U)U = hX \qquad \text{for all } X \in U^{\perp}, \tag{4}$$

where $U^{\perp}$ is the $(n-1)$-dimensional distribution orthogonal to U.

By computing, we have

Theorem 1 : *Under the local orthonormal frame field $\{e_A\}$ in which $e_n = U$, denoting $Q_A = \langle Q, e_A \rangle$, where $A = 1, 2, \ldots, n$, the conditions (3) and (4) hold if and only if*

$$\gamma_{pqrs} = k(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}) \tag{5a}$$

$$\gamma_{nqns} = h\delta_{qs} \tag{5b}$$

$$\gamma_{nqrs} = m(Q_r\delta_{qs} - Q_s\delta_{qr}), \tag{5c}$$

where $p, q, r, s = 1, \ldots, n-1$.

From (5a–c) and (2a–c) we obtain

Corollary : *The Riemannian manifolds satisfying* (3) *and* (4)

(i) *become spaces of quasi–constant curvature if and only if $m=0$ and*

(ii) *are conformally flat. Their Ricci tensor S may be decomposed as*

$$S = -[(n-2)k + h]g + (n-2)(k-h)u \otimes u - (n-2)m(q \otimes u + u \otimes q),$$

where u and q are 1–forms such that $u(X) = \langle U,X \rangle$ and $q(X) = \langle Q,X \rangle$ for each tangent vector X.

B.Y.Chen and K.Yano [2] introduced and studied the notion of *special conformally flat space*, for which the components of curvature tensor $K_{kji}{}^h$ can be written in the form

$$K_{kji}{}^h = \delta_j^h L_{ki} - \delta_k^h L_{ji} + L_j^h g_{ki} - L_k^h g_{ji},$$

where $L_k^h = L_{kt} g^{th}$, and

$$L_{ji} = -\frac{1}{2}(\lambda - \alpha^2)g_{ji} - \beta(\nabla_j \alpha)(\nabla_i \alpha)$$

for two functions α, β and some constant λ. C. C. Hwang [4] has proved that each special conformally flat space is a space of quasi–constant curvature. Hence, the Riemannian manifolds satisfying coditions (3) and (4) generalize the notion of the special conformally flat spaces as well as that of the spaces of quasi–constant curvature.

Now we denote by $K(X \wedge Y)$ the sectional curvature of a Riemannian manifold at its point p in the direction of section plane spanned by vectors X and Y.

Theorem 2 : *Let M be a Riemannian manifold satisfying conditions* (3) *and* (4) .

Then

$$K(X\wedge Y)=k \text{ for all } X,Y\in U^{\perp} \tag{6}$$

$$K(X\wedge U)=h \text{ for all } X\in U^{\perp}. \tag{7}$$

Proof : Since

$$K(X\wedge Y)=\frac{\langle R(X,\ Y)Y,\ X\rangle}{\langle X,\ X\rangle\langle Y,\ Y\rangle-\langle X,\ Y\rangle^2},$$

$$K(X\wedge U)=\frac{\langle R(X,\ U)U,\ X\rangle}{\langle X,\ X\rangle\langle U,\ U\rangle-\langle X,\ U\rangle^2},$$

by noting $\langle U,\ U\rangle=1$, $\langle X,\ U\rangle=0$, from (3) and (4) it follows (6) and (7) ∎

3 . Canonical metrics

It is known that a space of quasi–constant curvature can be foliated by a system of totally umbilical hypersurfaces of constant curvature [4]. As a generalization, we can consider those Riemannian manifolds which satisfy (3) and (4) and have a foliation by a system of totally umbilical hypersurfaces with normal vector field U. From the equations of Gauss, we know that each of the hypersurfaces is of constant curvature. Hence we can write the metrics locally in the form

$$\mathrm{d}s^2=\frac{1}{U^2}\Big\{\sum_{i=1}^{n-1}(\mathrm{d}x^i)^2+V^2(\mathrm{d}x^n)^2\Big\}. \tag{8a}$$

Using Theorem 1, we find

$$U=A\sum_{i=1}^{n-1}(x^i)^2+\sum_{i=1}^{n-1}B_ix^i+C \tag{8b}$$

$$V=a\sum_{i=1}^{n-1}(x^i)^2+\sum_{i=1}^{n-1}b_ix^i+c, \tag{8c}$$

where A, B_i, C, a, b_i and c are functions of x^n only, $UV\neq 0$.

Conversely, when (8a–c) hold, we have the equalities (5a–c) with

$$k=4AC-\sum_{i=1}^{n-1}B_i^2-\Big(\frac{1}{V}\frac{\partial U}{\partial x^n}\Big)^2$$

$$h=\frac{U^2}{V}\Big[\frac{\partial}{\partial x^n}\Big(\frac{1}{UV}\frac{\partial U}{\partial x^n}\Big)-2a\Big]+2AU+\frac{U}{V}\sum_{i=1}^{n-1}\frac{\partial U}{\partial x^i}\frac{\partial V}{\partial x^i}-\sum_{i=1}^{n-1}\Big(\frac{\partial U}{\partial x^i}\Big)^2$$

$$mQ_j=U\frac{\partial}{\partial x^j}\Big(\frac{1}{V}\frac{\partial U}{\partial x^n}\Big)\qquad (j=1,\ 2,\ \cdots,\ n-1)\ .$$

Noting the last equality and the Corollary of Theorem 1, we have the following

Theorem 3 : *The metrics* (8a–c) *satisfy the conditions* (3) *and* (4). *They will be the metrics of quasi–constant curvature if and only if*

$$\mathrm{d}\Big(\frac{1}{V}\frac{\partial U}{\partial x^n}\Big)\wedge \mathrm{d}x^n=0\ .$$

In particular, when $\partial U/\partial x^n=Vf(x^n)$, where $f(x^n)$ is a function of x^n only, we get Hwang's metrics of the spaces of quasi–constant curvature [4].

4. Characteristic classes

We consider now some global properties.

Theorem 4 : *Let M be a compact orientable Riemannian manifold of even dimension $n=2s$, $n\geqslant 4$. Suppose M satisfies* (3) *and* (4). *Then*

$$\chi(M)=\frac{(2s-1)!!}{(2\pi)^s}\int_M k^{s-2}\left(kh-\frac{2s-2}{2s-1}m^2\right)dV,$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M, but in the case $k=0$ and $2s=4$ the factor k^{s-2} must be omitted.

Proof : Make use of an orthonormal frame field $\{e_A\}$ such that $e_n=U$. It is known that [3]

$$\chi(M)=\frac{(-1)^s}{2^{2s}\pi^s s!}\int_M \Delta \tag{9a}$$

and

$$\Delta=\sum_{i_1,\cdots,i_{2s}}\varepsilon_{i_1 i_2\cdots i_{2s}}\Omega_{i_1 i_2}\wedge\cdots\wedge\Omega_{i_{2s-1}i_{2s}}, \tag{9b}$$

where Ω_{AB} are determined by the structure equations of M:

$$d\omega_A=\sum_B\omega_B\wedge\omega_{BA},\quad \omega_{AB}+\omega_{BA}=0$$

$$d\omega_{AB}=\sum_C\omega_{AC}\wedge\omega_{CB}+\Omega_{AB} \tag{10a}$$

$$\Omega_{AB}=\frac{1}{2}\sum_{C,D}\gamma_{ABCD}\omega_C\wedge\omega_D, \tag{10b}$$

where $i,\ j=1,\ \cdots,\ n-1$.

When $2s>4$, or $2s=4$ and $k\neq 0$, substituting (10a) and (10b) into (9b), and using $\langle Q,\ Q\rangle=1$, we see

$$\Delta=2^s s!(2s-1)!!(-1)^s k^{s-2}\left[hk-\frac{2(s-1)}{2s-1}m^2\right]dV, \tag{11}$$

$$dV=\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_{2s}.$$

When $2s=4$ and $k=0$, in a like manner, we have

$$\Delta=-16m^2 dV. \tag{12}$$

Substituting (11) or (12) into (9a), we find the statement ▮

Corollary : *Let M be a compact orientable Riemannian manifold of even dimension $n=2s$, $n\geqslant 4$. Suppose the curvature tensor of M satisfies conditions* (3) *and* (4).

i) *If $hk<0$ on M everywere, then $\chi(M)$ has the same sign as $-k^{\frac{1}{2}n}$.*

ii) *If k vanishes identically and $n>4$, then $\chi(M)=0$.*

iii) *If k vanishes identically and $n=4$, then $\chi(M)\leqslant 0$, and equality holds if and only* *if M is of quasi-constant curvature.*

iv) *If h vanishes identically but* $k \neq 0$, *then* $\chi(M)$ *is* 0 *or has the same sign as* $-k^{\frac{1}{2}n}$, *according as M is of quasi-constant curvature or not.*

The Corollary of Theorem 4 partially generalizes the Theorem 3 of Chern [3].

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Book review

H. BART, I. GOHBERG and M.A. KAASHOEK (eds): **Topics in Matrix and Operator Theory.** Workshop on Matrix and Operator Theory Rotterdam (The Netherlands), June 26 - 29, 1989 (Operator Theory: Advances and Applications: Vol. 50). Basel - Boston - Berlin: Birkhäuser Verlag 1991; 378 pp.

It is a matter of fact that modern operator theory is very much influenced by problems in electrical engineering, especially problems in networks, systems and control, and scattering theory. Taking this into account many workshops and conferences dedicated to the applications have been organized in the past. One of the most important and traditional meetings with this dedication are the biannual symposia on Mathematical Theory of Networks and Systems (MTNS). The MTNS meetings bring together mathematicians from different fields with