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Some Classes of Nonlinear Mixed Volterra and Singular Integral Equations

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Existence theorems for some classes of nonlinear mixed Volterra and singular integral or integro-differential equations are proved applying methods of monotone operator theory and Schauder's fixed point theorem, respectively. Moreover, uniqueness theorems for the solution are given.

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Some years ago the author dealt with nonlinear singular integral equations of Cauchy type using methods of monotone operator theory [4] and a novel application of Schauder's fixed point theorem [5], respectively. In the present note these methods are applied to some classes of nonlinear mixed Volterra and singular integral or integro-differential equations.

First, equations with an integral operator composed by a nonlinear Volterra operator and the sum of operators of fractional integration and logarithmic type are considered. By differentiation these equations are reduced to singular integral equations of Cauchy type which under suitable assumptions can be studied through a combination of the method of monotone operator theory in [4] and the iteration method.

Second, a quasi-linear Volterra integro-differential equation additionally involving the Cauchy singular integral operator and a related class of nonlinear Volterra integral equations additionally involving the logarithmic integral operator are dealt with. Again by differentiation these equations are reduced to singular integro-differential equations of Cauchy type treated in [5].

1. Equations of first type

At first we look for a function $u \in L_2(-a, a)$ and a constant $c \in \mathbb{R}$ satisfying the equation

$$\int_{-a}^{x} k(x,y,u(y))\,dy + \sum_{k=1}^{n} \mu_k \int_{-a}^{x} u(y)(x-y)^{\alpha_k} dy - \frac{\lambda}{\pi}\int_{-a}^{a} u(y)\ln|x-y|\,dy = f(x) + c \tag{1}$$

for a.a. $x \in [-a, a]$. In the following we write L_2 for the Lebesgue space $L_2(-a, a)$ and $\|\cdot\|$ for the norm in it.

The **Assumptions A** on the data are the following ones:

(i) f is absolutely continuous with $f' \in L_2$.

(ii) $\lambda \in \mathbb{R}$; $\mu_k \geq 0$ and $0 < \alpha_k < 1$ for $k = 1, \ldots, n$.

(iii) $k(x,y,u)$ is continuous on $[-a,a] \times [-a,a] \times \mathbb{R}$ $(= [-a,a]^2 \times \mathbb{R})$ and fulfils the strong monotonicity condition

$$(k(x,x,u) - k(x,x,v))(u - v) \geq q(u - v)^2 \quad (q > 0) \tag{2}$$

for $x \in [-a, a]$ and $u, v \in \mathbb{R}$.

(iv) For $(y,u) \in [-a,a] \times \mathbb{R}$ the function $k(x,y,u)$ is differentiable with respect to x. The derivative $k_x(x,y,u)$ is a measurable function of $(x,y) \in [-a,a] \times [-a,a]$ for all $u \in \mathbb{R}$, satisfies the Lipschitz condition

$$|k_x(x,y,u) - k_x(x,y,v)| \leq \gamma(x,y)|u - v| \tag{3}$$

for a.a. $(x,y) \in [-a,a] \times [-a,a]$ with $(x,y) \in \Delta := \{(x,y): -a \leq y \leq x \leq a\}$ and $u,v \in \mathbb{R}$, where $\gamma \in L_2(\Delta)$, i.e.

$$M^2 := \int_{-a}^{a} \int_{-a}^{x} \gamma^2(x,y)\,dy\,dx < \infty \tag{4}$$

and satisfies the additional condition

$$\int_{-a}^{x} |k_x(x,y,0)|\,dy \in L_2. \tag{5}$$

Differentiating (1) we obtain the equivalent operator equation

$$Au = Bu, \quad u \in L_2, \tag{6}$$

where the operator A is given by

$$(Au)(x) = k(x,x,u(x)) + \sum_{k=1}^{n} \mu_k \alpha_k (J_k u)(x) + \lambda (Su)(x)$$

with

$$(J_k u)(x) = \int_{-a}^{x} u(y)(x-y)^{\alpha_k - 1}\,dy \quad \text{and} \quad (Su)(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{u(y)}{y - x}\,dy$$

the operators of fractional integration and the Cauchy singular integral operator, respectively, and where the operator B is given by

$$(Bu)(x) = f'(x) - \int_{-a}^{x} k_x(x,y,u(y))\,dy. \tag{7}$$

In view of the assumption A/(i), (3) and (5) the operator B maps L_2 into itself. Further, analoguously as in Theorem II of [4] the auxiliary equation $Au = g$ has a unique solution $u \in L_2$ for any $g \in L_2$ since J_k and S are linear bounded positive operators in L_2 (cf. [3] and [4], respectively) and the Nemyzki operator of the function $k(x,x,u)$ in L_2 has a non-void domain and is coercive and strictly monotone due to the assumption (2). Therefore the operator equation (6) is equivalent to the *fixed point equation*

$$u = CBu, \quad u \in L_2, \tag{8}$$

with $C = A^{-1}$ the inverse operator to A.

From (2) there follows the Lipschitz continuity of C:

$$\|Cg_1 - Cg_2\| \leq {}^1\!/_q \|g_1 - g_2\| \text{ for } g_1, g_2 \in L_2. \tag{9}$$

Moreover, by (3) the operator B is Lipschitz continuous, too:

$$\|Bu_1 - Bu_2\| \leq M \|u_1 - u_2\| \text{ for } u_1, u_2 \in L_2. \tag{10}$$

Hence the operator CB satisfies a Lipschitz condition with Lipschitz constant M/q. Applying Banach's fixed point theorem to equation (8), we obtain

Theorem 1: *If the* Assumptions A *with* $M < q$ *are fulfilled, then the integral equation* (1) *possesses a unique solution pair* $(u, c) \in L_2(-a, a) \times \mathbb{R}$.

Remark 1: A corresponding existence theorem holds for the equation (1) with additional operators

$$-\int_x^a \tilde{k}(x,y,u(y))\,dy - \sum_{j=1}^m r_j \int_x^a u(y)(y-x)^{\beta_j} dy \quad (r_j \geq 0 \text{ and } 0 < \beta_j < 1 \text{ for } j = 1,\dots,m),$$

where k has to be replaced by $k + \tilde{k}$ in (2) and for $\tilde{k}$ there hold additional conditions analogously to (3) and (5) with a function $\tilde{\gamma}(x,y)$ in $\tilde{\Delta} = \{-a \leq x \leq y \leq a\}$. The constant M is then given by

$$M^2 = \int_{-a}^{a}\left(\int_{-a}^{x}\gamma^2(x,y)\,dy + \int_x^a \tilde{\gamma}^2(x,y)\,dy\right)dx.$$

Remark 2: In contrast to the case of a pure Volterra integral equation (cf. [3: 1.13]) we can prove the existence of a solution to equation (1) under the restriction $M < q$ only, since for $\lambda \neq 0$ the operator C is not of Volterra type and the Lipschitz condition (9) holds for the norm in L_2 but not pointwise as it would be for instance in the case $\lambda = 0$ and $\mu_k = 0$ for $k = 1,\dots,n$ (cp. [1] for an analogous approach in solving the nonlinear Abel's integral equation of the first kind).

Further we seek a solution pair (u, c) of an absolutely continuous function u and a constant $c \in \mathbb{R}$ to the *integral equation*

$$\int_{-a}^{x} k(x,y,u(y))\,dy - \frac{\lambda}{\pi}\int_{-a}^{a} u(y)\ln|x-y|\,dy - (Su)(x) = f(x) + c \tag{11}$$

for a.a. $x \in [-a, a]$ under the boundary conditions $u(-a) = u(a) = 0$.

The **Assumptions B** on the data are the following ones:

(i) f is absolutely continuous with $f' \in Y := L_2(r)$ with $r(x) = \sqrt{a^2 - x^2}$, i.e. $\|f'\|_Y^2 = \int_{-a}^{a}(a^2 - x^2)^{1/2} f'^2(x)\,dx < \infty$.

(ii) $\lambda \in \mathbb{R}$.

(iii) $k(x,y,u)$ is continuous on $(-a,a) \times (-a,a) \times \mathbb{R}$ and the function $k(x,x,u)$ is nondecreasing in u for all $x \in [-a,a]$ and fulfils the growth condition

$$|k(x,x,u)| \leq b(x) + dr^{-1}(x)|u| \quad (b \in Y, d > 0) \tag{12}$$

for all $x \in (-a, a)$ and $u \in \mathbb{R}$.

(iv) For $(y,u) \in [-a,a] \times \mathbb{R}$ the function $k(x,y,u)$ is differentiable with respect to x. The derivative $k_x(x,y,u)$ is a measurable function of $(x,y) \in [-a,a] \times [-a,a]$ for all $u \in \mathbb{R}$, satisfies the Lipschitz condition

$$|k_x(x,y,u) - k_x(x,y,v)| \leq \gamma(x,y) r^{-1}(y)|u-v| \tag{13}$$

for a.a. $x, y \in [-a,a]$ with $(x,y) \in \Delta$ and $u, v \in \mathbb{R}$, where

$$M_1^2 := \int_{-a}^{a}\sqrt{a^2 - x^2}\int_{-a}^{x}\left(\gamma^2(x,y)/\sqrt{a^2 - y^2}\right)dy\,dx < \infty, \tag{14}$$

and satisfies the additional condition

$$\int_{-a}^{x} |k_x(x,y,0)|\, dy \in Y. \tag{15}$$

Differentiating (11) we obtain the equivalent operator equation

$$A_1 u = Bu,\ u \in X := L_2(r^{-1}), \tag{16}$$

where the operator A_1 is given by

$$(A_1 u)(x) = k(x,x,u(x)) + \lambda(Su)(x) + (Tu)(x) \tag{17}$$

with the Cauchy integro-differential operator $Tu = -Su'$ and the operator B is given by the equation (7).

In view of the assumptions B/(i), (13) and (15) the operator B maps X into Y. Furthermore, analogously as in Theorem VI of [4] the auxiliary equation $A_1 u = h$ has a unique solution $u \in X$ with $u' \in Y$ and $u(-a) = u(a) = 0$ for any $h \in Y$ taking into account the assumptions on the function $k(x,x,u)$ in B/(iii). Therefore the operator equation (16) is equivalent to the *fixed point equation*

$$u = C_1 Bu,\ u \in X, \tag{18}$$

with $C_1 = A_1^{-1}: Y \to X$ the inverse operator to A_1.

From Schleiff's inequality (cf. [4])

$$(Tu, u) \ge \|u\|_X^2 = \int_{-a}^{a} (a^2 - x^2)^{-1/2} u^2(x)\, dx \tag{19}$$

analogously to (9) there follows the Lipschitz continuity of C_1:

$$\|C_1 h_1 - C_1 h_2\|_X \le \|h_1 - h_2\|_Y \quad (h_1, h_2 \in Y).$$

Moreover, analogously to (10) by (13) the operator B is Lipschitz continuous, too:

$$\|Bu_1 - Bu_2\|_Y \le M_1 \|u_1 - u_2\|_X \quad (u_1, u_2 \in X)$$

with M_1 defined by (14). Hence the operator $C_1 B: X \to X$ satisfies a Lipschitz condition with Lipschitz constant M_1. Then Banach's fixed point theorem applied to (18) yields

Theorem 2: *If the* Assumptions B *with* $M_1 < 1$ *are fulfilled, then the integral equation* (11) *possesses a unique solution pair* $(u,c) \in L_2(r^{-1}) \times \mathbb{R}$, $r(x) = \sqrt{a^2 - x^2}$, *with* $u' \in L_2(r)$ *and* $u(-a) = u(a) = 0$.

Remark: If $k(x,x,u)$ satisfies the strong monotonicity condition

$$(k(x,x,u) - k(x,x,v))(u - v) \ge q_1 r^{-1}(x)(u - v)^2 \quad (q_1 > 0)$$

for $x \in [-a,a]$ and $u, v \in \mathbb{R}$, then the condition $M_1 < 1$ can be weakened to $M_1 < 1 + q_1$.

Corollary: *If in the Assumptions* B *we have* $f' \in L_2$ *in* (i), $|k(x,x,u)| \le b(x) + d|u|$ $(b \in L_2, d > 0)$ *in* (iii), *and* (3) *with* (4) *in* (iv) *where* $aM < 1$, *then the integral equation* (11) *possesses a unique solution pair* $(u,c) \in L_2 \times \mathbb{R}$ *with* $u' \in L_2(r)$, $Tu \in L_2$ *and* $u(-a) = u(a) = 0$. *If further the additional assumption* (2) *is fulfilled, then the condition* $aM < 1$ *can be replaced by* $M < 1/a + q$.

The **Proof** can be done as that of Theorem 1 using the Corollary to Theorem VI in [4] and besides (10) the estimation $(Tu, u) \geq 1/a \, \|u\|^2$ following from (19) ∎

Remark: More generally, Theorem 2 holds for $f' \in L_2(r_0)$, $r_0 = (a^2 - x^2)^{\delta}$, $0 \leq \delta \leq 1/2$,

$$|k(x,x,u)| \leq b(x) + d r_0^{-1}(x)|u|$$

and

$$|k_x(x,y,u) - k_x(x,y,v)| \leq \gamma(x,y) r_0^{-1}(y)|u-v|$$

with

$$b \in L_2(r_0),\ d > 0 \ \text{ and } \ M_0^2 := \int_{-a}^{a} r_0(x) \int_{-a}^{x} r_0^{-1}(y)\gamma^2(x,y)\,dy\,dx < \infty,$$

where $u \in L_2(r_0^{-1})$, $c \in \mathbb{R}$ with $u' \in L_2(r)$, $Tu \in L_2(r_0)$ and $u(-a) = u(a) = 0$ under the condition $a^{1-2\delta} M_0 < 1$.

2. Equations of second type

We look for an absolutely continuous function u and a constant $c \in \mathbb{R}$ fulfilling the integro-differential equation

$$F(x,u(x)) + \int_{-a}^{x} k(y,u(y))u'(y)\,dy + \int_{-a}^{x} l(x,y,u(y))\,dy = (Su)(x) + c \tag{20}$$

for a.a. $x \in [-a,a]$ and the boundary conditions $u(-a) = u(a) = 0$.

The **Assumptions C** on the data are the following ones:

(i) $F(x,u)$ is a continuous function on $[-a,a] \times \mathbb{R}$, possesses a continuous derivative $F_u(x,u)$ there and a derivative $F_x(x,u)$ satisfying the Carathéodory condition (i.e. $F_x(x,u)$ is continuous with respect to u for a.a. x and measurable with respect to x for all u) and an estimation of the form

$$|F_x(x,u)| \leq l_0(a^2 - x^2)^{-\delta},\ 0 \leq \delta \leq 1/2. \tag{21}$$

(ii) $k(x,u)$ is a continuous function on $[-a,a] \times \mathbb{R}$.

(iii) $l(x,y,u)$ is a continuous function on $(-a,a) \times (-a,a) \times \mathbb{R}$ satisfying an estimation of the form

$$|l(x,y,u)| \leq l_1(a^2 - x^2)^{-\delta},\ 0 \leq \delta \leq 1/2 \tag{22}$$

and possesses a derivative $l_x(x,y,u)$ satisfying a corresponding estimation

$$\int_{-a}^{x} |l_x(x,y,u)|\,dy \leq l_2(a^2 - x^2)^{-\delta},\ 0 \leq \delta \leq 1/2. \tag{23}$$

(iv) The function $A(x,u) = F_u(x,u) + k(x,u)$ fulfils the inequality

$$-\alpha \leq A(x,u) \leq \beta \text{ with finite } \alpha, \beta \geq 0 \text{ and } \alpha\beta < 1. \tag{24}$$

Differentiating (20) leads to the equivalent equation

$$A(x,u(x))u'(x) - (Su')(x) = -F_x(x,u(x)) - l(x,x,u(x)) - \int_{-a}^{x} l_x(x,y,u(y))\,dy \tag{25}$$

for a.a. $x \in [-a,a]$. Applying to (25) Theorem 2 in [5] (as D. Oestreich observed, the case $\delta = 1/2$ in this Theorem is also admitted), we obtain

Theorem 3: *If the* Assumptions C *with* (24) *are fulfilled, then the integro-differential equation* (20) *possesses a solution pair* $(u,c) \in W_p^1(-a,a) \times \mathbb{R}$, $p < \min\{1/\delta, 1/\beta_1, 1/\beta_2\}$, *where* $\beta_1 = 1/2 - (1/\pi)\arctan A(-a,u)$ *and* $\beta_2 = 1/2 + (1/\pi)\arctan A(a,u)$ *with* $u(-a) = u(a) = 0$.

Remark 1: The uniform conditions (21) - (23) may be relaxed replacing them by corresponding ones in $|u| \le R$ with factors $l_k(R) = b_k + c_k R^{r_k}$, $r_k < 1$ for all $R > 0$ $(k = 0,1,2)$ (cp. [5]).

Remark 2: By differentiation the integral equation

$$\int_{-a}^{x} K(x,y,u(y))\,dy + \frac{1}{\pi}\int_{-a}^{a} u(y)\ln|x-y|\,dy = g(x) + cx + c_1 \tag{26}$$

with two free constants c, c_1 and boundary conditions $u(-a) = u(a) = 0$ can be reduced to a particular case of the integro-differential equation (20). The differentiated equation of (26) is

$$K(x,x,u(x)) + \int_{-a}^{x} K_x(x,y,u(y))\,dy = (Su)(x) + g'(x) + c$$

so that for the existence of a solution we make the following assumptions:

(i) g is continuously differentiable and possesses a second derivative g'' satisfying the estimate $|g''(x)| \le g_0(a^2 - x^2)^{-\delta}$, $0 \le \delta \le 1/2$.

(ii) $K(x,y,u)$ is continuous on $[-a,a] \times [-a,a] \times \mathbb{R}$ and $H(x,u) = K(x,x,u)$ fulfils the Assumption C/(i) on $F(x,u)$.

(iii) $K_x(x,y,u)$ fulfils the Assumption C/(iii) on $l(x,y,u)$.

(iv) There holds the inequality $-\alpha \le K_u(x,y,u) \le \beta$ with finite $\alpha, \beta \ge 0$ and $\alpha\beta < 1$.

Remark 3: If there are the additional terms

$$-\int_{x}^{a} \tilde{k}(y,u(y),u'(y))\,dy - \int_{x}^{a} \tilde{l}(x,y,u(y))\,dy$$

in (20), where $\tilde{k}$ and $\tilde{l}$ have analogous properties as k and l, respectively, then the function $A(x,u)$ in (24) is defined by $A(x,u) = F_u(x,u) + k(x,u) + \tilde{k}(x,u)$.

Finally, we prove two *uniqueness theorems* for the equation (20) in the particular case $F = G(u) - f(x)$ and $l(x,y,u) = 0$, i.e. for the corresponding differentiated equation (25)

$$A(x,u(x))u'(x) - (Su')(x) = f'(x) \tag{27}$$

under the boundary conditions $u(-a) = u(a) = 0$.

At first we assume that the function $A(x,u) = G'(u) + k(x,u)$ has continuous derivatives with respect to u and x, i.e. $G(u)$ is twice continuously differentiable and $k(x,u)$ is continuously differentiable with respect to x and u. The difference $u = u_1 - u_2$ of two solutions u_1 and u_2 of (27) satisfies the equation

$$(Tu)(x) + A(x,u_1(x))u'(x) + B(x)u_2'(x)u(x) = 0, \tag{28}$$

where $Tu = -Su'$ as above and

$$B(x) = \int_0^1 A_u(x, u_2(x) + t\,u(x))\,dt. \tag{29}$$

Multiplying (28) with u and integrating over $[-a,a]$, we obtain the relation

$$(Tu, u) + \int_{-a}^{a} A(x, u_1(x))u(x)u'(x)\,dx + \int_{-a}^{a} B(x)u_2'(x)u^2(x)\,dx = 0$$

and integrating the second integral by parts we obtain

$$(Tu, u) - \frac{1}{2}\int_{-a}^{a} C(x)u^2(x)\,dx + \int_{-a}^{a} B(x)u_2'(x)u^2(x)\,dx = 0, \tag{30}$$

where $C(x) = A_x(x, u_1(x)) + A_u(x, u_1(x))\,u_1'(x)$. Let now be

$$\sup_{x,u} |A_u(x, u)| = M_1, \quad \sup_{x,u}\left[r_0(x)|A_x(x, u)|\right] = M_2 \tag{31}$$

and

$$|u_1'(x)|,\ |u_2'(x)| \le N r_0^{-1}(x), \ r_0(x) = (a^2 - x^2)^{\delta}, \ 0 \le \delta \le 1/2.$$

Then observing Schleiff's inequality (19) the left-hand side of (30) can be estimated from below through the expression

$$\left[a^{2\delta-1} - \frac{1}{2}M_2 - \frac{3}{2}M_1 N\right]\int_{-a}^{a} r_0^{-1}(x)u^2(x)\,dx.$$

Hence there follows $u = 0$ if

$$3M_1 N + M_2 < 2/a^{1-2\delta}. \tag{32}$$

Theorem 4: *In the particular case* $F = G(u) - f(x)$, $l = 0$ *there exists at most one solution pair* (u, c), $c \in \mathbb{R}$, *to the equation* (20) *with absolutely continuous function* u satisfying $u(-a) = u(a) = 0$ *and* $|u'(x)| \le N(a^2 - x^2)^{-\delta}$, $0 \le \delta \le 1/2$, *if the inequality* (32) *holds, where* M_1, M_2 *are defined by* (31) *with the function* $A(x, u) = G'(u) + k(x, u)$.

Remark: If $F = -f(x)$, $k = k(x)$ and $l = 0$, then the equation (27) is linear with $A(x, u) = k(x)$ and has a unique absolutely continuous solution u satisfying $u(-a) = u(a) = 0$.

Furthermore, we suppose that the function $A(x, u)$ has continuous derivatives A_u, A_{ux} and A_{uu} and consider solutions u to (20) with continuous first and second order derivatives. Multiplying (28) by u' and integrating over $[-a, a]$, we obtain

$$\int_{-a}^{a} A(x, u_1(x))u'^2(x)\,dx + \int_{-a}^{a} B(x)u_2'(x)u(x)u'(x)\,dx = 0,$$

where B is given by (29) again. Integrating the second integral by parts yields the relation

$$\int_{-a}^{a} A(x, u_1(x))u'^2(x)\,dx - \frac{1}{2}\int_{-a}^{a} D(x)u^2(x)\,dx = 0, \tag{33}$$

where $D(x) = B'(x)u_2'(x) + B(x)u_2''(x)$. Let now be

$$A(x, u) \ge A_0 > 0, \tag{34}$$

$$\sup_{x,u} |A_{ux}(x, u)| = M_3, \quad \sup_{x,u} |A_{uu}(x, u)| = M_4, \tag{35}$$

$$|u_k'(x)| \le N_1, \ |u_k''(x)| \le N_2 \ (k = 1, 2). \tag{36}$$

Then the left-hand side of (33) can be estimated from below through the expression

$$A_0 \int_{-a}^{a} u'^2(x)\,dx - \tfrac{1}{2} A_1 \int_{-a}^{a} u^2(x)\,dx \quad \text{with} \quad A_1 = (M_3 + M_4 N_1) N_1 + M_1 N_2 . \tag{37}$$

Moreover, by Wirtinger's inequality we have

$$\int_{-a}^{a} u^2(x)\,dx \le \frac{4a^2}{\pi^2} \int_{-a}^{a} u'^2(x)\,dx \quad \text{for } u \in W_2^1(-a,a) \text{ with } u(-a) = u(a) = 0 .$$

Hence there follows the lower bound $\left(A_0 - (2a^2/\pi^2)A_1\right)\int_{-a}^{a} u'^2(x)\,dx$ for the left-hand side of (33) and we obtain $u = 0$ if

$$2a^2 A_1 < \pi^2 A_0 . \tag{38}$$

Theorem 5: *In the particular case* $F = Gu - f(x)$, $l = 0$ *with* $A(x,u) = G'(u) + k(x,u)$ *satisfying* (34) *there exists at most one solution pair* (u,c) *to equation* (20) *with continuous second derivative* u'' *and* $u(-a) = u(a) = 0$ *satisfying* $|u'(x)| \le N_1$, $|u''(x)| \le N_2$ *if the inequality* (38) *holds, where* A_0 *and* A_1 *are given by* (34) *and* (37), *respectively, with* M_1, M_3 *and* M_4 *given by* (31) *and* (35).

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