Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11 (1992) 1, 43 – 55

A Remark on Interpolation with Generalized Parameters

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We study a generalized form of the Lions-Peetre interpolation spaces $(A_0, A_1)_{\Theta q}$, where the parameter Θ and q are substituted in a natural way by suitable sequences u and lattices $\mathbf{a}$, respectively. A reiteration theorem is proved and applications to generalized Lorentz spaces are given.

Key words: *Interpolation spaces, Lorentz spaces*
AMS subject classification: 46 M, 46 E

1. Introduction

In this paper we study a generalized form of the Lions-Peetre real interpolation spaces $(A_0, A_1)_{\Theta q}$. By discretization these spaces can be defined to consist of all $x \in \Sigma(\bar{A}) = A_0 + A_1$ such that

$$\left\{ \sum_{-\infty}^{\infty} |2^{-\Theta k} K(2^k, x, \bar{A})|^q \right\}^{1/q} = \|(2^{-\Theta k} K(2^k, x, \bar{A}))_{k \in \mathbf{Z}} | l_q(\mathbf{Z})\| < \infty,$$

where $K(t, x, \bar{A}) = K(t, x, A_0, A_1) = \inf_{x = x_0 + x_1}(\|x_0|A_0\| + t\|x_1|A_1\|)$ is the usual K-functional (see [1,p.41]). If we replace the sequence $(2^{-\Theta k})_{k \in \mathbf{Z}}$ and the sequence space $l_q(\mathbf{Z})$ by more general and appropriate sequences $u = (u_k)_{k \in \mathbf{Z}}$ and sequence spaces $\mathbf{a}$, respectively, then we obtain more general interpolation spaces $(A_0, A_1)_{u,\mathbf{a}}$, which consist of all $x \in \Sigma(\bar{A})$ such that $\|(u_k K(2^k, x, \bar{A}))_{k \in \mathbf{Z}} | \mathbf{a}\| < \infty$. The parameter u plays the role of the main parameter, whereas $\mathbf{a}$ is some kind of fine parameter.

Several authors have studied generalized interpolation spaces. Let us mention the interpolation with a parameter function φ (cf.[4, 5, 11]). In our setting this method corresponds to the special case $u = (1/\varphi(2^k))$ and $\mathbf{a} = l_q(\mathbf{Z})$. A more general form of interpolation — including our definition — was treated by NILSSON (cf.[6]) and BRUDNYI AND KRUGLJAK (cf.[2, 3]). We use their fundamental result concerning reiteration.

The main result of the paper is a reiteration theorem (see Theorem 3.9), which, in its turn, is a special case of the general theorem of BRUDNYI AND KRUGLJAK and NILSSON (see Theorem 3.8). Moreover, we discuss the equivalence theorem, the power theorem, interpolation between intersection and sum and give an application of our results to generalized Lorentz spaces.

CONVENTIONS. If no confusion can occur we use the notation $u = (u_k)$ for any sequence with $\mathbf{N}$ or $\mathbf{Z}$ as indexing set, otherwise they will be written as $u = (u_k)_{k \in \mathbf{N}}$, $u = (u_k)_{k \in \mathbf{Z}}$, or $u = (u_k)_{k\geq 0}$, respectively. The equivalence $a_k \asymp b_k$ means that $c_1 a_k \leq b_k \leq c_2 a_k$ for all k and some positive constants c_1 and c_2. Two quasi-normed spaces A and B are considered as equal and we write $A = B$, whenever their quasi-norms are equivalent. Finally we will write log instead of log_2.

2. Interpolation spaces

Let $\mathbf{a}$ be a quasi-Banach space of real valued sequences with $\mathbf{Z}$ (resp. $\mathbf{N}$) as index set.This space is called a $\mathbf{Z}$-*lattice* (resp. $\mathbf{N}$-*lattice*) if it has the following monotonicity property:

$$\|(x_k)|\mathbf{a}\| \leq \|(y_k)|\mathbf{a}\| \quad \text{whenever} \quad |x_k| \leq |y_k| \quad \text{for all} \quad k \in \mathbf{Z} \quad (\text{resp. } k \in \mathbf{N}).$$

By $\mathbf{a}(u)$, where $u = (u_k)$ is a positive sequence, we denote the space of all sequences $(x_k) \in \mathbf{a}$, such that $(x_k u_k) \in \mathbf{a}$. When equipped with the quasi-norm $\|(x_k)|\mathbf{a}(u)\| = \|(x_k u_k)|\mathbf{a}\|$ the space $\mathbf{a}(u)$ becomes a Z-lattice. Let $\mathbf{a}$ and $\mathbf{b}$ be Z-lattices. A sequence (x_k) belongs to the *quotient* $\mathbf{a}^{-1}\circ\mathbf{b}$ if $(x_k y_k) \in \mathbf{b}$ for all $(y_k) \in \mathbf{a}$. When equipped with the quasi-norm

$$\|(x_k)|\mathbf{a}^{-1}\circ\mathbf{b}\| = \inf\{c : \|(x_k y_k)|\mathbf{b}\| \leq c\|(y_k)|\mathbf{a}\|\}$$

the space $\mathbf{a}^{-1}\circ\mathbf{b}$ becomes a Z-lattice. A sequence (x_k) belongs to the *product* $\mathbf{a}\circ\mathbf{b}$ if there are $(y_k) \in \mathbf{a}$ and $(z_k) \in \mathbf{b}$ such that $x_k = y_k z_k$. When equipped with the quasi-norm

$$\|(x_k)|\mathbf{a}\circ\mathbf{b}\| = \inf\{\|(y_k)|\mathbf{a}\|\,\|(z_k)|\mathbf{b}\| : x_k = y_k z_k\}$$

the space $\mathbf{a}\circ\mathbf{b}$ becomes a Z-lattice. Note the following simple fact:

$$\mathbf{a}(u) \subseteq \mathbf{b}(v) \iff (u_k^{-1} v_k) \in \mathbf{a}^{-1}\circ\mathbf{b}. \tag{1}$$

The use of the notation $\mathbf{a}(u)$ will be normalized by $\|e_k|\mathbf{a}\| \asymp 1$. By $\mathbf{a} = \mathbf{a}^- \oplus \mathbf{a}^+$ we denote the canonical decomposition of a Z-lattice $\mathbf{a}$ into the sum of two N-lattices. This decomposition is generated by the following conventions concerning sequences:

Let $x = (x_k)_{k\geq 0}, y = (y_k)_{k\geq 0}$ be N-sequences, then we put

$$x \oplus y = (z_k)_{k\in\mathbf{Z}}, \text{ where } z_k = \begin{cases} y_k & \text{for } k \geq 0 \\ x_{-k-1} & \text{for } k < 0 \end{cases},$$

and vice versa, if $x = (x_k)_{k\in\mathbf{Z}}$ we put $x^+ = (x_k)_{k\geq 0}$ and $x^- = (x_{-k-1})_{k\geq 0}$. Obviously $x = x^- \oplus x^+$.

By S_+ (resp. S_-) we denote the *right* (resp.*left*) *shift operator* acting on the lattice $\mathbf{a}$, i.e. $\mathsf{S}_+((x_k)) = (x_{k-1})$ (resp. $\mathsf{S}_-((x_k)) = (x_{k+1})$). Observe, that in the case of an N-lattice we have to set $x_{-1} = 0$. Let $\mathbf{a}$ be an N-lattice. By D we denote the *double sequence operator* $\mathsf{D}((x_k)) = (x_{[k/2]})$ and by T_φ the *subsequence operator* $\mathsf{T}_\varphi((x_k)) = (x_{\varphi(k)})$, where $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ satisfies $0 < \varphi(k+1) - \varphi(k) \leq n_0$ for all $k \in \mathbf{N}$.

Definition 2.1: A Z-lattice (resp. N-lattice) a is called *admissible* if the

(i) right and the left shift operators have spectral radius 1.

(ii) double sequence operator D and all subsequence operators T_φ are bounded in $\mathbf{a}^+$ and $\mathbf{a}^-$, respectively.

Observe that $\mathbf{a} = l_q(\mathbf{Z})$ is an admissible Z-lattice.

A positive sequence $v = (v_k)$ is said to be *quasi-geometric* if the sequences (v_k/v_{k+1}) and (v_{k+1}/v_k) are bounded from above. We further assume that $v_0 = 1$. The set of these sequences will be denoted by $Q_{\mathbf{Z}}$ and $Q_{\mathbf{N}}$ if the indexing set is $\mathbf{Z}$ or $\mathbf{N}$, respectively. For quasi-geometric sequences the following quantities make sense:

(i) Let $v = (v_k) \in Q_{\mathbf{Z}}$. We put $\hat{v}_k = \sup_{l\in\mathbf{Z}} \frac{v_{k+l}}{v_l}$. Observe that $\hat{v}_{k+j} \le \hat{v}_k\hat{v}_j$. The *Boyd indices* are defined by

$$\alpha(v) = \lim_{k\to-\infty} \frac{\log \hat{v}_k}{k} \quad \text{and} \quad \beta(v) = \lim_{k\to\infty} \frac{\log \hat{v}_k}{k}.$$

(ii) Furthermore, let $v = (v_k) \in Q_{\mathbf{N}}$. We put $\overline{v} = (\overline{v}_k)$, where $\overline{v}_k = \sup_{l\ge 0} \frac{v_{k+l}}{v_l}$ and $\underline{v} = (\underline{v}_k)$, where $\underline{v}_k = \sup_{l\ge 0} \frac{v_l}{v_{k+l}}$. Observe that $\underline{v}_k = \overline{v_k^{-1}}$, $\overline{v}_{k+j} \le \overline{v}_k\overline{v}_j$, and $\underline{v}_{k+j} \le \underline{v}_k\underline{v}_j$ $(k, j \ge 0)$. Moreover, we introduce

$$\overline{\gamma}(v) = \lim_{k\to\infty} \frac{\log \overline{v}_k}{k} \quad \text{and } \underline{\gamma}(v) = \lim_{k\to\infty} \frac{\log \underline{v}_k}{k},$$

respectively. Obviously we have

$$\lim_{k\to\infty} \underline{v}_k = 0 \iff \underline{\gamma}(v) < 0 \text{ and } \lim_{k\to\infty} \overline{v}_k = 0 \iff \overline{\gamma}(v) < 0.$$

Now we fix some standard basic notations concerning interpolation spaces. Let A_0, A_1 be two quasi-Banach spaces. We say that (A_0, A_1) is a *compatible pair* if there is a Hausdorff topological vector space $\mathcal{A}$ such that the injections $A_i \longrightarrow \mathcal{A}$ (i=1,2) are continuous. A compatible pair will be denoted by $\bar{A} = (A_0, A_1)$. For a pair $\bar{A}$ we put $\Sigma(\bar{A}) = A_0 + A_1$, $\Delta(\bar{A}) = A_0 \cap A_1$. Furthermore, for $x \in \Sigma(\bar{A})$ we define the K- *functional*

$$K(t, x, \bar{A}) = \inf_{x=x_0+x_1} (\|x_0|A_0\| + t\|x_1|A_1\|)$$

and for $x \in \Delta(\bar{A})$ we define the J-*functional*

$$J(t, x, \bar{A}) = \max(\|x|A_0\|, t\|x|A_1\|).$$

Definition 2.2: Let a be an admissible Z-lattice and $u \in Q_{\mathbf{Z}}, -1 < \alpha(u) \le \beta(u) < 0$.

(i) For a pair $\bar{A} = (A_0, A_1)$ of quasi-Banach spaces we define the K-*space* $\bar{A}_{u,\mathbf{a};K}$ to consist of all $x \in \Sigma(\bar{A})$ such that $(K(2^k, x, \bar{A})) \in \mathbf{a}(u)$. Put $\|x|\bar{A}_{u,\mathbf{a};K}\| = \|(K(2^k, x, \bar{A}))|\mathbf{a}(u)\|$.

(ii) The J-*space* $\bar{A}_{u,\mathbf{a};J}$ is defined to consist of all $x \in \Sigma(\bar{A})$ that may be written as $x = \sum_k x_k, x_k \in \Delta(\bar{A})$ (convergence in $\Sigma(\bar{A})$) with $(J(2^k, x_k, \bar{A})) \in \mathbf{a}(u)$. We put $\|x|\bar{A}_{u,\mathbf{a};J}\| = \inf_{x=\sum_k x_k} \|(J(2^k, x_k, \bar{A}))|\mathbf{a}(u)\|$.

Remark 2.3: (i) The imposed conditions $-1 < \alpha(u) \leq \beta(u) < 0$ guarantee that the spaces are non-trivial. (ii) For $u = (2^{-\theta k})$ and $\mathbf{a} = l_q$, $0 < \theta < 1$, $0 < q \leq \infty$, we recover the $\bar{A}_{\theta,q}$-scale of Lions-Petree (see [1]). (iii) Let $Q(0,1)$ denote the class of functions ρ on $(0,\infty)$ such that for some $\epsilon > 0$, $\rho(t)\, t^{-\epsilon}$ is non-decreasing and $\rho(t)\, t^{-1+\epsilon}$ is non-increasing. The interpolation space with the parameter function $\rho \in Q(0,1)$ is defined to be

$$\bar{A}_{\rho,q;K} = \left\{ x \in \Sigma(\bar{A}) : \left(\int_0^\infty (K(t,x,\bar{A})/\rho(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

(cf. [11]). These spaces also generalize the Lions-Peetre real interpolation spaces $\bar{A}_{\theta q}$ (put $\rho(t) = t^\theta$). On the other hand, an easy computation shows that

$$\int_{2^k}^{2^{k+1}} (K(t,x,\bar{A})/\rho(t))^q \frac{dt}{t} \asymp (u_k K(2^k,x,\bar{A}))^q,$$

where $u_k = 1/\rho(2^k)$ and $u = (u_k) \in Q_{\mathbf{Z}}$, $-1 < \alpha(u) \leq \beta(u) < 0$. This implies that $\bar{A}_{\rho,q;K} = \bar{A}_{(1/\rho(2^k)),q;K}$. The interpolation spaces $\bar{A}_{\mathbf{u},\mathbf{a};K}$ recover also the interpolation spaces with a parameter function.

Lemma 2.4: *Let* $\mathbf{a}$ *be an admissible* $\mathbf{Z}$*-lattice. Let* ρ *and* q *with* $0 < \rho < 1$ *and* $0 < q \leq 1$, *then there is a constant* C *such that*

(i) $\|(\{\sum_j \rho^{|j-k|}|x_j|^q\}^{1/q})_{k\in\mathbf{Z}}|\mathbf{a}\| \leq C\, \|(x_k)_{k\in\mathbf{Z}}|\mathbf{a}\|$

(ii) $\|(\{\sum_{j=0}^\infty \rho^{|j-k|}|x_j|^q\}^{1/q})_{k\in\mathbf{Z}}|\mathbf{a}\| \leq C\, \|(x_k)_{k\in\mathbf{N}}|\mathbf{a}^+\|$.

PROOF: Put $p = 1/q$ and assume $\mathbf{a}$ to be r-normed. Then it follows from

$$\{\sum_j \rho^{|j-k|}|x_j|^q\}^{1/q} = \{\sum_j \rho^{(1-q)|j|}(\rho^{|j|}|x_{j+k}|)^q\}^{1/q} \leq \{\sum_j \rho^{(1-q)p'|j|}\}^{1/p'q}\{\sum_j \rho^{|j|}|x_{j+k}|\}$$

that

$$\|(\{\sum_j \rho^{|j-k|}|x_j|^q\}^{1/q})|\mathbf{a}\| \leq C_1\, \|(\sum_j \rho^{|j|}|x_{j+k}|)|\mathbf{a}\| \leq C_1\, \{\sum_j (\rho^{|j|}\|(x_{j+k})|\mathbf{a}\|)^r\}^{1/r}.$$

Choose ρ_0 with $1 < \rho_0 < 1/\rho$. Since the spectral radius of the shift operator is 1 we can find a constant C_2 such that $\max(\|S_+^j\|, \|S_-^j\|) \leq C_2\, \rho_0^j$ for each $j \in \mathbf{N}$. Consequently,

$$\begin{aligned} \|(\{\sum_j \rho^{|j-k|}|x_j|^q\}^{1/q})|\mathbf{a}\| &\leq C_1 C_2\, \{\sum_j (\rho^{|j|}\rho_0^{|j|}\|(x_k)|\mathbf{a}\|)^r\}^{1/r} \\ &= C_1 C_2\, \{\sum_j (\rho\rho_0)^{r|j|}\}^{1/r}\|(x_k)|\mathbf{a}\| \\ &\leq C\, \|(x_k)_{k\in\mathbf{Z}}|\mathbf{a}\|. \end{aligned}$$

To prove (ii) observe that $\{\sum_{j=0}^\infty \rho^{|j-k|}|x_j|^q\}^{1/q} = \rho^{-k/q}\{\sum_{j=0}^\infty \rho^j|x_j|^q\}^{1/q}$ for $k < 0$. Combined with (i), (ii) follows ■

Using Lemma 2.4 and the definition of the Boyd indices a straight- forward calculation shows that

$$\|(\{\sum_j (\min(1,2^{k-j})|x_j|)^q\}^{1/q})|\mathbf{a}(u)\| \le C\ \|(x_k)|\mathbf{a}(u)\|, \quad 0 < q \le 1.$$

Applying this statement the proof of the equivalence of the K- and J-method given in [1, Theorem 3.11.3] immediately carries over (see also [6, Lemma 2.5]).

Theorem 2.5 (The equivalence theorem): *Let* $\mathbf{a}$ *be an admissible* $\mathbf{Z}$*-lattice and* $u \in Q_{\mathbf{Z}}$, $-1 < \alpha(u) \le \beta(u) < 0$. *Then we have* $\bar{A}_{u,\mathbf{a};K} = \bar{A}_{u,\mathbf{a};J}$.

In the sequel we shall write $\bar{A}_{u,\mathbf{a}}$ instead of $\bar{A}_{u,\mathbf{a};K}$ or $\bar{A}_{u,\mathbf{a};J}$. In the following theorem we summerize some properties of $\bar{A}_{u,\mathbf{a}}$.

Theorem 2.6: *Let* $\bar{A} = (A_0, A_1)$ *be a given couple. Then*

(i) $(A_0, A_1)_{u,\mathbf{a}^- \oplus \mathbf{a}^+} = (A_1, A_0)_{u^*,\mathbf{a}^+ \oplus \mathbf{a}^-}$, *where* $u^* = (2^{-k} u_{-k})$.

(ii) $\bar{A}_{u,\mathbf{a}} \subseteq \bar{A}_{v,\mathbf{a}}$ *if* $(u_k^{-1} v_k) \in \mathbf{a}^{-1} \circ \mathbf{b}$.

(iii) $\bar{A}_{u,\mathbf{a}}$ *is a quasi-Banach space.*

PROOF: (i) is an immediate consequence of $K(2^k, x, A_0, A_1) = 2^k K(2^{-k}, x, A_1, A_0)$ and the boundness of the right shift operator. (ii) follows from (2.1). (iii) Without loss of generality we may suppose that $A_0, A_1, \Sigma(\bar{A}), \mathbf{a}$, and $\bar{A}_{u,\mathbf{a}}$ are q-normed ($0 < q \le 1$). Let $\sum_1^\infty \|x_j|\bar{A}_{u,\mathbf{a}}\|^q < \infty$. It suffices to show that $\sum_1^\infty x_j$ converges in $\bar{A}_{u,\mathbf{a}}$. Since $\Sigma(\bar{A})$ is complete and $\|x_j|\Sigma(\bar{A})\| \le \|x_j|\bar{A}_{u,\mathbf{a}}\|$ there is $x = \sum_1^\infty x_j \in \Sigma(\bar{A})$. We choose a subsequence (n_k) such that $\{\sum_{n_k+1}^{n_{k+1}} \|x_j|\bar{A}_{u,\mathbf{a}}\|^q\}^{1/q} \le 1/2^k$. For fixed $\epsilon > 0$, we find $y_k^{0,l}$ and $y_k^{1,l}$ with $\sum_{n_k+1}^{n_{k+1}} x_j = y_k^{0,l} + y_k^{1,l}$, $\|y_k^{0,l}|A_0\| \le \frac{1+\epsilon}{u_l} \frac{1}{2^k}$ and $\|y_k^{1,l}|A_1\| \le \frac{1+\epsilon}{2^l u_l} \frac{1}{2^k}$. Hence there are $y^{0,l} = \sum_{k \ge k_0} y_k^{0,l} \in A_0$, $y^{1,l} = \sum_{k \ge k_0} y_k^{1,l} \in A_1$ and we get $K(2^l, \sum_{n_{k_0}+1}^\infty x_j, \bar{A}) \le \|y^{0,l}|A_0\| + 2^l \|y^{1,l}|A_1\|$, where k_0 is arbitrary, but fixed. A straight-forward calculation gives $\|(u_l K(2^l, \sum_{n_{k_0}+1}^\infty x_j, \bar{A}))|\mathbf{a}\| \le \frac{C}{2^{k_0/2}}$, where the constant C is independent of k_0 and the proof is complete ■

3. The reiteration theorem

As an easy consequence of the boundness of the double sequence operator D and the subsequence operators T_φ we obtain

Lemma 3.1: *Let* $\mathbf{a}$ *be an admissible* $\mathbf{N}$*-lattice, and let* $\mathbf{N} = \bigcup_0^\infty A_k, A_k \ne \emptyset$ *be any disjoint decomposition of* $\mathbf{N}$ *with* $\max(A_k)$ *increasing and* $\max(A_k) - \min(A_k) < n_0$ *for all* $k \in \mathbf{N}$. *If*

$$c_1 x_k \le y_j \le c_2 x_k \textit{ for all } j \in A_k \textit{ and } k \in \mathbf{N},$$

then

$$\tilde{C}_1 \|(x_k)_{k \in \mathbf{N}}|\mathbf{a}\| \le \|(y_j)_{j \in \mathbf{N}}|\mathbf{a}\| \le \tilde{C}_2 \|(x_k)_{k \in \mathbf{N}}|\mathbf{a}\|,$$

where $\tilde{C}_1, \tilde{C}_2$ *are constants independent of* $(x_k)_{k \in \mathbf{N}}$ *and* $(y_j)_{j \in \mathbf{N}}$.

Lemma 3.2: *Let* $u = (u_k) \in Q_{\mathbf{N}}$ *with* $\underline{\gamma}(u) < 0$. *Put* $A_k = \{j \in \mathbf{N} : \sigma k \leq \log u_j < \sigma(k+1)\}$, *where* $\sigma = \log \overline{u}_1$. *Then we have*

(i) $\sigma > 0$.

(ii) $A_k \neq \emptyset$ *for all* $k \in \mathbf{N}$.

(iii) $\varphi(k) = \max(A_k)$ *is increasing.*

(iv) $\max(A_k) - \min(A_k) \leq n_0$, *where* n_0 *is independent of* k.

PROOF: **(i)** $\sigma > 0$ is an immediate consequence of $\lim_{k\to\infty} u_k = +\infty$. **(ii)** Suppose, on the contrary, that $A_{k_0} = \emptyset$ for some $k_0 \in \mathbf{N}$. Since the number $j_0 = \max\{j : \log u_j < \sigma k_0\}$ exists, we can infer $\log u_{j_0} < \sigma k_0 < \sigma(k_0+1) \leq \log u_{j_0+1}$. This yields the desired contradiction $\sigma < \log \frac{u_{j_0+1}}{u_{j_0}} \leq \log \overline{u}_1 = \sigma$. **(iii)** can be proved by similar arguments. **(iv)** Put $\psi(k) = \min(A_k)$. Since $\sigma k \leq \log u_{\psi(k)}, \log u_{\varphi(k)} < \sigma(k+1)$ we obtain $\log u_{\varphi(k)}/u_{\psi(k)} < \sigma$ and hence we get $0 < 1/\overline{u}_1 < u_{\psi(k)}/u_{\varphi(k)} \leq \underline{u}_{\varphi(k)-\psi(k)}$. Because $\lim_{j\to+\infty} \underline{u}_j = 0$ there is a constant n_0 such that $\varphi(k) - \psi(k) \leq n_0$ for all $k \in \mathbf{N}$ ■

Remark 3.3: If we suppose $\overline{\gamma}(u) < 0$, the same statement as in the preceeding lemma holds true for $A_k = \{j \in \mathbf{N} : \sigma k \leq -\log u_j < \sigma(k+1)\}$ where $\sigma = \log \underline{u}_1$. To see this consider u^{-1} and apply Lemma 3.2 .

Lemma 3.4: *Let* $\mathbf{a}$ *be an admissible* $\mathbf{N}$*-lattice, and* $u, v \in Q_{\mathbf{N}}$, $\underline{\gamma}(u) < 0$. *Moreover, let* $K : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *be an increasing function satisfying* $K(2x) \leq 2K(x)$. *Then* $(v_k K(2^k))_{k\in\mathbf{N}} \in \mathbf{a}$ *implies*

$$C_1 \, \|(v_k K(2^k))_{k\in\mathbf{N}}|\mathbf{a}\| \leq \|(v_{[\log u_k]} K(u_k))_{k\in\mathbf{N}}|\mathbf{a}\| \leq C_2 \, \|(v_k K(2^k))_{k\in\mathbf{N}}|\mathbf{a}\|,$$

where C_1, C_2 *are constants independent of* K.

PROOF: By Lemma 3.2 it turns out that

$$A_k = \{j \in \mathbf{N} : \sigma k \leq \log u_j < \sigma(k+1)\} \neq \emptyset, \qquad \sigma = \log \overline{u}_1.$$

For $j \in A_k$ we have

$$m_v v_{[\sigma k]} K(2^{[\sigma k]}) \leq v_{[\log u_j]} K(u_j) \leq M_v v_{[\sigma k]} K(2^{[\sigma k]}),$$

where $m_v = (\max_{0\leq j\leq 1+[\sigma]} \underline{v}_j)^{-1}$, $M_v = \max_{0\leq j\leq 1+[\sigma]} 2^{1+[\sigma]} \, \overline{v}_j$. Applying Lemma 3.1 we obtain

$$\tilde{C}_1 m_v \|(v_{[\sigma k]} K(2^{[\sigma k]}))|\mathbf{a}\| \leq \|(v_{[\log u_k]} K(u_k))|\mathbf{a}\| \leq \tilde{C}_2 M_v \|(v_{[\sigma k]} K(2^{[\sigma k]}))|\mathbf{a}\|.$$

Repeating the same reasoning we can substitute $\|(v_{[\sigma k]} K(2^{[\sigma k]}))|\mathbf{a}\|$ by $\|(v_k K(2^k))|\mathbf{a}\|$ and the desired result is proved ■

Proposition 3.5: *Let* $\mathbf{b}$ *be an admissible* $\mathbf{Z}$*-lattice, and* $\mathbf{a}_0(u^0), \mathbf{a}_1(u^1)$ *be* $\mathbf{Z}$*-lattices. Moreover, let* $v \in Q_{\mathbf{Z}}$, $-1 < \alpha(v) \leq \beta(v) < 0$. *Put* $u = (u_k) = (u_k^0/u_k^1)$, *and* $w = (w_k)_{k\in\mathbf{Z}} = (u_k^0 v_{[\log u_k]})_{k\in\mathbf{Z}}$. *Then we have*

$$(\mathbf{a}_0(u^0), \mathbf{a}_1(u^1))_{v,\mathbf{b}} = \mathbf{c} \oplus \mathbf{d}(w),$$

where

$$\mathbf{c} = \begin{cases} \mathbf{b}^+ & \text{for } \underline{\gamma}(u^-) < 0 \\ \mathbf{b}^- & \text{for } \overline{\gamma}(u^-) < 0 \\ (\mathbf{a}_0^-, \mathbf{a}_1^-)_{v,\mathbf{b}} & \text{for } u_k^- \asymp 1 \end{cases}, \quad \text{and} \quad \mathbf{d} = \begin{cases} \mathbf{b}^+ & \text{for } \underline{\gamma}(u^+) < 0 \\ \mathbf{b}^- & \text{for } \overline{\gamma}(u^+) < 0 \\ (\mathbf{a}_0^+, \mathbf{a}_1^+)_{v,\mathbf{b}} & \text{for } u_k^+ \asymp 1 \end{cases}.$$

PROOF: Note that $K(t, x, \mathbf{a}_0, \mathbf{a}_1) \asymp K(t, x^-, \mathbf{a}_0^-, \mathbf{a}_1^-) + K(t, x^+, \mathbf{a}_0^+, \mathbf{a}_1^+)$ immediately implies $(\mathbf{a}_0^- \oplus \mathbf{a}_0^+, \mathbf{a}_1^- \oplus \mathbf{a}_1^+)_{u,\mathbf{b}} = (\mathbf{a}_0^-, \mathbf{a}_1^-)_{u,\mathbf{b}} \oplus (\mathbf{a}_0^+, \mathbf{a}_1^+)_{u,\mathbf{b}}$. Therefore it is sufficient to prove that

$$(\mathbf{a}_0(u^0), \mathbf{a}_1(u^1))_{v,\mathbf{b}} = \begin{cases} \mathbf{b}^+(w) & \text{for } \underline{\gamma}(u) < 0 \\ \mathbf{b}^-(w) & \text{for } \overline{\gamma}(u) < 0 \\ (\mathbf{a}_0, \mathbf{a}_1)_{v,\mathbf{b}}(w) & \text{for } u_k \asymp 1 \end{cases},$$

where $\mathbf{a}_0(u^0)$, and $\mathbf{a}_1(u^1)$ are $\mathbf{N}$-lattices. (i) We consider the case $\underline{\gamma}(u) < 0$. According to Lemma 3.2 we put $A_k = \{j \in \mathbf{N} : \sigma k \leq \log u_j < \sigma(k+1)\}$. Let $x \in (\mathbf{a}_0(u^0), \mathbf{a}_1(u^1))_{v,\mathbf{b}}$. From the definition of the K-functional it follows $CK(2^{\sigma k}, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)) \geq |u_j^0 x_j|$ for all $j \in A_k$. Therefore we get

$$\tilde{C}\, \|(v_{[\sigma k]} K(2^{\sigma k}, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)))|\mathbf{b}\| \geq \|(w_j x_j)|\mathbf{b}^+\|$$

and by Lemma 3.4 it follows $x \in \mathbf{b}^+(w)$. On the other hand, let $x \in \mathbf{b}^+(w)$. For some $0 < r \leq 1$ we have $K(t, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)) \leq \{\sum_{j=0}^{\infty} K(t, x^j, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1))^r\}^{1/r}$, where $x^j = (x_l)_{l\in A_j} = (\chi_{A_j}(l) x_l)_{l\in\mathbf{N}}$. Using the estimate

$$\begin{aligned} K(2^{\sigma k}, x^j, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)) &= \inf_{x^j = y^0 + y^1} \{\|((u_l^0 y_l^0)_{l\in A_j})|\mathbf{a}_0\| + 2^{\sigma k}\|((u_l^1 y_l^1)_{l\in A_j})|\mathbf{a}\|\} \\ &\asymp \inf_{x^j = y^0 + y^1} \{\|((u_l^0 y_l^0)_{l\in A_j})|\mathbf{a}_0\| + 2^{\sigma(k-j)}\|((u_l^0 y_l^1)_{l\in A_j})|\mathbf{a}\|\} \\ &= K(2^{\sigma(k-j)}, (u_l^0 x_l)_{l\in A_j}, \mathbf{a}_0, \mathbf{a}_1) \\ &\leq \min(1, 2^{\sigma(k-j)})\|((u_l^0 x_l)_{l\in A_j})|\Delta(\bar{\mathbf{a}})\| \\ &\asymp \min(1, 2^{\sigma(k-j)}) \max_{l\in A_j} |u_l^0 x_l| \end{aligned}$$

we obtain

$$\begin{aligned} \|(v_{[\sigma k]} K(2^{\sigma k}, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)))|\mathbf{b}\| &\leq C\, \|(\{\sum_{j=0}^{\infty}(v_{[\sigma k]} \min(1, 2^{\sigma(k-j)}) \max_{l\in A_j} |u_l^0 x_l|)^r\}^{1/r})|\mathbf{b}\| \\ &\leq C\, \|(\{\sum_{j=0}^{\infty}(\overline{v}_{[\sigma(k-j)]} \min(1, 2^{\sigma(k-j)}) \max_{l\in A_j} |w_l x_l|)^r\}^{1/r})|\mathbf{b}\|. \end{aligned}$$

Since $\overline{v}_{[\sigma(k-j)]} \min(1, 2^{\sigma(k-j)}) < \tilde{C}\rho^{|k-j|}$ for some $\rho, 0 < \rho < 1$ it follows from Lemma 2.4/(ii) combined with Lemma 3.4 that

$$\|(v_k K(2^k, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)))|\mathbf{b}\| \leq C\, \|(\max_{l\in A_j} |w_l x_l|)_j|\mathbf{b}^+\| \leq \hat{C}\, \|x|\mathbf{b}^+(w)\|.$$

(ii) The case $\overline{\gamma}(u) < 0$ follows from (i) combined with

$$\|(v_k K(2^k, x, \mathbf{a}_0(u^0), \mathbf{a}_1(u^1)))|\mathbf{b}^- \oplus \mathbf{b}^+\| \asymp \|(2^k v_k K(2^{-k}, x, \mathbf{a}_1(u^1), \mathbf{a}_0(u^0)))|\mathbf{b}^+ \oplus \mathbf{b}^-\|$$

and $u_k^1\, 2^{-\log u_k^1/u_k^0}\, v_{-[\log u_k^1/u_k^0]} \asymp u_k^0 v_{[\log u_k]} = w_k$. **(iii)** The case $u_k \asymp 1$ is obvious ■

Remark 3.6: In the cases $\mathbf{a}_0 \neq \mathbf{a}_1$ and $\underline{\gamma}(u^-) = \overline{\gamma}(u^-) = 0$ (resp. u^+) no simple description of $(\mathbf{a}_0(u^0), \mathbf{a}_1(u^1))_{v,\mathbf{b}}$ is known.

Remark 3.7: By Proposition 3.5 we have $(l_\infty, l_\infty(2^{-k}))_{v,\mathbf{b}} = \mathbf{b}(v)$. Note that in this case there holds $u = (2^k)_{k\in\mathbf{Z}}$, $\overline{\gamma}(u^-) = \underline{\gamma}(u^+) = -1$.

Therefore, $\mathbf{b}(v)$ is an interpolation space between $\bar{l}_\infty = (l_\infty, l_\infty(2^{-k}))$. Moreover, for $(x_k) \in \mathbf{b}(v)$ it is easy to check that $\lim_{k\to-\infty} x_k = \lim_{k\to+\infty} 2^{-k}x_k = 0$, which implies that $\mathbf{b}(v) \subset \Sigma(\bar{c}_0)$, where $\bar{c}_0 = (c_0, c_0(2^{-k}))$.

Let E be an interpolation space with respect to the pair $\bar{l}_\infty = (l_\infty, l_\infty(2^{-k}))$. Then we say that $x \in \bar{A}_{E;K}$ whenever $\|(K(2^k, x, \bar{A}))_{k\in\mathbf{Z}}|E\| < \infty$. Now we are in position to deduce the desired reiteration result as a special setting of an importent theorem independently discovered by BRUDNYI AND KRUGLJAK[2, 3] and NILSSON[6, p.301]. Their result reads as follows.

Theorem 3.8: *Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach pair and $\bar{E} = (E_0, E_1)$ any pair of interpolation spaces between $\bar{l}_\infty$. Furthermore let us assume that $E_i \subset \Sigma(\bar{c}_0)$, $i = 0, 1$. Then for all $t > 0$ and $x \in \Sigma(\bar{A})$ we have*

$$K(t, x, \bar{A}_{E_0;K}, \bar{A}_{E_1;K}) \asymp K(t, (K(2^k, x, \bar{A}))_{k\in\mathbf{Z}}, \bar{E}).$$

Observe that this formula implies $(\bar{A}_{E_0;K}, \bar{A}_{E_1;K})_{F;K} = \bar{A}_{(E_0,E_1)_{F;K}}$, where F is any interpolation space between $\bar{l}_\infty$. Applying Proposition 3.5, Remark 3.7, and Theorem 3.8 we can infer the following reiteration theorem.

Theorem 3.9 (The reiteration theorem): *Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}$ be admissible $\mathbf{Z}$-lattices and $u^0, u^1, v \in Q_{\mathbf{Z}}$, $-1 < \alpha(u^0), \alpha(u^1), \alpha(v), \beta(u^0), \beta(u^1), \beta(v) < 0$. Put $u = (u_k) = (u_k^0/u_k^1)$ and $w = (w_k)_{k\in\mathbf{Z}} = (u_k^0 v_{[\log u_k]})_{k\in\mathbf{Z}}$. Then we have*

$$(\bar{A}_{u^0,\mathbf{a}_0}, \bar{A}_{u^1,\mathbf{a}_1})_{v,\mathbf{b}} = \bar{A}_{w,\mathbf{c}\oplus\mathbf{d}},$$

where

$$\mathbf{c} = \begin{cases} \mathbf{b}^+ & \text{for } \underline{\gamma}(u^-) < 0 \\ \mathbf{b}^- & \text{for } \overline{\gamma}(u^-) < 0 \\ (\mathbf{a}_0^-, \mathbf{a}_1^-)_{v,\mathbf{b}} & \text{for } u_k^- \asymp 1 \end{cases}, \quad \text{and} \quad \mathbf{d} = \begin{cases} \mathbf{b}^+ & \text{for } \underline{\gamma}(u^+) < 0 \\ \mathbf{b}^- & \text{for } \overline{\gamma}(u^+) < 0 \\ (\mathbf{a}_0^+, \mathbf{a}_1^+)_{v,\mathbf{b}} & \text{for } u_k^+ \asymp 1 \end{cases}.$$

4. Generalized Lorentz spaces

In this section we deal with generalized Lorentz spaces. It is of interest to know that they form a "scale" of spaces in the sense that they are stable with respect to some method of interpolation. We restrict our consideration to the typical discrete and continuous case, namely the Lorentz sequence spaces and the Lorentz spaces defined on the unit interval $(0, 1)$ with Lebesgue measure. These spaces show different behaviours.

Definition 4.1: Let $v \in Q_{\mathbf{N}}$ and let $\mathbf{a}$ be an admissible $\mathbf{N}$-lattice. The *generalized Lorentz spaces* are defined as follows:

(i) $\lambda_{v,\mathbf{a}} = \{x = (x_k)_{k\geq 0} : \|(v_k x^*_{2^k-1})_{k\geq 0}|\mathbf{a}\| < \infty\}$, with $\underline{\gamma}(v) < 0$, where (x^*_k) denotes the non-increasing rearrangement of (x_k).

(ii) $\Lambda_{v,\mathbf{a}} = \{f : \|(v_k f^*(2^{k-1}))_{k\geq 0}|\mathbf{a}\| < \infty\}$, with $\overline{\gamma}(v) < 0$, where f^* denotes the non-increasing rearrangement of the measurable function f on the measure space $((0,1), dx)$.

Observe that $\lambda_{(2^{k/p}),l_q} = l_{pq}$ and $\Lambda_{(2^{-k/p}),l_q} = L_{pq}(0,1)$. Using (1) it immediately follows

Remark 4.2: If $(v_k/u_k) \in \mathbf{a}^{-1}\circ\mathbf{b}$, then $\lambda_{u,\mathbf{a}} \subset \lambda_{v,\mathbf{b}}$ and $\Lambda_{u,\mathbf{a}} \subset \Lambda_{v,\mathbf{b}}$.

To prove that the Lorentz spaces can be generated by interpolation we need a preparing lemma.

Lemma 4.3: *Let* $\mathbf{a}$ *be an admissible* $\mathbf{N}$*-lattice. Let* $0 < q < 1$ *and* $u = (u_k) \in Q_{\mathbf{N}}$.

(i) *If* $\overline{\gamma}(u) < 1/q$, *then* $\|(u_k\{\sum_{j=0}^{k} 2^{j-k}|x_j|^q\}^{1/q})_{k\in\mathbf{N}}|\mathbf{a}\| \asymp \|(u_k x_k)_{k\in\mathbf{N}}|\mathbf{a}\|$.

(ii) *If* $\underline{\gamma}(u) < 1/q$, *then* $\|(u_k\{\sum_{j=k}^{\infty} 2^{j-k}|x_j|^q\}^{1/q})_{k\in\mathbf{N}}|\mathbf{a}\| \asymp \|(u_k x_k)_{k\in\mathbf{N}}|\mathbf{a}\|$.

PROOF: First observe that $\overline{\gamma}(u) < 1/q$ (resp. $\underline{\gamma}(u) < 1/q$) implies the existence of numbers $C > 0$ and ρ with $0 < \rho < 1$ such that $\overline{u}_m 2^{-m/q} \leq C\,\rho^{m/q}$ for $m > 0$ (resp. $\underline{u}_m 2^{-m/q} \leq C\,\rho^{m/q}$). To prove (i) consider the estimation

$$\begin{aligned} u_k\{\sum_{j=0}^{k} 2^{j-k}|x_j|^q\}^{1/q} &= \{\sum_{j=0}^{k}(\frac{u_k}{u_j}2^{-(j-k)/q}u_j|x_j|)^q\}^{1/q} \\ &\leq \{\sum_{j=0}^{k}(\overline{u}_{k-j}2^{-(j-k)/q}|u_j x_j|)^q\}^{1/q} \\ &\leq C\{\sum_{j=0}^{k}(\rho^{|j-k|}|u_j x_j|)^q\}^{1/q}. \end{aligned}$$

The assertion then follows from Lemma 2.4. (ii) can be proved in a similar way ■

Lemma 4.4: *Let* $\mathbf{a}$ *be an admissible* $\mathbf{Z}$*-lattice,* $0 < q \leq 1$, *and* $u \in Q_{\mathbf{Z}}$ *with* $-1 < \alpha(u) \leq \beta(u) < 0$. *Then we have*

(i) $(l_q, l_\infty)_{u,\mathbf{a}^-\oplus\mathbf{a}^+} = \lambda_{v^+,\mathbf{a}^+}$, *where* $v^+ = (2^{k/q}u_{[k/q]})_{k\in\mathbf{N}}$.

(ii) $(L_q, L_\infty)_{u,\mathbf{a}^-\oplus\mathbf{a}^+} = \Lambda_{v^-,\mathbf{a}^-}$, *where* $v^- = (2^{-k/q}u_{[-k/q]})_{k\in\mathbf{N}}$.

PROOF: (i) First we note that

$$K(2^k, x, l_q, l_\infty) \asymp \begin{cases} \{\sum_0^{2^{[qk]}-1}|x_j^*|^q\}^{1/q} & \text{for } k \geq 0 \\ 2^k\|x|l_\infty\| & \text{for } k < 0 \end{cases}.$$

Hence

$$u_k K(2^k,x,l_q,l_\infty) \asymp \begin{cases} v_{[qk]}\{\frac{1}{2^{[qk]}}\sum_0^{2^{[qk]}-1}|x_j^*|^q\}^{1/q} & \text{for} \quad k\ge 0 \\ 2^k\, u_k\|x|l_\infty\| & \text{for} \quad k<0 \end{cases}.$$

Applying Lemma 3.4 we get

$$\begin{aligned} \|(u_kK(2^k,x,l_q,l_\infty))_{k\in\mathbf{Z}}|a\| &\asymp \|(v^+_{[qk]}\{\frac{1}{2^{[qk]}}\sum_0^{2^{[qk]}-1}|x_j^*|^q\}^{1/q})_{k\ge 0}|\mathbf{a}^+\| + \|(2^{-k}u_{-k}\|x|l_\infty\|)_{k\ge 0}|\mathbf{a}^-\| \\ &\asymp \|(v_k^+\{\frac{1}{2^k}\sum_0^{2^k-1}|x_j^*|^q\}^{1/q})_{k\ge 0}|\mathbf{a}^+\| + \|(2^{-k}u_{-k})_{k\ge 0}|\mathbf{a}^-\|\;\|x|l_\infty\| \\ &\asymp \|(v_k^+\{\sum_{j=0}^{k}2^{j-k}|x^*_{2^j-1}|^q\}^{1/q})_{k\ge 0}|\mathbf{a}^+\|. \end{aligned}$$

Since $\overline{\gamma}(v^+)\le\beta(u)/q+1/q<1/q$ it follows from Lemma 4.3 that

$$\|(u_kK(2^k,x,l_q,l_\infty))_{k\in\mathbf{Z}}|a\| \asymp \|(v_k^+x^*_{2^k-1})_{k\ge 0}|\mathbf{a}^+\|.$$

(ii) can be proved in the same way ■

Now we are in position to formulate and prove the interpolation theorem for Lorentz spaces.

Theorem 4.5: *Let $\lambda_{u^i,\mathbf{a}_i}$ and $\Lambda_{u^i,\mathbf{a}_i}$ $(i=0,1)$ be generalized Lorentz spaces. Suppose $v\in Q_{\mathbf{Z}}$ with $-1<\alpha(v)\le\beta(v)<0$ and let $\mathbf{b}$ to be an admissible Z-lattice. Put $u=(u_k)=(u_k^0/u_k^1)$, $\mathbf{c}=(\mathbf{a}^0,\mathbf{a}^1)_{v,\mathbf{b}}$ and $w=(w_k)_{k\in\mathbf{N}}=(u_k^0v_{[\log u_k]})_{k\in\mathbf{N}}$. Then we have*

$$\text{(i)}\qquad (\lambda_{u^0,\mathbf{a}_0},\lambda_{u^1,\mathbf{a}_1})_{v,\mathbf{b}} = \begin{cases} \lambda_{w,\mathbf{b}^+} & \textit{for} \quad \underline{\gamma}(u)<0 \\ \lambda_{w,\mathbf{b}^-} & \textit{for} \quad \overline{\gamma}(u)<0 \\ \lambda_{w,\mathbf{c}} & \textit{for} \quad u_k\asymp 1 \end{cases}.$$

$$\text{(ii)}\qquad (\Lambda_{u^0,\mathbf{a}_0},\Lambda_{u^1,\mathbf{a}_1})_{v,\mathbf{b}} = \begin{cases} \Lambda_{w,\mathbf{b}^+} & \textit{for} \quad \underline{\gamma}(u)<0 \\ \Lambda_{w,\mathbf{b}^-} & \textit{for} \quad \overline{\gamma}(u)<0 \\ \Lambda_{w,\mathbf{c}} & \textit{for} \quad u_k\asymp 1 \end{cases}.$$

PROOF: (i) Choose q with $0<q<\min(1,1/\overline{\gamma}(u^0),1/\overline{\gamma}(u^1))$, θ with $0<\theta<1$ and $1+q\max(\underline{\gamma}(u^0),\underline{\gamma}(u^1))<\theta$. Now we put $s^i=(s_k^i)_{k\in\mathbf{Z}}$, with $s_k^i=\begin{cases} 2^{-k}u^i_{[qk]} & \text{for} \quad k\ge 0 \\ 2^{-\theta k} & \text{for} \quad k>0 \end{cases}$.
A straight-forward calculation shows that Lemma 4.4 can be applied and we get

$$\lambda_{u^i,\mathbf{a}_i} = (l_q,l_\infty)_{s^i,\mathbf{a}_i\oplus\mathbf{a}_i} \qquad (i=0,1).$$

By Theorem 3.9 it follows that

$$((l_q,l_\infty)_{s^0,\mathbf{a}_0\oplus\mathbf{a}_0},(l_q,l_\infty)_{s^1,\mathbf{a}_1\oplus\mathbf{a}_1})_{v,\mathbf{b}} = \begin{cases} (l_q,l_\infty)_{t,\mathbf{c}\oplus\mathbf{b}^+} & \text{for} \quad \underline{\gamma}(s^+)<0 \\ (l_q,l_\infty)_{t,\mathbf{c}\oplus\mathbf{b}^-} & \text{for} \quad \overline{\gamma}(s^+)<0 \\ (l_q,l_\infty)_{t,\mathbf{c}\oplus\mathbf{c}} & \text{for} \quad s_k^+\asymp 1 \end{cases},$$

where $s=(s_k)=(s_k^0/s_k^1)$, and $t=(t_k)=(s_k^0\,v_{[\log u_k]})_{k\in\mathbf{Z}}$. Note that we have $\underline{\gamma}(s^+)=q\underline{\gamma}(u)$, $\overline{\gamma}(s^+)=q\overline{\gamma}(u)$, and $s_k^+\asymp u_k\asymp 1$, respectively. The proof will be completed by applying again

Lemma 4.4. Observe that $w_k = 2^{k/q}\, t_{[k/q]} = 2^{k/q}\, s^0_{[k/q]} v_{[\log s_{[k/q]}]} = u^0_k\, v_{[\log u_k]}$. (ii) can be proved by similar arguments ■

In [12] A. PIETSCH has proposed to introduce Lorentz sequence spaces depending on a finite number of indices $0 < r_1, \dots, < \infty$. In the present setting these spaces can inductively be defined as follows. The starting point is the relation $l_{p_1 p_2} = \lambda_{(2^{k/p_1}), l_{p_2}}$ for the usual Lorentz sequence spaces which is well known. Now we put $l_{p_1 \dots p_n} = \lambda_{(2^{k/p_1}), l_{p_2 \dots p_n}}$. By Remark 4.2 it is clear that the spaces are lexicographically ordered. Note that the generalized Lorentz sequence spaces are admissible. Next we give some interpolation results for the Lorentz sequence spaces $l_{p_1 \dots p_n}$ which we will formulate in terms of the classical interpolation scale of Lions-Peetre, i. e. we use $\bar{A}_{\theta,q} = \bar{A}_{(2^{-\theta k}), l_q}$, $0 < \theta < 1$, $0 < q \le \infty$. The following results are immediate consequences of Theorem 4.5.

Corollary 4.6: *We have*

(i) $(l_{p_1 p_2}, l_{q_1 q_2})_{\theta,r} = l_{sr}$, *for* $p_1 < q_1, \frac{1}{s} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}$

(i*) $(l_{p_1 \dots p_n}, l_{q_1 \dots q_m})_{\theta,r} = l_{sr}$, *for* $p_1 < q_1, \frac{1}{s} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}$

(ii) $(l_{p_1 p_2}, l_{p_1 q_2})_{\theta,r} = l_{p_1 s r}$, *for* $p_2 < q_2, \frac{1}{s} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}$

(ii*) $(l_{p_1 \dots p_k p_{k+1} \dots p_n}, l_{p_1 \dots p_k q_{k+1} \dots q_m})_{\theta,r} = l_{p_1 \dots p_k s r}$, *for* $p_{k+1} < q_{k+1}, \frac{1}{s} = \frac{1-\theta}{p_{k+1}} + \frac{\theta}{q_{k+1}}$.

5. Miscellaneous results

In order to generalize a special case of the power theorem, we introduce the admissible $\mathbf{Z}$-lattice $\mathbf{a}_p = \{(x_k) : \|(|x_k|^p)|\mathbf{a}\| < \infty\}$, which is equipped with the quasi-norm $\|(x_k)|\mathbf{a}_p\| = \|(|x_k|^p)|\mathbf{a}\|^{1/p}$. By the Lemma 3.4 and the same arguments as given in [1, p.68-69] we deduce

Theorem 5.1 (The power theorem): *If* $0 < p < \infty$, *then*

$$(A_0^p, A_1^p)_{u,\mathbf{a}} = (A_0, A_1)^p_{v,\mathbf{a}_p}, \quad \textit{where} \quad v = (v_k) = (u^{1/p}_{[pk]}).$$

The duality theorem reads as follows.

Theorem 5.2 (The duality theorem): *Let (A_o, A_1) be a Banach pair, let* $\mathbf{a}$ *be an admissible Banach* $\mathbf{Z}$*-lattice, and let $u \in Q_{\mathbf{Z}}, -1 < \alpha(u) \le \beta(u) < 0$. Moreover, suppose that $\Delta(\bar{A})$ is dense in A_0, A_1, and $\bar{A}_{u,\mathbf{a}}$. Then we have*

$$(A_0, A_1)'_{u,\mathbf{a}^- \oplus \mathbf{a}^+} = (A_0', A_1')_{v,\mathbf{a}_*^+ \oplus \mathbf{a}_*^-}, \quad \textit{where} \quad v = (v_k) = (1/u_{-k}) \quad \textit{and} \quad \mathbf{a}_* = \mathbf{a}^{-1} \circ l_1.$$

PROOF: Apply Theorem 2.6/(i), the equivalence theorem, and carry out a generalized form of the proof given in [1, p.54] ■

Note, if the finite sequences are dense in $\mathbf{a}$, then $\Delta(\bar{A})$ is dense in $\bar{A}_{u,\mathbf{a}}$, too.

Finally we want to mention that the interpolation between the sum and the intersection can be treated in the same way as pointed out in [11, p.218].The result follows immediately from the estimates

$$K(2^k, x, \Sigma(\bar{A}), \Delta(\bar{A})) \asymp \begin{cases} K(2^k, x, \bar{A}) + 2^k K(2^{-k}, x, \bar{A}) & \text{for} \quad k \le 0 \\ K(1, x, \bar{A}) & \text{for} \quad k > 0 \end{cases},$$

and

$$K(2^k, x, \Sigma(\bar{A}), A_1) \asymp \begin{cases} K(2^k, x, \bar{A}) & \text{for} \quad k \le 0 \\ K(1, x, \bar{A}) & \text{for} \quad k > 0 \end{cases}.$$

Theorem 5.3: *Let* **a** *be an admissible* **Z**-*lattice and* **b**,**c** *be admissible* **N**-*lattices,* $u \in Q_{\mathbf{Z}}$, $-1 < \alpha(u) \le \beta(u) < 0$. *Then we have*

$$(\Sigma(\bar{A}), \Delta(\bar{A}))_{u,\mathbf{a}^- \oplus \mathbf{a}^+} = (\Sigma(\bar{A}), A_0)_{u,\mathbf{a}^- \oplus \mathbf{b}} \cap (\Sigma(\bar{A}), A_1)_{u,\mathbf{a}^- \oplus \mathbf{c}}.$$

In the literature the problem is considered to characterize interpolation spaces by means of the intersection or the union of a family of spaces indexed by functions belonging to some classes (see [10]). The next result gives a contribution to an abstract version of this problem.

Let **a** be an admissible **Z**-lattice. We put $Q_{\mathbf{a},\epsilon} = \{u \in \mathbf{a} \cap Q_{\mathbf{Z}} : -\epsilon \le \alpha(u) \le \beta(u) \le \epsilon\}$. Observe the following fact. Let $(x_k) \in \mathbf{a}$ and define

$$u_k = \sum_j \rho^{|k-j|} |x_j|, \quad \text{where} \quad 0 < \rho < 1. \tag{2}$$

By Lemma 2.4 we have $(u_k) \in \mathbf{a}$ and $\|(x_k)|a\| \le \|(u_k)|a\| \le C_\rho \|(x_k)|a\|$. Moreover, since $\rho \le u_{k+1}/u_k \le \rho^{-1}$ we get $(u_k) \in Q_{\mathbf{a},\epsilon}$, for $2^{-\epsilon} < \rho < 1$.

Theorem 5.4: *Let* $\bar{A} = (A_0, A_1)$ *be a given couple,* **a** *and* **b** *admissible* **Z**-*lattices, and* $v \in Q_{\mathbf{Z}}$, $-1 < \alpha(v) \le \beta(v) < 0$. *Fix* $\epsilon > 0$. *Then it holds*

(i) $\bar{A}_{v,\mathbf{a} \circ \mathbf{b}} = \bigcup_{u \in Q_{\mathbf{a},\epsilon}} \bar{A}_{v/u,\mathbf{b}}$.

(ii) $\bar{A}_{v,\mathbf{a}^{-1} \circ \mathbf{b}} = \bigcap_{u \in Q_{\mathbf{a},\epsilon}} \bar{A}_{vu,\mathbf{b}}$.

PROOF: We prove only the first assertion. The remaining one can be treated in a similar way. It is sufficient to prove that $\mathbf{a} \circ \mathbf{b} = \bigcup_{u \in Q_{\mathbf{a},\epsilon}} \mathbf{b}(u^{-1})$. Obviously, we have $\|z|\mathbf{a} \circ \mathbf{b}\| \le \|u|\mathbf{a}\| \, \|z/u|\mathbf{b}\|$ for all $z \in \mathbf{b}(u^{-1})$. Hence $\bigcup_{u \in Q_{\mathbf{a},\epsilon}} \mathbf{b}(u^{-1}) \subset \mathbf{a} \circ \mathbf{b}$.

On the other hand, for given $z \in \mathbf{a} \circ \mathbf{b}$ and $\delta > 0$ there is a decomposition $z = (x_k y_k)$ such that $(1+\delta)\|z|\mathbf{a} \circ \mathbf{b}\| \ge \|(x_k)|\mathbf{a}\| \, \|(y_k)|\mathbf{b}\| \ge \|(x_k)|\mathbf{a}\| \, \|(\frac{x_k}{u_k} y_k)|\mathbf{b}\| \ge C \, \|u|\mathbf{a}\| \, \|z|\mathbf{b}(u^{-1})\|$, where $u = (u_k) \in Q_{\mathbf{a},\epsilon}$ is constructed by (2), and so the proof is complete ∎

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Received 22.05.1990; in revised form 05.03.1991

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Book reviews

H.-J. SCHELL: **Unendliche Reihen** (Mathematik für Ingenieure, Naturwissenschaftler, Ökonomen, Landwirte: Bd. 3). Leipzig: B. G. Teubner Verlagsges. 1990; 8. ed., 116 pp, 21 fig.

As indicated in the bibliographical data this book is the third volume from the MINÖL series ("Mathematik für Ingenieure, Naturwissenschaftler, Ökonomen und Landwirte" - Mathematics for Engineers, Natural Scientists, Economists and Agriculturists) designed by various authors in a module-like system during the 1970s. It was aimed to support the university education of "non-mathematicians", obviously mainly that of engineers, and is also intended to support correspondence courses and private studies.

Besides of a scheme of its logical structure (interdependence graph) and a motivating introduction the book contains the following sections:

2. Series with constant elements (including the integrability criterion) - 17 pages
3. Function series (general theory including uniform convergence) - 17 pages
4. Power series (covering: Abel's theorem, Taylor series, solution of differential equations, asymptotic power series, substitution of power series into anothers and their reversion) - 37 pages
5. Fourier series (including their complex form, numerical Fourier analysis, Gibb's phenomenon, least square approximation)
6. Fourier integrals (including Fourier transformation).

Every section closes with exercises. Their solutions are given at the end of the book. Indexes on names and subjects and a short list of standard literature exists, too.