On Certain Singular Ordinary Differential Equations of the First Order in Banach Spaces

H. Ugowski

We consider in the interval $(0, T)$ certain linear and nonlinear singular ordinary differential equations of the first order, where the unknown function takes values in a Banach space and 0 is a singular point. Under suitable assumptions we prove that for each of these equations there exists a unique solution of the class C^1 in $(0, T)$ which is continuous at 0 or bounded in a right-hand neighbourhood of 0. Moreover, in the linear case there is introduced another version of assumptions which guarantees that every solution of the considered linear equations has the properties stated above.

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1. Preliminaries

Singular ordinary differential equations have been extensively investigated (see, for instance, [3,4,6–8,10–12]). In this paper we consider certain singular ordinary differential equations of the first order in Banach spaces. We discuss the existence of solutions of class C^1 of these equations which are continuous at a singular point or bounded in a neighbourhood of this point. In order to formulate the problems in question precisely we introduce some notation.

Let B be a real Banach space with norm $\| \cdot \|_B$. The limit and continuity of functions with values in B are understood in the strong sense. The same also concerns derivatives and integrals of functions of real variables with values in B. Let $D \subset \mathbb{R}$ be any interval. By $C(D, B)$ we denote the linear space of all continuous functions $x : D \to B$ and $C^1(D, B)$ denotes the linear space of all functions $x : D \to B$ which have derivative $x' \in C(D, B)$. Of course, $C^1(D, B) \subset C(D, B)$. If $x \in C(D, B)$ is a bounded function, then we define the norm

$$\| x \|_{D,B} = \sup [\| x(t) \|_B : t \in D].$$

The set $C_0(D, B)$ of all bounded continuous functions with the above norm is a Banach space. Clearly, $C_0(D, B) = C(D, B)$ for any closed interval D. In case $B = \mathbb{R}$ we omit this symbol in the above notation.

By $L(B)$ we denote Banach space of all bounded linear operators from B into B, where the norm $\| \cdot \|_{L(B)}$ is defined in the usual way. The zero element of $L(B)$ is denoted by I_0. Finally, $C(D, L(B))$ and $C^1(D, L(B))$ denote the spaces of operator functions which are the counterparts of the spaces $C(D, B)$ and $C^1(D, B)$, respectively.

In section 2 we consider the nonlinear equation

$$\frac{d}{dt}[a(t) x(t)] = f(t, x(t)), \quad \forall t \in (0, T) \tag{1.1}$$

(T being a positive constant or $T = \infty$), where x and f take values in B, a takes values in $L(B)$ and $t = 0$ is a singular point (i.e., $a(0) = I_0$). We prove the existence of a unique solution $x \in C^1((0,T), B)$ of this equation which is continuous at 0 or bounded in a right-hand neighbourhood of 0.

In Section 3 the results of Section 2 are formulated for the linear equation

$$\frac{d}{dt}[a(t)\,x(t)] = b(t)\,x(t) + c(t), \quad \forall t \in (0,T), \tag{1.2}$$

where b and c take values in $L(B)$ and B, respectively. Next equation (1.2) and the equation

$$a(t)\,x'(t) = b(t)\,x(t) + c(t), \quad \forall t \in (0,T) \tag{1.3}$$

are considered in the case where a and b are real functions. In this case we obtain more general results than these ones mentioned above.

Finally, in Section 4 there are formulated and proved two Hospital rules for the ratio f/g in the case where g is a real function and f takes values in a normed space. These rules were used in Sections 2 and 3.

The employment of a Banach space B instead of the Euclidean space $\mathbb{R}$ is justified because this enables us to formulate the results concerning equations (1.1)—(1.3) for various particular cases. We give, for instance, the following examples.

1. Finite or infinite systems of singular scalar equations. In this case we take $B = \mathbb{R}^n$ or B is a suitable Banach space of infinite sequences.

2. Certain random singular equations. Then we introduce the complete probability space $(\Omega, \mathcal{F}, P)$ and we define B as the Banach space consisting of all random variables $\xi :$ $\Omega \to \mathbb{R}$ with finite norm

$$\| \xi \|_B = \left[\int_\Omega |\xi(\omega)|^q P(d\omega) \right]^{1/q}, \quad q \in [1, \infty) \quad or \quad \| \xi \|_B = \operatorname*{ess\,sup}_{\omega \in \Omega} |\xi(\omega)|.$$

Systems of random singular equations may be also considered. Theory concerning nonsingular random equations can be found, for instance, in monographs [2,14,15].

3. Various classes of singular equations with parameter, for instance the case where $B = C[\lambda_0, \lambda_1]$. Of course, these equations involve random equations as particular cases.

General assumptions concerning equations (1.1)–(1.3) introduced in Sections 2 and 3 enable us to formulate assumptions for the above examples without difficulty. Therefore we shall not further discuss the examples mentioned.

There are many papers devoted to singular equations in Banach spaces which are interpreted as parabolic equations (see, for instance, [7,8,10] and the references therein). There are made general assumptions which require advanced theory (in particular semigroups and interpolation spaces). Applying the results of those papers to concrete examples we have to show that appropriate assumptions are satisfied. The present paper is devoted to certain classical singular ordinary differential equations and we make simple assumptions which can be easily verified in concrete examples. The results of the paper are proved with the aid of the classical methods of mathematical analysis.

2. Nonlinear singular equation

In this section we consider equation (1.1) under the following assumptions.

(2.I) $a \in C^1([0,T), L(B))$, $a(0) = I_0$ and there exist inverse operators $[a'(0)]^{-1}$, $[a(t)]^{-1} \in L(b)$ for all $t \in (0,T)$.

(2.II) $f : [0,T) \times B \to B$ is a continuous function which satisfies the Lipschitz condition

$$\| f(t,x) - f(t,y) \|_B \leq M(t) \| x - y \|_B, \quad \forall t \in [0,T), \quad x, y \in B,$$

where $M : [0,T) \to [0,\infty)$ is a continuous function such that

$$0 \leq M(0) < \left(\| [a'(0)]^{-1} \|_{L(B)} \right)^{-1} = \alpha. \tag{2.1}$$

It follows from assumption (2.I) that

$$a^{-1} \in C((0,T), L(B)) \tag{2.2}$$

(see [13 : p.33]). Moreover, for the function $a_0 : [0,T) \to L(B)$ given by the formula $a_0(t) = a(t)/t$ for $t \in (0,T)$ and $a_0(0) = a'(0)$ we have $a_0 \in C([0,T), L(B))$ and there exists the inverse

$$a_0^{-1} \in C([0,T), L(B)), \tag{2.3}$$

where $a_0^{-1}(t) = ta^{-1}(t)$ for $t \in (0,T)$ and $a_0^{-1}(0) = [a'(0)]^{-1}$. This implies that

$$\lim_{t \to 0^+} \| ta^{-1}(t) - [a'(0)]^{-1} \|_{L(B)} = 0. \tag{2.4}$$

Now we formulate and prove the following theorem.

Theorem 1. *If assumptions (2.I) and (2.II) are satisfied, then there exists a unique solution x of equation (1.1) in the set*

$$C([0,T), B) \cap C^1((0,T), B). \tag{2.5}$$

Moreover, for the above solution we have

$$x(0) = x_0, \tag{2.6}$$

where x_0 is a unique solution of the equation

$$x_0 = [a'(0)]^{-1} f(0, x_0). \tag{2.7}$$

Proof. Assumption (2.II) and the Banach fixed-point theorem imply that there exists a unique solution $x_0 \in B$ of the equation (2.7). It is clear that in space $C([0,T), B)$ equation (1.1) is equivalent to the equation

$$x(t) = [a(t)]^{-1} \int_0^t f(s, x(s)) \, ds, \quad \forall t \in (0,T). \tag{2.8}$$

Let x be a solution of equation (1.1) in the set (2.5). Then, taking into account (2.8), (2.4) and the relation

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t f(s, x(s)) \, ds = f(0, x(0)) \tag{2.9}$$

(following from the Hospital rule - Theorem 8, Sec. 4), we find that $z(0) = \lim_{t \to 0+} z(t) = [a'(0)]^{-1} f(0, z(0))$. This implies condition (2.6).

It remains to prove the existence of a unique solution of equation (1.1) in the set (2.5). First consider the equation

$$\frac{d}{dt}[a(t)z(t)] = f(t, z(t)), \quad \forall t \in (0, t_1], \tag{2.10}$$

where $t_1 \in (0, T)$ is a constant which will be specified later. In space $C([0, t_1], B)$ this equation is equivalent to the equation

$$z(t) = [a(t)]^{-1} \int_0^t f(s, z(s)) \, ds, \quad \forall t \in (0, t_1]. \tag{2.11}$$

Denote $K_0 = \{z \in C([0, t_1], B) : z(0) = z_0\}$ and for any $z \in K_0$ define the function $X : [0, t_1] \to B$ by the formula

$$X(t) = [a(t)]^{-1} \int_0^t f(s, z(s)) \, ds, \quad \forall t \in (0, t_1], \qquad X(0) = z_0.$$

In virtue of (2.2),(2.4),(2.7),(2.9) and assumption (2.II) we have

$$X \in C((0, t_1], B) \quad \text{and} \quad \lim_{t \to 0+} X(t) = z_0$$

which implies that $X \in K_0$. Hence, setting $Zz = X$ for $z \in K_0$ we define an operator $Z : K_0 \to K_0$. This definition and assumption (2.II) yield the inequality

$$\| (Zx - Zy)(t) \|_B \leq M(t) \| a_0^{-1}(t) \|_{L(B)} \sup_{s \in [0, t]} \| z(s) - y(s) \|_B, \quad \forall x, y \in K_0, \quad t \in (0, t_1]. \tag{2.12}$$

Using condition (2.1) take any $\delta \in (M(0)/\alpha, 1)$. Then, by (2.3), there exists a $t_1 \in (0, T)$ such that $M(t) \| a_0^{-1}(t) \|_{L(B)} \leq \delta$ for $t \in [0, t_1]$. Consequently, by (2.12), we have

$$\| Zx - Zy \|_{[0, t_1], B} \leq \delta \| x - y \|_{[0, t_1], B}, \quad \forall x, y \in K_0.$$

According to the Banach fixed-point theorem there exists a unique solution $z_1 \in K_0$ of equation $x = Zx$. At the same time z_1 is a unique solution of equation (2.10) in the set $C([0, t_1], B) \cap C^1((0, t_1], B)$.

Now consider the problem

$$\frac{d}{dt}[a(t)z(t)] = f(t, z(t)), \quad \forall t \in [t_1, t_2], \quad z(t_1) = z_1(t_1),$$

where $t_2 \in (t_1, T)$ is arbitrarily fixed. We can write it in the form

$$z'(t) = [a(t)]^{-1}[f(t, z(t)) - a'(t)z(t)], \quad \forall t \in [t_1, t_2], \quad z(t_1) = z_1(t_1).$$

By the well-known existence and uniqueness theorem (see [5 : Sec. 1.1]) there exists a unique solution $z_2 \in C^1([t_1, t_2], B)$ of the above problem. In general, in the n-th step ($n \geq 2$) we obtain a unique solution $z_n \in C^1([t_{n-1}, t_n], B)$ of the problem

$$z'(t) = [a(t)]^{-1}[f(t, z(t)) - a'(t)z(t)], \quad \forall t \in [t_{n-1}, t_n],$$
$$z(t_{n-1}) = z_{n-1}(t_{n-1}),$$

where $t_n \in (t_{n-1}, T)$ is arbitrarily fixed. Suppose that $t_n \to T$ as $n \to \infty$ and define the function $x : [0, T) \to B$ by the formula

$$x(t) = x_n(t), \quad \forall t \in [t_{n-1}, t_n], \quad n = 1, 2, \dots \quad (t_0 = 0).$$

It is clear that x is a unique solution of equation (1.1) in the set (2.5) $\square$

Let us denote by $K_1(B)$ the set of all functions $x \in C((0, T), B)$ which are bounded in a right-hand neighbourhood of 0. We consider the existence of a unique solution of equation (1.1) in the set

$$K_1(B) \cap C^1((0, T), B). \tag{2.13}$$

The following assumptions are needed:

(2.III) $a \in C([0, T), L(B)) \cap C^1((0, T), L(B))$, $a(0) = I_0$, and for any $t \in (0, T)$ there exists an inverse $[a(t)]^{-1} \in L(B)$ such that

$$\| [a(t)]^{-1} \|_{L(B)} \leq \beta/t, \quad \forall t \in (0, \beta'), \tag{2.14}$$

where $\beta > 0$ and $\beta' \in (0, T)$ are certain constants.

(2.IV) $f : (0, T) \times B \to B$ is continuous function which satisfies the Lipschitz condition

$$\| f(t, x) - f(t, y) \|_B \leq M(t) \| x - y \|_B, \quad \forall t \in (0, T), \quad x, y \in B$$

and $f(\cdot, \vartheta) \in K_1(B)$, where ϑ is the zero element of B and $M \in K_1 = K_1(\mathbb{R})$ (i.e. $B = \mathbb{R}$) is a non-negative function such that

$$\gamma = \limsup_{t \to 0+} M(t) < 1/\beta. \tag{2.15}$$

Theorem 2. *If assumptions (2.III) and (2.IV) are satisfied, then there exists a unique solution of equation (1.1) in the set (2.13).*

Proof. We proceed like in the proof of Theorem 1. Namely, using (2.15) and (2.14), take any $\delta \in (\beta\gamma, 1)$ an select $t_1 \in (0, \beta')$ in such way that

$$0 < tM(t) \| [a(t)]^{-1} \|_{L(B)} \leq \delta, \quad \forall t \in (0, t_1]. \tag{2.16}$$

It follows from assumption (2.IV) that for any $x \in K_2 = C_0((0, t_1], B)$ the function $u : (0, t_1] \to B$ defined by the formula

$$u(t) = f(t, x(t)), \quad \forall t \in (0, t_1]$$

belongs to K_2 as well. This implies that $v \in K_2$ for all $x \in K_2$, where

$$v(t) = \int_0^t f(s, x(s)) \, ds, \quad \forall t \in (0, t_1].$$

Let $x \in K_2 \cap C^1((0, t_1], B)$ be a solution of equation (2.10). Then, using the relation

$$a(t)x(t) - a(\tau)x(\tau) = \int_\tau^t f(s, x(s)) \, ds, \quad \forall \tau, t \in (0, t_1],$$

we find that

$$a(t)x(t) = \int_0^t f(s, x(s)) \, ds, \quad \forall t \in (0, t_1].$$

This means that in space K_2 the equation (2.10) is equivalent to equation (2.11). For any $x \in K_2$ define Zx by the formula

$$(Zx)(t) = [a(t)]^{-1} \int_0^t f(s, x(s)) \, ds, \quad \forall t \in (0, t_1].$$

Using (2.III), (2.IV) and (2.16) and proceeding like in the proof of Theorem 1 we find that $Z : K_2 \to K_2$ and

$$\| Zx - Zy \|_{(0,t_1],B} \le \delta \| x - y \|_{(0,t_1],B}, \quad \forall x, y \in K_2.$$

According to the Banach fixed-point theorem there exists a unique solution $x_1 \in K_2$ of equation $x = Zx$. At the same time x_1 is a unique solution of equation (2.10) in the space $K_2 \cap C^1((0, t_1], B)$. The further proceeding is the same as in the proof of Theorem 1 □

Remark 1. *Notice that assumptions concerning the function f in equation (1.1) are in some sense similar to those of [11 : Sec. 3.6].*

Remark 2. *Taking into considerations Lemmas 3.5–3.7 of [13 : p. 22,23] and the proof of the Theorem 1 one can find that the condition $a \in C^1([0, T), L(B))$ may be replaced by the following weaker one : the function $a : [0, T) \to L(B)$ is continuously differentiable in the strong sense (i.e., for any $x \in B$ there exists the derivative $[a(\cdot)x]' \in C([0, T), B))$ and there holds the inequality (2.14). Then condition (2.1) should have the form $0 \le M(0) < \min(\alpha, 1/\beta)$.*

Remark 3. *Analoguosly to Remark 2 the condition*

$$a \in C([0, T), L(B)) \cap C^1((0, T), L(B))$$

in assumption (2.III) may be replaced by the following weaker one : the function $a : [0, T) \to L(B)$ is continuously differentiable in $(0, T)$ in the strong sense and $\lim_{t \to 0^+} \| a(t) \|_{L(B)} = 0$.

3. Linear singular equations

The results of Section 2 can be applied to the linear singular equation (1.2). As a corollary of Theorem 1 we obtain the following one.

Theorem 3. *Let assumption (2.I) be satisfied. Suppose that $c \in C([0, T), B)$ and let $b : [0, T) \to L(B)$ be strongly continuous such that*

$$\sup \left\{ \| b(t) \|_{L(B)} : t \in [0, \gamma] \right\} \le \| a'(0) \|_{L(B)} \quad \text{for some} \quad \gamma \in (0, T).$$

Then there exists a unique solution x of equation (1.2) in the set (2.5). Moreover, this solution satisfies the condition $x(0) = [a'(0) - b(0)]^{-1} c(0)$.

Proof. Indeed, there exists a function $M \in C[0,T)$ such that

$$M(0) < \| a'(0) \|_{L(B)}, \quad M(t) \geq \| b(t) \|_{L(B)}, \quad \forall t \in [0,T).$$

Hence it follows that the function f defined by the formula

$$f(t,z) = b(t)z + c(t), \quad \forall t \in [0,T), \, z \in B$$

satisfies assumption (2.II) which implies the assertion $\square$

In a similar way we obtain the following theorem as a corollary of Theorem 2.

Theorem 4. *Let assumption (2.III) be satisfied. Suppose that $c \in K_1(B)$ and let $b :$ $(0,T) \to L(B)$ be strongly continuous such that the function $\| b(t) \|_{L(B)}$ is bounded in right-hand neighbourhood of 0 and $\limsup_{t \to 0+} \| b(t) \|_{L(B)} < 1/\beta$. Then there exists a unique solution of equation (1.2) in the set (2.13).*

Now we consider the linear singular equation (1.3) in the case where a and b are real functions, whereas z and c take values in the Banach space B. The following assumptions are needed:

(3.I) $a, b \in C[0,T), \quad c \in C(0,T), B)$,

(3.II) $a(0) = 0, \quad a(t) > 0, \quad \forall t \in (0,T)$,

(3.III) $a(t) \leq \alpha t, \quad \forall t \in (0,\beta)$, where $\alpha > 0$, $\beta \in (0,T)$ are certain constants.

Theorem 5. *If assumptions (3.I)–(3.III) are satisfied, then the following assertions hold.*

$1°$ *If $b(0) < 0$, then there exists a unique solution x of equation (1.3) in the set (2.5). Moreover, this solution satisfies the condition*

$$x(0) = -c(0)/b(0). \tag{3.1}$$

$2°$ *If $b(0) > 0$, then in the space $C^1((0,T), B)$ there exists a one-parameter family K of solutions of equation (1.3) (i.e., the general solution of this equation). Moreover, we have $\lim_{t \to 0+} x(t) = -c(0)/b(0)$ for all $x \in K$.*

Proof of $1°$. Let us introduce the functions

$$g(t) = b(t)/a(t), \quad \forall t \in (0,T), \tag{3.2}$$

$$G(t) = \int_\beta^t g(s)\,ds, \quad \forall t \in (0,T). \tag{3.3}$$

The general solution of equation (1.3) is given by the formula

$$x(t) = x_0(t)(P_0(t) + N), \quad \forall t \in (0,T), \tag{3.4}$$

where $N \in B$ and

$$x_0(t) = \exp(G(t)), \quad \forall t \in (0,T), \tag{3.5}$$

$$P_0(t) = \int_0^t \frac{c(s)}{a(s)} \exp(-G(s))\,ds, \quad \forall t \in (0,T). \tag{3.6}$$

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In order to verify this fact we should show the convergence of the integral (3.6). Indeed, in view of (3.I),(3.III),(3.2) and $b(0) < 0$ we have

$$t\,g(t) < b(0)/(2\alpha) = \gamma < 0, \quad \forall t \in (0,\beta_1),$$

where $\beta_1 \in (0,\beta)$ is a constant. Hence, by (3.3), it follows that

$$G(t) \geq N_1 + \gamma \ln t, \quad \forall t \in (0,\beta_1), \tag{3.7}$$

$N_1 \in \mathbb{R}$ being a constant. Relations (3.7) and (3.5) imply that

$$\lim_{t \to 0^+} G(t) = \infty, \qquad \lim_{t \to 0^+} x_0(t) = \infty. \tag{3.8}$$

Taking into account the inequality

$$[a(t)]^{-1} \exp(-G(t)) \leq N_2 t^{-1-\gamma}, \quad \forall t \in (0,\beta_1), \quad N_2 = \text{const} > 0$$

(following from the (3.III) and (3.7)) we conclude that the integral (3.6) is convergent and

$$\lim_{t \to 0^+} P_0(t) = \vartheta. \tag{3.9}$$

The Hospital rule (Theorem 8, Sec. 4) and the relations (3.9),(3.8),(3.5),(3.6), (3.2),(3.3) imply that $\lim_{t \to 0^+} x_0(t)P_0(t) = -c(0)/b(0)$. Hence, by (3.8), it follows that function (3.4) with $N = \vartheta$, i.e.

$$x(t) = x_0(t)P_0(t), \quad \forall t \in (0,T), \qquad x(0) = -c(0)/b(0) \tag{3.10}$$

is a unique solution of equation (1.3) in the set (2.5) and this solution satisfies (3.1).

Proof of 2°. Retaining the definitions (3.2),(3.3) and (3.5) let us introduce the function

$$P_1(t) = \int_\beta^t \frac{c(s)}{a(s)} \exp(-G(s))\,ds, \quad \forall t \in (0,T). \tag{3.11}$$

Then the one-parametric family K of functions

$$x(t) = x_0(t)(P_1(t) + N), \quad \forall t \in (0,T), \quad (N \in B) \tag{3.12}$$

is the general solution of equation (1.3). Like in the proof of 1° we get $\lim_{t \to 0^+}[x_0(t)]^{-1} = \infty$. Hence, using (3.12),(3.11),(3.2),(3.3),(3.5) and applying the Hospital rule (Theorem 9, Sec. 4), we conclude that $\lim_{t \to 0^+} x(t) = -c(0)/b(0)$ □

Now we consider equation (1.2) under the following assumptions:

(**3.IV**) $a \in C^1[0,T], \quad a(0) = 0, \quad a'(0) \neq 0, \quad a(t) > 0, \quad \forall t \in (0,T).$
(**3.V**) $b \in C[0,T], \quad c \in C([0,T), B).$

Of course, assumption (3.IV) implies that $a'(0) > 0$. Writing equation (1.2) in the form

$$a(t)x'(t) = [b(t) - a'(t)]x(t) + c(t), \quad \forall t \in (0,T)$$

and applying to this equation Theorem 5 we obtain the following theorem.

Theorem 6. *If assumptions (3.IV),(3.V) are satisfied, then assertions 1° and 2° of Theorem 5 remain valid if we replace $b(0)$ by $b(0) - a'(0)$.*

Now we prove the following theorem concerning equation (1.2).

Theorem 7. *We assume that $a \in C[0,T) \cap C^1(0,T)$, $a(t) > 0$ for all $t \in (0,T)$, $a(0) = 0$, $b \in K_1$, $c \in K_1(B)$, where K_1 and $K_1(B)$ are defined in Section 2. Then the following assertions hold.*

1°   *If $a(t) \geq \alpha t$, $b(t) \leq \delta$ for all $t \in (0,\beta)$ $(\alpha > 0$, $\beta \in (0,T)$, $\delta \in (0,\alpha)$ being constants), then there exists a unique solution of equation (1.2) in the set (2.13).*

2°   *Let $\alpha t \leq a(t) \leq \alpha' t$, $b(t) \geq \delta$ for all $t \in (0,\beta)$, where $\beta \in (0,T)$, $\alpha > 0$, $\alpha' > \alpha$, $\delta > \alpha'$ are certain constants. Then in the space $C^1((0,T), B)$ there exists the general solution of equation (1.2) and, moreover, this solution belongs to $K_1(B)$.*

Proof of 1°. Let us introduce the function P_0 by the formula

$$P_0(t) = \int_0^t c(s) \exp(-G(s)) \, ds, \quad \forall t \in (0,T), \qquad P_0(0) = 0, \tag{3.13}$$

where G is defined by (3.2) and (3.3). In view of the inequalities

$$t\, g(t) \leq \gamma = \delta/\alpha, \quad \forall t \in (0,\beta) \quad (0 < \gamma < 1),$$

$$\exp(-G(t)) \leq N_3 t^{-\gamma}, \quad \forall t \in (0,\beta) \quad (N_3 = \text{const} > 0)$$

we have

$$\lim_{t \to 0^+} P_0(t) = \vartheta, \tag{3.14}$$

which implies that $P_0 \in C([0,T), B) \cap C^1((0,T), B)$. The function

$$x_0(t) = [a(t)]^{-1} \exp(G(t)), \quad \forall t \in (0,T) \tag{3.15}$$

belongs to $C^1(0,T)$ and is a particular solution of the scalar equation

$$\frac{d}{dt}[a(t)x(t)] = b(t)x(t), \quad \forall t \in (0,T). \tag{3.16}$$

Consequently, the general solution of equation (1.2) is given by the formula

$$x(t) = x_0(t)[P_0(t) + N] \quad (N \in B) \tag{3.17}$$

and this solution belongs to $C^1((0,T), B)$.

Now we show that $x_0 \notin K_1$. Indeed, suppose that a non-negative function $y \in K_1 \cap C^1(0,T)$ is a solution of equation (3.16). This yields the relation

$$a(t)y(t) = \int_0^t b(s)y(s) \, ds, \quad \forall t \in (0,T). \tag{3.18}$$

Take arbitrarily fixed $\bar{t} \in (0,\beta)$. Then, applying to (3.18) the integral mean value theorem, we obtain

$$y(\bar{t}) \leq [a(\bar{t})]^{-1} \int_0^{\bar{t}} \delta y(s) \, ds = \delta \bar{t} [a(\bar{t})]^{-1} y(\bar{t}_1) \text{ for some } \bar{t}_1 \in (0, \bar{t})$$

which implies that $y(\bar{t}) \leq \gamma y(\bar{t}_1)$. In the same way we find that

$$y(\bar{t}_1) \leq \gamma y(\bar{t}_2) \text{ for some } \bar{t}_2 \in (0, \bar{t}_1)$$

and consequently $y(\bar{t}) \leq \gamma^2 y(\bar{t}_2)$. In general we obtain a decreasing sequence $\bar{t}_1, \bar{t}_2, \ldots$ with elements from the interval $(0, \bar{t})$ such that

$$y(\bar{t}) \leq \gamma^n y(\bar{t}_n) \leq \gamma^n \sup[y(t) : t \in (0, \bar{t})], \quad n = 1, 2, \ldots$$

Hence, by condition $0 < \gamma < 1$, we get $y(\bar{t}) = 0$. At the same time we have proved that $y(t) = 0$ for all $t \in (0, \beta)$ which implies that $y(t) = 0$ for all $t \in (0, T)$. This fact and the inequality $x_0(t) > 0$ for all $t \in (0, T)$ yield the relation $x_0 \notin K_1$.

For the function (3.17) we have $\| x(t) \|_B \geq x_0(t) \left| \| N \|_B - \| P_0(t) \|_B \right|$. Hence, in view of (3.14) it follows that $x \notin K_1(B)$ for all $N \in B \setminus \{\vartheta\}$. It remains to consider the function

$$x(t) = x_0(t) P_0(t), \quad \forall t \in (0, T). \tag{3.19}$$

Using the relations

$$G(t) - G(s) = \int_s^t g(z)dz \leq \gamma \ln \frac{t}{s}, \quad \exp[G(t) - G(s)] \leq \left(\frac{t}{s} \right)^\gamma, \quad \forall t \in (0, \beta), \ s \in (0, t)$$

we obtain

$$\begin{aligned}
\| x(t) \|_B &\leq [a(t)]^{-1} \exp(G(t)) \int_0^t \| c(s) \|_B \exp(-G(s)) \, ds \\
&\leq \frac{1}{1-\gamma} \frac{t}{a(t)} \| c \|_{(0,\beta),B} \leq \frac{1}{\alpha - \delta} \| c \|_{(0,\beta),B}, \quad \forall t \in (0, \beta).
\end{aligned}$$

This means that the function (3.19) belongs to $K_1(B)$ which completes the proof of 1°.

Proof of 2°. The general solution of equation (1.2) is given by the formula

$$x(t) = x_0(t)[P_1(t) + N] \quad (N \in B), \tag{3.20}$$

where

$$P_1(t) = \int_\beta^t c(s) \exp(-G(s)) \, ds, \quad \forall t \in (0, T)$$

and $x_0(t)$ is defined by (3.15). One can show that

$$\exp(G(t)) \leq N_4 t^\gamma, \quad \forall t \in (0, \beta), \quad \gamma = \delta/\alpha' > 1,$$

$N_4 > 0$ being a constant. Hence, it follows that

$$x_0(t) \leq N_4 \alpha^{-1} t^{\gamma - 1}, \quad \forall t \in (0, \beta)$$

and consequently $\lim_{t \to 0+} x_0(t) = 0$. Further we have

$$\exp[G(t) - G(s)] \leq \left(\frac{s}{t} \right)^{-\gamma}, \quad \forall t \in (0, \beta), \ s \in (t, \beta),$$

which implies that

$$\begin{aligned}
\| x_0(t) P_1(t) \|_B &\leq [a(t)]^{-1} \int_t^\beta \| c(s) \|_B \exp(G(t) - G(s)) \, ds \\
&\leq [a(t)]^{-1} \| c \|_{(0,\beta),B} (\gamma - 1)^{-1} (t - t^\gamma \beta^{1-\gamma}) \leq [\alpha(\gamma - 1)]^{-1} \| c \|_{(0,\beta),B}
\end{aligned}$$

for any $t \in (0, \beta)$. At the same time we have proved that for any $N \in B$ the function (3.20) belongs to $K_1(B)$ $\square$

4. Hospital rules

In this section we state and prove two Hospital rules for the ratio f/g in the case where g is a real function and f takes values in a real normed space X with norm $\| \cdot \|$. These rules were used in Sections 2 and 3 of this paper. The proofs of the rules in question are based on the corollary from the Hahn-Banach extension theorem (see, for instance, [16 : p. 108]) and on the methods used in the case where f and g are real functions (see, for instance, [9 : Sec. 150,151]). Notice that the first Hospital rule formulated below is stated in [1 : p.241] as an exercise.

Theorem 8 (see [1]). *Let the following assumptions be satisfied:*

1° *The function $f : (a,b) \to X$ is strongly continuous, has a weak derivative $f'_w : (a,b) \to X$ and there exists the strong limit $\lim_{t \to b-} f(t) = \vartheta$ (ϑ being the zero element of X).*

2° *For the function $g : (a,b) \to \mathbb{R}$ there exists the derivative $g' : (a,b) \to \mathbb{R} \setminus \{0\}$ and $\lim_{t \to b-} g(t) = 0$.*

3° *There exists the strong limit $\lim_{t \to b-} f'_w(t)[g'(t)]^{-1} = z$.*

Then there exists the strong limit

$$\lim_{t \to b^-} f(t)/g(t) = z. \tag{4.1}$$

Proof. First of all notice that assumption 2° yields $g(t) \neq 0$ for all $t \in (a,b)$ and the strict monotonicity of g. Let us denote

$$h(t) = f(t) - z\, g(t), \quad \forall\, t \in (a,b).$$

Then, by 3°, for any $\varepsilon > 0$ there exists a $\delta \in (0, b-a)$ such that

$$\| h'_w(t)/g'(t) \| \leq \varepsilon, \quad \forall\, t \in [b-\delta, b). \tag{4.2}$$

We show that for any $\tau, s \in [b-\delta, b)$, $\tau < s$ there holds the inequality

$$\| [h(\tau) - h(s)][g(\tau) - g(s)]^{-1} \| \leq \varepsilon. \tag{4.3}$$

Indeed, if $h(\tau) = h(s)$, then (4.3) is true. In the case where $h(\tau) \neq h(s)$ there exists a bounded linear functional F in X with norm $\| F \| = 1$ such that

$$F\left\{ [h(\tau) - h(s)][g(\tau) - g(s)]^{-1} \right\} = \| [h(\tau) - h(s)][g(\tau) - g(s)]^{-1} \| . \tag{4.4}$$

In view of (4.2) the real function $H = Fh$ satisfies the inequality $|H'(t)/g'(t)| \leq \varepsilon$ for all $t \in [b-\delta, b)$. Hence, by the Cauchy theorem, it follows that

$$\left| [H(\tau) - H(s)][g(\tau) - g(s)]^{-1} \right| = |H'(t)/g'(t)| \leq \varepsilon \text{ for some } t \in [\tau, s).$$

Consequently, by (4.4), we get (4.3) which implies that

$$\| [f(\tau) - f(s)][g(\tau) - g(s)]^{-1} - z \| \leq \varepsilon. \tag{4.5}$$

Passing to the limit in (4.5) as $s \to b^-$ we obtain, by 1° and 2°, $\| f(\tau)[g(\tau)]^{-1} - z \| \leq \varepsilon$. Since this inequality has been shown for any $\tau \in [b-\delta, b)$, relation (4.1) holds □

Theorem 9. *Let the following assumptions be satisfied:*

1° *The function $f : (a,b) \to X$ has the weak derivative $f'_\omega : (a,b) \to X$.*

2° *For the function $g : (a,b) \to \mathbb{R}$ there exists the derivative $g' : (a,b) \to \mathbb{R} \setminus \{0\}$ and $\lim_{t \to b-} g(t) = \infty$ or $\lim_{t \to b-} g(t) = -\infty$.*

3° *There exists the strong limit $\lim_{t \to b-} f'_\omega(t)/g'(t) = z$.*

Then there exists the strong limit $\lim_{t \to b-} f(t)/g(t) = z$.

Proof. It suffices to consider the case

$$\lim_{t \to b-} g(t) = \infty. \tag{4.6}$$

Then there exists a $t_0 \in [a,b)$ such that $g(t) > 0$ for all $t \in (t_0, b)$ and we have $g'(t) > 0$ for all $t \in (a,b)$ which implies that g is an increasing function. Further we use the proof of Theorem 8, namely the relations (4.2)–(4.5). Hence it follows that

$$\| [f(s) - f(t_1)][g(s) - g(t_1)]^{-1} - z \| \le \epsilon, \quad \forall s \in (t_1, b), \tag{4.7}$$

where $t_1 = \max\{t_0, b - \delta\}$. Taking advantage of the relations

$$f(s)[g(s)]^{-1} - z = [f(t_1) - zg(t_1)][g(s)]^{-1}$$
$$+ \; [1 - g(t_1)/g(s)][(f(s) - f(t_1))(g(s) - g(t_1))^{-1} - z], \quad 0 < g(t_1)/g(s) < 1, \; \forall s \in (t_1, b)$$

we obtain the inequality

$$\| f(s)[g(s)]^{-1} - z \| \; \le \; \| [f(t_1) - zg(t_1)] \| \, [g(s)]^{-1}$$
$$+ \; \| f(s) - f(t_1)[(g(s) - g(t_1)]^{-1} - z \|, \quad \forall s \in (t_1, b). \tag{4.8}$$

By (4.6) there exists a $t_2 \in (t_1, b)$ such that

$$\| [f(t_1) - zg(t_1)] \| \, /g(s) \le \epsilon, \quad \forall s \in (t_2, b).$$

Hence by (4.7) and (4.8), we have $\| [f(s)[g(s)]^{-1} - z \| \le 2\epsilon$ for all $s \in (t_2, b)$. At the same time we have proved the relation (4.1) and the proof is completed $\square$

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Dr. Henryk Ugowski
Technical University of Gdańsk
Department of Differential Equations
ul. Majakowskiego 11/12
P — 80-952 Gdańsk

Book reviews

I. GOHBERG (ed.): **Extension and Interpolation of Linear Operators and Matrix Functions** (Operator Theory: Advances and Applications: Vol. 47). Basel - Boston - Berlin : Birkhäuser Verlag 1990, 305 pp.

The anthology, which is dedicated to the great Soviet mathematician M. G. Krein, who died on October 17th, 1989, contains papers on special aspects of Schur analysis. The present interest in the topic is documented on one hand by a great number of single publications during the last decade, on the other hand by the following monographs published lately or to appear:

- H. Dym: *Contractive Matrix Functions. Reproducing Kernel Hilbert Spaces and Interpolation* (CBMS Regional Conf. Ser. Math.: Vol. 71). Providence: Amer. Math. Soc. 1989.
- C. Foias and A. E. Frazho: *The Commutant Lifting Approach to Interpolation Problems* (Operator Theory: Advances and Applications: Vol. 44). Basel: Birkhäuser Verlag 1990.
- J. A. Ball, I. Gohberg, and L. Rodman: *Interpolation of Rational Matrix Functions* (Operator Theory: Advances and Applications: Vol. 45). Basel: Birkhäuser Verlag 1990.
- V. K. Dubovoj, B. Fritzsche, and B. Kirstein: *The Matricial Version of the Classical Schur Problem* (Teubner-Texte zur Mathematik). Leipzig: B.G. Teubner Verlagsges. 1992.