

Convergent Solutions of Ordinary and Functional-Differential Pendulum-Like Equations

G. A. LEONOV, V. REITMANN and V. B. SMIRNOVA

Sufficient frequency-domain conditions for complete stability of ordinary and functional-differential equations on the cylinder are given. For a class of phase-controlled systems, periodic Lyapunov functions and Popov functionals are constructed.

Key words: *Pendulum like equations, integro-differential equations, global convergence, boundedness, PLL-systems*

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1. Introduction

In this paper, we investigate the asymptotic behaviour of some classes of ordinary and functional-differential equations with periodic non-linearities, which are called phase-controlled or pendulum-like systems [6, 10, 12, 14, 19]. Such systems describe models of phase synchronization [16], coupled Josephson junctions [20] and other systems with angular co-ordinates. The set of equilibrium points of a phase system is either empty or infinite. It follows that a central issue of investigating such systems is to provide sufficient conditions for the solutions to converge (the so-called *complete stability*). Note that complete stability of differential phase systems does not imply Lyapunov stability of any of its equilibrium points. Moreover, for autonomous differential phase equations there are always Lyapunov unstable equilibria. It is easy to state that it is meaningless to use standard Lyapunov functions of the type "quadratic form plus integral of the non-linearity" for the stability analysis of such systems. A different technique is required for this class of systems.

One way to obtain results concerning dissipativity, boundedness and convergence is using positively invariant quadratic cones [6, 9, 12, 15, 17, 19, 20].

Another approach to the problem of complete stability is the non-local reduction method [6, 11, 14, 20]. By this method, stability criteria for n -dimensional systems are expressed in terms of the frequency domain response of the linear system component and in terms of solutions of second-order comparison systems.

For certain classes of phase-controlled systems with a gradient non-linearity on a non-compact manifold, the convergence of a solution is equivalent to its boundedness [6, 20].

In the paper [1] Yu.N. Bakaev and A.A. Guzh have shown the possibility of proving the convergence of solutions of autonomous phase-controlled systems by means of periodic Lyapunov functions. This kind of applying Lyapunov functions or functionals is called by us Bakaev-Guzh technique. Using this technique and the Yakubovich-Kalman theorem [6, 7] for the solvability of special matrix inequalities, convergence criteria were proved in [6, 8, 12, 15, 20].

In his paper [3] R.W. Brockett proposes a method of investigating the convergence of differential equations on a flat Riemannian manifold. The author poses certain conditions on the period of a closed one-form which guarantee the existence of a continuous Lyapunov function on the ω -limit set of a bounded solution.

In the present paper we prove some new results concerning the Bakaev-Guzh technique for differential equations which are regarded as feedback equations with a single non-linearity. For this class of equations we get also the results of [3]. Furthermore, we give efficient convergence results for integro-differential equations by means of the Bakaev-Guzh technique and the Popov a priori integral estimates method, which continue the results of [13 - 15].

2. Convergence of dynamical systems in metric spaces

Let us consider in a complete metric space M the dynamical system

$$\dot{x} : M \times \mathbf{R} \rightarrow M. \quad (1)$$

The function x satisfies the conditions following [2]:

- (i) x is continuous
- (ii) $x(0, a) = a \quad (a \in M)$
- (iii) $x(t_1 + t_2, a) = x(t_1, x(t_2, a))$ for all $t_1, t_2 \in \mathbf{R}, a \in M$.

For fixed a the function $x(\cdot, a)$ is called a *motion* of system (1) and the set $\{x(t, a), t \geq 0\}$ its *positive semi-orbit* through a . A point $a \in M$ which satisfies $x(t, a) = a$ for all $t \in \mathbf{R}$ will be called an *equilibrium point*. Suppose that system (1) has multiple equilibrium points which are isolated. The motion $x(\cdot, a)$ (or the positive semi-orbit of $x(\cdot, a)$) is said to be *convergent* if $x(t, a) \rightarrow p$ for $t \rightarrow +\infty$, where p is an equilibrium point. The motion $x(\cdot, a)$ is *bounded* (for $t \geq 0$) if it has a relatively compact positive semi-orbit. The following theorem of Lyapunov type gives sufficient conditions for all bounded motions of (1) to converge. It was proved in [6] for the case of differential equations in \mathbf{R}^n .

Lemma 1: *Suppose there is a continuous function $V : M \rightarrow \mathbf{R}$ such that for all motions of (1) we have*

- (i) $V(x(t_1, a)) \leq V(x(t_2, a))$ for $t_1 \geq t_2$
- (ii) if $V(x(t, a)) \equiv \text{const}$, then $x(t, a) \equiv \text{const}$ on $[0, +\infty)$.

Then every bounded motion of system (1) is convergent.

Proof: Let $x(\cdot, x_0)$ be a bounded motion of (1). Property (i) of the function V allows us to conclude that there exists the limit

$$\lim_{t \rightarrow \infty} V(x(t, x_0)) = \alpha(x_0).$$

It follows from boundedness that the set Ω_{x_0} of ω -limit points of $x(\cdot, x_0)$ is not empty. Consider now a motion $x(\cdot, y)$, where $y \in \Omega_{x_0}$ for all $t \geq 0$. The set Ω_{x_0} is positively invariant [2], so $x(t, y) \in \Omega_{x_0}$ for all $t \geq 0$. Then $V(x(t, y)) = \alpha(x_0)$ for all $t \geq 0$. From property (ii) of the function V it follows that $x(t, y) \equiv y \in \Omega_{x_0}$. Thus all the ω -limit points of the trajectory $x(t, x_0)$ are equilibrium points of (1). As equilibrium points of (1) are isolated, it is clear that y is a single ω -limit point of $x(\cdot, x_0)$ ■

3. Frequency-domain stability criterion for a feedback system with periodic non-linearity

Let us consider a feedback system in the second canonical form [3, 6]

$$\begin{aligned} \dot{z} &= Az + b\varphi(\sigma) \\ \dot{\sigma} &= c^*z + \rho\varphi(\sigma), \end{aligned} \quad (2)$$

where A is an $n \times n$ matrix all of whose characteristic roots have negative real parts, b and c are n -column vectors, ρ is a constant ($\rho \neq c^*A^{-1}b$). The asterisk denotes Hermitian conjugate (in particular, transposition in the case of real vectors and matrices and complex conjugate in the case of numbers). Furthermore, $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ is continuously differentiable, periodic with period Δ , and has only finitely many zeroes on $[0, \Delta)$. Then

$$\mu_1 \leq \varphi'(\sigma) \leq \mu_2 \quad \text{for all } \sigma \in \mathbf{R}, \tag{3}$$

where μ_i with $\mu_1\mu_2 < 0$ are either certain numbers or $\pm\infty$. If $\mu_1 = -\infty$, we will take $\mu_1^{-1} = 0$; if $\mu_2 = +\infty$, we will take $\mu_2^{-1} = 0$. We suppose that the pairs (A, b) and (A^*, c) are controllable. (The pair (P, q) , where P is an $m \times m$ matrix and q is an m -vector, is called *controllable* if $\det [q, Pq, \dots, P^{m-1}q] \neq 0$.) Let us define the function

$$K(p) = -\rho + c^*(A - pI_n)^{-1}b \tag{4}$$

for $p \in \mathbf{C}$ with $\det(A - pI_n) \neq 0$. (I_n is the $n \times n$ -unit matrix.) The main result of this section is formulated in terms of the function K and has the character of a frequency inequality. Now we will establish the following statement, which is a slight modification of the Yakubovich-Kalman frequency theorem [6, 7].

Lemma 2: *Suppose that A is an $n \times n$ matrix all eigenvalues of which have negative real parts, b and c are n -column vectors, $\rho, \tau, \delta, \varepsilon, \mu_1 \neq 0, \mu_2 \neq 0$ are scalars. Suppose that (A, b) is controllable. Then there exists a real $(n + 1) \times (n + 1)$ matrix $H = H^*$ such that*

$$2[z^*, \eta]H \begin{bmatrix} Az + b\eta \\ \xi \end{bmatrix} \leq \begin{aligned} & -(c^*z + \rho\eta)\eta \\ & - \tau[(c^*z + \rho\eta) - \mu_1^{-1}\xi][-\mu_2^{-1}\xi + (c^*z + \rho\eta)] \\ & - \delta\eta^2 - \varepsilon(c^*z + \rho\eta)^2 \end{aligned} \tag{5}$$

for all $z \in \mathbf{R}^n, \eta, \xi \in \mathbf{R}$ if and only if the inequality

$$\operatorname{Re}\{K(i\omega)\} - \tau[K(i\omega) + \mu_1^{-1}i\omega]^*[K(i\omega) + \mu_2^{-1}i\omega] - \varepsilon|K(i\omega)|^2 \geq \delta \tag{6}$$

holds for all $\omega \in \mathbf{R}$. Let the matrix H from (5) be of the form $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ where H_{ij} is an $n \times n$ matrix. If, in addition, $\tau + \eta > 0$ and (A^*, c) is controllable, then $H_{11} > 0$.

Proof: Let us consider new objects

$$Q = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} c \\ \rho \end{bmatrix}, \quad l = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbf{R}^{n+1}, \quad y = \begin{bmatrix} z \\ \eta \end{bmatrix}.$$

Inequality (5) is equivalent to the inequality

$$2y^*H[Qy + l\xi] \leq \begin{aligned} & y^*ld^*y - \tau[-d^*y + \mu_1^{-1}\xi][\mu_2^{-1}\xi - d^*y] \\ & - \delta(l^*y)^2 - \varepsilon(d^*y)^2 \quad (y \in \mathbf{R}^{n+1}, \xi \in \mathbf{R}). \end{aligned}$$

Because of the controllability of (A, b) the pair (Q, l) is controllable [6]. Let us consider the Hermitian form

$$G(y, \xi) = \varepsilon y^*dd^*y + y^*ld^*y + \tau[d^*y - \mu_1^{-1}\xi]^*[-\mu_2^{-1}\xi + d^*y] + \delta y^*ll^*y \tag{7}$$

for all $y \in \mathbf{C}^{n+1}, \xi \in \mathbf{C}$.

By the Yakubovich-Kalman theorem [6,7] there exists an $(n + 1) \times (n + 1)$ matrix H such that the inequality (7) is true if and only if the inequality

$$G[(i\omega I_{n+1} - Q)^{-1}l\xi, \xi] \leq 0$$

is true for all $\omega \in \mathbf{R}$ and $\xi \in \mathbf{C}$. The latter coincides with (6). Thus, the main statement of Lemma 2 is proved.

Let in (5) $\eta = 0$ and $\xi = 0$. Then

$$2z^* H_{11} Az \leq -(\tau + \varepsilon)(c^* z)^2 \quad (z \in \mathbf{R}^n).$$

Hence, because the eigenvalues of the matrix A have negative real parts and (A^*, c) is controllable it follows [6] that $H_{11} > 0$ if $\tau + \varepsilon > 0$ ■

Lemma 3: Suppose that for the matrix A , the column vectors b, c and scalars ρ, μ_1, μ_2 from (2) and (3) there exists an $(n + 1) \times (n + 1)$ matrix H and scalars $\delta > 0, \tau \geq 0, \varepsilon \geq 0$ such that (5) and at least one of the following two conditions holds:

- (i) $4\tau\delta > \nu^2$, where $\nu = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta \sqrt{(1 - \mu_1^{-1} \varphi'(\sigma))(1 - \mu_2^{-1} \varphi'(\sigma))} |\varphi(\sigma)| d\sigma}$
- (ii) $4\varepsilon\delta > \nu_0^2$, where $\nu_0 = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta |\varphi(\sigma)| d\sigma}$.

Then there exists a smooth scalar function $V(z, \sigma)$ ($z \in \mathbf{R}^n, \sigma \in \mathbf{R}$) with the following properties:

- 1) $V(z, \sigma + \Delta) = V(z, \sigma)$ ($z \in \mathbf{R}^n, \sigma \in \mathbf{R}$)
- 2) V does not increase along solutions of system (2)
- 3) if for a solution $z(\cdot), \sigma(\cdot)$ of (2) there holds $V(z(t), \sigma(t)) \equiv \text{const}$ ($t \geq 0$), then $z(t) \equiv \text{const}$ ($t \geq 0$).

Proof: We consider the case of $4\tau\delta > \nu^2$ because the other case was investigated in [6]. Let us define the function

$$W(z, \varphi(\sigma)) = [z^*, \varphi(\sigma)] H \begin{bmatrix} z \\ \varphi(\sigma) \end{bmatrix},$$

where the $(n + 1) \times (n + 1)$ matrix H satisfies (5). Note that $W(z, \varphi(\sigma))$ is periodic in σ with period Δ . From Lemma 2 it follows that

$$W(z, \sigma) = z^* H_{11} z \quad (z \in \mathbf{R}^n) \tag{8}$$

where H_{11} is a positive definite matrix. Let us calculate the derivative $\dot{W}(z(t), \varphi(\sigma(t)))$ along solutions of (2). With the aid of (5) it is easy to establish that

$$\begin{aligned} \frac{d}{dt} W(z(t), \varphi(\sigma(t))) &\leq -\varphi(\sigma(t))\dot{\sigma}(t) - \delta\varphi^2(\sigma(t)) \\ &\quad - \tau\dot{\sigma}^2(t)[1 - \mu_1^{-1}\varphi'(\sigma(t))][1 - \mu_2^{-1}\varphi'(\sigma(t))]. \end{aligned}$$

Its right part may be transformed with the help of new functions

$$\varphi_1(\sigma) = \sqrt{(1 - \mu_1^{-1}\varphi'(\sigma))(1 - \mu_2^{-1}\varphi'(\sigma))}$$

and

$$F_1(\sigma) = (\alpha\varphi_1(\sigma) - 1)\varphi(\sigma), \quad \alpha := \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta \varphi_1(\sigma)\varphi(\sigma)d\sigma}$$

in such a way that the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} W(z(t), \varphi(\sigma(t))) &\leq -\alpha\varphi(\sigma(t))\varphi_1(\sigma(t))\dot{\sigma}(t) - \delta\varphi^2(\sigma(t)) \\ &\quad - \tau\varphi_1^2(\sigma(t))\dot{\sigma}^2(t) + F_1(\sigma(t))\dot{\sigma}(t). \end{aligned} \tag{9}$$

Notice that because of the special choice of α we have $\int_0^\Delta F_1(\sigma)d\sigma = 0$. Let us consider the number

$$\nu_1 = \frac{\int_0^\Delta \varphi_1(\sigma)\varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi_1(\sigma)\varphi(\sigma)|d\sigma}$$

and construct the new function

$$F_2(\sigma) = \alpha\varphi_1(\sigma)\varphi(\sigma) - \nu_1|\alpha\varphi_1(\sigma)\varphi(\sigma)|.$$

It can be easily checked that $\int_0^\Delta F_2(\sigma)d\sigma = 0$. Consider the function

$$V(z, \sigma) = W(z, \varphi(\sigma)) + \int_0^\sigma [F_2(\sigma) - F_1(\sigma)]d\sigma.$$

This function is Δ -periodic because of the special properties of F_1 and F_2 . In virtue of (9) and condition (i) of the lemma the following chain of relations takes place:

$$\begin{aligned} \frac{d}{dt} V(z(t), \sigma(t)) &= \frac{d}{dt} W(z(t), \varphi(\sigma)) + [F_2(\sigma(t)) - F_1(\sigma(t))]\dot{\sigma}(t) \\ &\leq -\delta\varphi^2(\sigma(t)) - \tau\dot{\sigma}^2(t)\varphi_1^2(\sigma(t)) - \nu_1|\alpha||\varphi_1(\sigma(t))||\varphi(\sigma(t))|\dot{\sigma}(t) \\ &\leq -\beta_1\dot{\sigma}^2(t)\varphi_1^2(\sigma(t)) - \beta_2\varphi^2(\sigma(t)), \end{aligned}$$

where β_1, β_2 are certain positive numbers. Hence it follows that $V(z(t), \sigma(t))$ does not increase along trajectories of system (2). Furthermore, if $V(z(t), \sigma(t)) \equiv const$, then $\varphi(\sigma(t)) \equiv 0$.

As the zeroes of the function φ are isolated and $\sigma(\cdot)$ is continuous, it is clear that $\sigma(t) \equiv \sigma_0$, where σ_0 is a zero of φ . Then it follows that $W(z(t), \varphi(\sigma_0)) = W(z(t), 0) \equiv const$. As all eigenvalues of the matrix A have negative real parts, it follows from system (1) that in our situation $|z(t)| \rightarrow 0$ as $t \rightarrow \infty$. Hence $W(z(t), 0) = 0$. Then it follows from (8) that $z(t) \equiv 0$ ■

Theorem 1: : Consider the system (2) and the function $K(\cdot)$ defined by (4). Let μ_1, μ_2 be such that (3) is satisfied. Suppose there exist numbers $\delta > 0, \varepsilon > 0, \tau \geq 0$ such that the inequality (6) is true for all $\omega \in \mathbb{R}$ and at least one of the conditions (i) or (ii) of Lemma 3 is fulfilled. Then all solutions of the system (2) are convergent.

Proof: Under the above propositions the right part of system (2) is invariant with respect to the transformation $\begin{bmatrix} z \\ \sigma \end{bmatrix} \rightarrow \begin{bmatrix} z \\ \sigma + z \end{bmatrix}$ so that this system can be regarded as being defined on the cylinder $\mathbb{R}^{n+1}/\Gamma \cong \mathbb{R}^n \times S^1$, with $\Gamma = \{jd, j \in \mathbb{Z}\}$ and $d = \begin{bmatrix} 0 \\ \Delta \end{bmatrix}$. We define a metric on $\mathbb{R}^n \times S^1$ induced by the metric $ds^2 = dz_1^2 + \dots + dz_n^2 + d\sigma^2$, where $(z, \sigma \pmod{\Delta})$ are the co-ordinates in $\mathbb{R}^n \times S^1$. Because of $\rho \neq c^*A^{-1}b$ the set of equilibria of (2) on $\mathbb{R}^n \times S^1$ is $\{z = 0, \sigma = \bar{\sigma} \text{ with } \varphi(\bar{\sigma}) = 0, \bar{\sigma} \in [0, \Delta)\}$ and consists of isolated points only. Since A is a Hurwitzian matrix and φ is bounded, any solution of (2) is bounded with respect to z . According to Lemmas 2 and 3 on the Riemannian manifold $\mathbb{R}^n \times S^1$ there exists a continuous function V that satisfies all the conditions

of Lemma 1 ■

Example: In order to compare Theorem 1 with the results of the paper [3] consider the scalar second-order equation

$$\ddot{x} + a\dot{x} + \varphi(x) = 0, \tag{10}$$

with $a > 0$, $\varphi(x) = \varphi(x + 1)$, φ continuously differentiable and $\varphi'(x) \leq a/k$ for $x \in \mathbf{R}$ and some $k > 0$. Suppose that φ has a finite number of zeroes in $[0, 1]$. For equation (10) we have $K(p) = (p + a)^{-1}$ and inequality (5) takes the form

$$\omega^2\left(\tau\frac{k}{a} - \delta\right) + (a - \varepsilon - \tau - \delta a^2) \geq 0 \quad (\omega \in \mathbf{R}).$$

Let us choose $\varepsilon = 0$, $\delta = k(ak + 1)^{-1}$, $\tau = a(ak + 1)^{-1}$. Then the latter inequality is true for all $\omega \in \mathbf{R}$ and the only condition of global stability of (10) is the condition (i) from Lemma 3, i.e.

$$\frac{4ka}{1 + ka^2} > \left(\frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta \sqrt{1 - \frac{k}{a}\varphi'(\sigma)}|\varphi(\sigma)| d\sigma} \right)^2.$$

The latter coincides with the global stability conditions obtained in paper [3]. R.W. Brockett also considered system (2) in the case when the inequalities (3) were not taken into consideration (i.e. $\tau = 0$). He established complete stability conditions that coincide with the conditions obtained by means of Theorem 1 in the case (ii) (comp. [3]).

4. Complete stability of functional-differential equations

In this section we extend the results of the previous section to the Volterra integro-differential equations

$$\dot{\sigma}(t) = \alpha(t) + \rho\varphi(\sigma(t - h)) - \int_0^t \gamma(t - h)\varphi(\sigma(\tau))d\tau \tag{11}$$

with constants $\rho, h \geq 0$, a continuous function α such that

$$\alpha \in L_1[0, \infty], \quad \alpha(t) \rightarrow 0 \text{ as } t \rightarrow \infty \tag{12}$$

and with a function γ such that

$$\gamma(t)e^{\kappa t} \text{ belongs to } L_2[0, \infty] \text{ for some } \kappa > 0. \tag{13}$$

The function φ is assumed to be Δ -periodic and to satisfy all conditions for φ in section 2. For equation (11) an initial function is defined by

$$\sigma(t) = \sigma_0(t) \text{ for } t \in [-h, 0] \text{ with } \sigma^0 \in C^1([-h, 0]). \tag{14}$$

The linear part of (11) is characterized by the function

$$K(p) = -\rho e^{-hp} + \int_0^\infty \gamma(t)e^{-pt} dt \quad (p \in \mathbf{C}, \text{Re } p > -\kappa). \tag{15}$$

Notice that the system (1) can be easily reduced to equation (11) with a function (3) being identically equal to function (15).

Theorem 2: Consider equation (11) and the function K defined by (15). Let μ_1, μ_2 be such that (3) is satisfied. Suppose there exist numbers $\delta > 0, \epsilon > 0, \tau \geq 0$ such that the inequality

$$S(\omega) := \operatorname{Re}\{K(i\omega) - \tau[K(i\omega) + \mu_1^{-1}i\omega]^*[K(i\omega) + \mu_2^{-1}i\omega] - \epsilon|K(i\omega)|^2\} \geq \delta \tag{16}$$

is true for all $\omega \in \mathbf{R}$ and at least one of the inequalities (i) $4\tau\delta > \nu^2$ or (ii) $4\epsilon\delta > \nu_0^2$ is fulfilled, where the numbers ν and ν_0 are defined in Lemma 3. Then any solution of (11) approaches a zero of the function φ and the derivative of this solution tends to zero as $t \rightarrow \infty$.

Proof: Here we use the method of a priori integral estimates [5]. The case when condition (ii) holds was considered in the paper [13]. So we will suppose that condition (i) is fulfilled. Let σ be a solution of (11) with initial condition (14). Let us denote $\eta(t) = \varphi(\sigma(t))$ and consider for arbitrary $T > 1$ the auxiliary function

$$\nu_T(t) = \begin{cases} \mu(t)\eta(t) & \text{if } t \leq T \\ \nu_T(T)e^{\beta(T-t)} & \text{if } t > T \end{cases}, \text{ with } \mu(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \in [0, 1] \\ 1 & \text{if } t > 1 \end{cases},$$

where β is a positive constant. So, on the interval $[1, T]$ the function η_T coincides with the function η . Let us also introduce the function

$$\sigma_T(t) = \rho\eta_T(t-h) - \int_0^t \gamma(t-\tau)\eta_T(\tau)d\tau.$$

Notice that

$$\sigma_T(t) = \dot{\sigma}(t) + \sigma_0(t), \quad t \in [1, T] \tag{17}$$

where

$$\sigma_0(t) = -\alpha(t) + \rho[\mu(t-h) - 1]\eta(t-h) + \int_0^t \gamma(t-\tau)[1 - \mu(\tau)]\eta(\tau)d\tau.$$

Because of the special form of η_T and the property (13) the functions $\eta_T, \dot{\eta}_T$ and σ_T are in $L_2[0, +\infty)$ for each $T > 0$. So, they have the $L_2(-\infty, +\infty)$ -Fourier transforms $\tilde{\eta}_T(i\omega), \dot{\tilde{\eta}}_T(i\omega)$, and $\tilde{\sigma}_T(i\omega)$, respectively. Consider the functional

$$R_T = \int_0^\infty \{\sigma_T\eta_T + \delta\eta_T^2 + \epsilon\sigma_T^2 + \tau(\mu_1^{-1}\dot{\eta}_T - \sigma_T)(\mu_2^{-1}\dot{\eta}_T - \sigma_T)\}dt.$$

By virtue of the Parsival equality we get

$$R_T = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \{\tilde{\sigma}_T^*(i\omega)\tilde{\eta}_T(i\omega) + \delta|\tilde{\eta}_T(i\omega)|^2 + \tau(\mu_1^{-1}\dot{\tilde{\eta}}_T^*(i\omega) - \tilde{\sigma}_T^*(i\omega))(\mu_2^{-1}\dot{\tilde{\eta}}_T(i\omega) - \tilde{\sigma}_T(i\omega))\}d\omega.$$

Taking now into consideration that

$$\tilde{\sigma}_T(i\omega) = K(i\omega)\tilde{\eta}_T(i\omega) \quad \text{and} \quad \dot{\tilde{\eta}}_T(i\omega) = i\omega\tilde{\eta}_T(i\omega) \tag{18}$$

we get

$$R_T = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} (S(\omega) - \delta)|\tilde{\eta}_T(i\omega)|^2d\omega.$$

Inequality (16) guarantees that

$$R_T \leq 0. \tag{19}$$

With the help of (17) the functional R_T can be represented in the form

$$R_T = I_T + R_{0T} + R_{1T} + R_{2T} + R_{3T},$$

where

$$\begin{aligned}
 I_T &= \int_0^T \{ \dot{\sigma}\eta + \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau(\mu_1^{-1}\dot{\eta} - \dot{\sigma})(\mu_2^{-1}\dot{\eta} - \dot{\sigma}) \} dt \\
 R_{0T} &= - \int_0^1 \{ \dot{\sigma}(\eta - \eta_T) + \delta(\eta^2 - \eta_T^2) + \tau\mu_1^{-1}\mu_2^{-1}(\dot{\eta}^2 - \dot{\eta}_T^2) \\
 &\quad - \tau\dot{\sigma}(\mu_2^{-1} + \mu_1^{-1})(\dot{\eta} - \dot{\eta}_T) \} dt \\
 R_{1T} &= \int_0^T \{ \sigma_0\eta_T + (\varepsilon + \tau)\sigma_0^2 + 2(\varepsilon + \tau)\dot{\sigma}\sigma_0 - \tau\dot{\eta}_T\sigma_0(\mu_2^{-1} + \mu_1^{-1}) \} dt \\
 R_{2T} &= \int_T^\infty \{ \sigma_T\eta_T + \delta\eta_T^2 + \tau\mu_1^{-1}\mu_2^{-1}\dot{\eta}_T^2 - \tau\dot{\eta}_T\sigma_T(\mu_2^{-1} + \mu_1^{-1}) \} dt \\
 R_{3T} &= \int_T^\infty (\varepsilon + \tau)\sigma_T^2 dt .
 \end{aligned}$$

Note that on the interval $[0, T]$ the function σ_T is substituted by its expression according to formula (17). The functional R_{1T} contains all the terms with the function σ_0 . And at least on the interval $[0, T]$ the function η_T is substituted by η . But these two functions do not coincide on the interval $[0, 1]$. That is why the functional R_{0T} appears. Let us consider each of the functionals R_{kT} ($k = 0, 1, \dots, 3$) separately. It is evident that $|R_{0T}| < C_0$, where C_0 does not depend on T . Notice that η and $\dot{\sigma}$ are bounded on $[0, +\infty)$ and $\sigma_0 \in L_1[0, +\infty)$ because of the properties of α and γ . It gives us the opportunity to affirm

$$\left| \int_0^T \{ \sigma_0\eta_T + 2(\varepsilon + \tau)\dot{\sigma}\sigma_0 - \tau\eta_T\sigma_0(\mu_1^{-1} + \mu_2^{-1}) \} dt \right| < C_1$$

and consequently

$$R_{1T} > -C_1 + \int_0^T (\varepsilon + \tau)\sigma_0^2 dt > -C_1 ,$$

where C_1 does not depend on T . From the form of η_T it is obvious that for each $T > 0$ the functions η_T and $\dot{\eta}_T$ are bounded from above by the function $e^{-\beta t}$. That is why $|R_{2T}| < C_2$, with C_2 not depending on T . Notice that at least $R_{3T} > 0$. Taking account of all the above estimates we obtain from (19) that

$$I_T < C , \tag{20}$$

where C does not depend on T .

Let us now transform the functional I_T with the help of the functions φ_1 and F_1 , introduced in the previous section. Notice that

$$(\mu_1^{-1}\dot{\eta} - \dot{\sigma})(\mu_2^{-1}\dot{\eta} - \dot{\sigma}) = \dot{\sigma}^2\varphi_1^2(\sigma(t)) .$$

Then

$$\begin{aligned}
 I_T &= \int_0^T \{ \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau\eta_1^2\dot{\sigma}^2 + \alpha\eta_1\eta\dot{\sigma} \} dt + \int_0^T (\sigma\eta - \alpha\eta_1\eta\dot{\sigma}) dt \\
 &= \int_0^T \{ \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau\eta_1^2\dot{\sigma}^2 + \alpha\eta_1\eta\dot{\sigma} \} dt - \int_{\sigma(0)}^{\sigma(T)} F_1(\sigma) d\sigma ,
 \end{aligned}$$

where $\eta_1(t) = \varphi_1(\sigma(t))$ and α is defined in the previous section. The last integral is bounded by a constant independent of T because $\int_0^\Delta F_1(\sigma) d\sigma = 0$. So it follows from (20) that

$$I_{T1} := \int_0^T \{ \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau\eta_1^2\dot{\sigma}^2 + \alpha\eta_1\eta\dot{\sigma} \} dt < C_4 \tag{21}$$

where C_4 does not depend on T . Let us now use the function F_2 from the previous section and write

$$\begin{aligned}
 I_{T1} &= \int_0^T \{ \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau\eta_1^2\dot{\sigma}^2 + \nu_1\alpha|\eta_1\eta|\dot{\sigma} \} dt + \int_0^T \{ \alpha\eta_1\eta - \nu_1\alpha|\eta_1\eta| \} dt \\
 &= \int_0^T \{ \varepsilon\dot{\sigma}^2 + \delta\eta^2 + \tau\eta_1^2\dot{\sigma}^2 + \nu_1\alpha|\eta_1\eta|\dot{\sigma} \} dt + \int_{\sigma(0)}^{\sigma(T)} F_2(\sigma) d\sigma .
 \end{aligned}$$

As $\int_{\sigma(0)}^{\sigma(T)} F_1(\sigma) d\sigma$ is bounded by a constant independent of T , we get from (21) that

$$\int_0^T \{\varepsilon \dot{\sigma}^2 + \delta \eta^2 + \tau \eta_1^2 \dot{\sigma}^2 + \nu_1 \alpha |\eta_1 \eta| \dot{\sigma}\} dt < C_5,$$

where C_5 is independent of T . Hence, because of (i) it follows that

$$\dot{\sigma}(\cdot) \in L_2[0, +\infty) \quad \text{and} \quad \varphi(\sigma(\cdot)) \in L_2[0, +\infty). \quad (22)$$

It is easy to show that the functions $\dot{\sigma}(\cdot)$ and $\varphi(\sigma(\cdot))$ are uniformly continuous on $[0, +\infty)$. Indeed, $\varphi(\sigma(t))$ has a bounded derivative as $\dot{\sigma}(t)$ and $\varphi'(\sigma)$ are bounded. As to $\dot{\sigma}(t)$ we need only to prove the uniform continuity of $\int_0^t \gamma(t-\tau)\eta(\tau) d\tau$ which immediately follows from (13), the uniform continuity of η and the absolute continuity of the integral. Now it follows from (22) according to Barbalat's lemma [18] that $\dot{\sigma}(t) \rightarrow 0$ as $t \rightarrow +\infty$ and $\varphi(\sigma(t)) \rightarrow 0$ as $t \rightarrow +\infty$. Since, by assumption, the zeroes of φ are isolated, the foregoing relation implies $\sigma(t) \rightarrow \sigma_0$ as $t \rightarrow +\infty$ where $\varphi(\sigma_0) = 0$ ■

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Prof. Dr. Gennadii Alekseevich Leonov
Dr. Vera Borisovna Smirnova
Leningrad State University
Chair of Theoretical Cybernetics
Bibliotechnaya pl. 2.
198901 Petrodvoretz, USSR

Dr. Volker Reitmann
TU Dresden, Institute of Analysis
Mommssenstr. 13
0-8027 Dresden, Germany