Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11(1992)2, 153-166

Singular Integral Operators with Operator-Valued Kernels on Spaces of Homogeneous Type

B. BORDIN[1)] and D. L. FERNANDEZ[2)]

We give a generalized form of the Hörmander-Schwartz-Triebel theorem for singular integral operators of potential type as well as a sequential version. Applications include fractionary maximal inequalities of Hardy-Littlewood and Fefferman-Stein type.

Key words: *Singular integral operators, homogeneous spaces, vector-valued inequalities, fractionary maximal functions*

AMS subject classification: Primary 42 B 25, 42 B 20

1. Introduction

In the classical paper [8], L. Hörmander considers singular integral operators of potential type, i.e., operators which map L^p into L^q, with a kernel K satisfying

$$\left(\int_{|x|>2|y|} |K(x-y) - K(x)|^{\gamma} dx \right)^{1/\gamma} \le c,$$

where $1/p - 1/q = 1/\gamma$ (actually Hörmanders condition is slightly more general). The well-known potential operator of order γ is a particular case of these operators. Afterwards, J. Schwartz [15] and H. Triebel [16] obtained vectorial versions of Hörmander's theorem by considering operator-valued kernels. Before Triebel's work, Benedek, Calderón and Panzone [2] had also considered operator-valued kernels, but in the case $p = q$ and $\gamma = 1$.

Motivated by the elegant treatment of the Benedek-Calderón-Panzone theorem given by Rubio de Francia, Ruiz and Torrea [12], we shall present here an up-dated and generalized version of the Hörmander- Schwartz-Triebel theorem. We shall work with variable kernels and in the framework of the spaces of homogeneous type.

In Section 2, we give some preliminary definitions and a version for spaces of homogeneous type of an inequality of C. Fefferman and E. Stein, proved by R. Macias (see [10]). As a consequence we are able to state an interpolation theorem of Marcinkiewicz-Riviere type in this framework. In Section 3, we prove a generalized form of the Hörmander-Schwartz-Triebel theorem (see [8, 15, 16]) for singular integral operators of potential type as well as a sequential version. Section 4 is devoted mainly to obtain norm inequalities for fractionary maximal operators (of F. Zo's type) in spaces of homogeneous type. The fractionary maximal inequalities of Hardy-Littlewood and Fefferman-Stein type are derived in Section 5. In this section, we

1) Supported in part by CNPq - Proc. N° 301170/79

2) Supported in part by CNPq - Proc. N° 301600/78

also give an application of the sequential inequality of Fefferman and Stein, to obtain a version of an L^p-L^q inequality for a function of Marcinkiewicz type $J_{r,\varepsilon}(x)$ (see [7]), in normalized homogeneous spaces. Finally in Section 6 we state a theorem of Littlewood-Paley type.

2. Preliminary definitions and results

In this section we give some definitions and a version for spaces of homogeneous type of an inequality of C. Fefferman and E. Stein.

Definition 2.1: Let X be an arbitrary set. A map d: $X \times X \to \mathbb{R}$ is called *quasi-distance* provided

(i) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq k\big(d(x,z) + d(z,y)\big)$ for some constant $k \geq 1$, independent of x, y and z.

A quasi-distance $d: X \times X \to \mathbb{R}$ defines a uniform structure on X. The balls $B(x,r) = \{y \in X: d(x,y) < r\}$, $r > 0$, form a basis of neighbourhoods of $x \in X$ for the topology induced by the uniformity on X. This topology referred to as the *d-topology* of X is a metric one since the uniform structure associated to d has a countable basis.

Definition 2.2: Let X be a set endowed with a quasi-distance d and with a positive measure μ, defined on a σ-algebra of subsets of X which contains the d-open subsets and all balls $B(x,r)$. The triple (X, d, μ) is said a *space of homogeneous type*, or short a *homogeneous space*, if there exist two finite constants, $\alpha > 1$ and $A > 0$, such that

$$0 < \mu(B(x,\alpha r)) \leq A\mu(B(x,r)) < \infty$$

holds for all $x \in X$ and $r \in \mathbb{R}_+$.

Definition 2.3: A space of homogeneous type X will be called ρ-*normal* if there is a positive continuous function ρ on $\mathbb{R}_+$ and positive constants A_1 and A_2 such that

$$A_1 \leq \rho(r)\mu(B(x,r)) \leq A_2$$

holds for all $x \in X$ and $r \in \mathbb{R}_+$.

In particular if we take $\rho(r) = r^{-1}$, we obtain the so-called normalized homogeneous space (see, e.g., [4,5]). For examples of spaces of homogeneous type we refer to [4].

Let X be a space of homogeneous type and G a Banach space. We shall use the following notations in this paper:

$L^p_G = L^p(X,G)$, for the space of G-valued p-integrable functions on X

$L^0_G = L^0(X,G)$, for the space of G-valued strongly measurable functions on X

$L^\infty_{G,c} = L^\infty_c(X,G)$, for the space of G-valued bounded functions with compact support on X.

In any case, when $G = \mathbb{R}$ we shall drop the index G. The capital letters E and F always indicate Banach spaces.

Definition 2.4: If $f \in L^1_{loc}(X, E)$, the *Hardy-Littlewood maximal function* or simply the *maximal functon* of f is defined at $x \in X$ by setting

$$M_E f(x) = \sup \frac{1}{\mu(B)} \int_B \|f(y)\|_E \, d\mu(y),$$

where the supremum is taken over all balls B containing x.

The following result, due to A.P. Calderón, can be found in [3].

Theorem 2.1: *Let $f \in L^p(X, E)$ with $1 < p \le \infty$. Then $M_E f$ is finite a.e., $M_E f \in L^p(X, \mathbb{R})$ and*

(i) $\|M_E f\|_{L^p} \le c\|f\|_{L^p_E}$

(ii) $\|f(x)\|_E \le M_E f(x)$ *for a.a.* $x \in X$.

Definition 2.5: If $f \in L^r_{loc}(X, E)$, the *sharp* maximal function $M^\#_{E,r} f$, $1 \le r < \infty$, is defined at $x \in X$ by setting

$$M^\#_{E,r} f(x) = \sup \left(\frac{1}{\mu(B)} \int_B \|f(y) - f_B\|^r_E \, d\mu(y) \right)^{1/r},$$

where the supremum is taken over all the open balls B containing x, and f_B denotes the mean value of f on the ball B.

Definition 2.6: Let $f \in L^r_{loc}(X, E)$. We say that $f \in BMO^r(X, E) = BMO^r_E$, if $M^\#_{E,r} f \in L^\infty$, i.e., if $\|f\|_{BMO^r_E} = \|M^\#_{E,r} f\|_{L^\infty} < \infty$.

As in the case $E = \mathbb{R}$, we have $BMO^r_E = BMO^1_E$ with norm equivalence. We shall use this fact in the proof of our main result. We shall write $M^\#_E$ instead of $M^\#_{E,1}$ and BMO_E instead of BMO^1_E.

The following result, due to R. Macias (see [10]), is a version for spaces of homogeneous type of an inequality of C. Fefferman and E. Stein.

Theorem 2.2: *Let $q_0 \in \mathbb{R}$ be given such that $1 < q_0 \le \infty$. If $g \in L^{q_0}(X, E)$, then for all $q \ge q_0$ we have*

$$\|M_E g\|_{L^q} \le C \begin{cases} \|M^\#_E g\|_{L^q} & \text{if } \mu(X) = \infty \\ \|M^\#_E g\|_{L^q} + \|g\|_{L^1_E} & \text{if } \mu(X) < \infty \end{cases} \tag{2.1}$$

where the constant is independent of g.

Actually, Theorems 2.1 and 2.2 are established for scalar-valued functions, but the proofs in the vectorial case follow in the same lines.

As a consequence of the inequalities (2.1) a Marcinkiewicz-Rivière interpolation theorem in spaces of homogeneous type can be established.

Theorem 2.3: *Let X be a space of homogeneous type, let E and F be Banach spaces and $T: L^\infty_c(X, E) \to L^0(X, F)$ a sublinear operator such that, for some $p_0, q_0, r \in \mathbb{R}$ with $1 \le p_0 \le r \le \infty$ and $1 < q_0 < \infty$, we have*

(i) $\|Tf\|_{L^{q_0}_F} \le C_0 \|f\|_{L^{p_0}_E}$ (ii) $\|Tf\|_{BMO_F} \le C_1 \|f\|_{L^r_E}$

for all $f \in L_c^\infty(X,E)$. *Then* T *has a bounded extension from* $L^p(X,E)$ *into* $L^q(X,F)$, *where* $1/p = (1-\vartheta)/p_0 + \vartheta/r$ *and* $1/q = (1-\vartheta)/q_0$, $0 < \vartheta < 1$. *Moreover, if* $1/p_0 - 1/q_0 = 1/r$, *we also have* $1/p - 1/q = 1/r$.

Proof: Let us consider the sublinear operator $M_E^\# \circ T$ and $f \in L_c^\infty(X,E)$. Since $q_0 > 1$, by the maximal Theorem 2.1 and (i) we get

$$\|M_E^\# \circ Tf\|_{L^{q_0}} \le 2\|M_E \circ Tf\|_{L^{q_0}} \le 2\|Tf\|_{L_F^{q_0}} \le C_0'\|f\|_{L_E^{p_0}}$$

and the definition of BMO_E yields at once that

$$\|M_E^\# \circ Tf\|_{L^\infty} \le C\|f\|_{L_E^r}.$$

Having both these inequalities at hand, the Marcinkiewicz interpolation theorem assures that $M_E^\# \circ T$ is bounded from $L^p(X,E)$ *into* $L^q(X,\mathbb{R})$, where $1/p = (1-\vartheta)/p_0 + \vartheta/r$ *and* $1/q = (1-\vartheta)/q_0$, $0 < \vartheta < 1$. Now, if $\mu(X) = \infty$, by (2.1) we have for $f \in L_c^\infty(X,E)$

$$\|Tf\|_{L_F^q} \le \|M_E \circ Tf\|_{L^q} \le C\|M_E^\# \circ Tf\|_{L^q} \le C\|f\|_{L_E^p}.$$

On the other hand, if $\mu(X) < \infty$, by (2.1) and applying Hölder's inequality twice, we obtain

$$\begin{aligned}
\|Tf\|_{L_F^q} &\le C_1\big(\|M_E^\# \circ Tf\|_{L^q} + \|Tf\|_{L_F^1}\big)\\
&\le C_1\big(C'\|f\|_{L_E^p} + \mu(X)^{1/q_0'}\|Tf\|_{L_F^{q_0}}\big)\\
&\le C_1\big(C'\|f\|_{L_E^p} + C''\mu(X)^{1/q_0'}\|f\|_{L_E^{p_0}}\big)\\
&\le C_1\big(C'\|f\|_{L_E^p} + C''\mu(X)^{1/q_0'+1-p_0/p}\|f\|_{L_E^p}\big),
\end{aligned}$$

since $p > p_0$. The density of $L_c^\infty(X,E)$ in $L^p(X,E)$, $1 < p < \infty$, leads finally to the desired result ∎

3. Singular integral operators

Let E and F be Banach spaces and let X be a space of homogeneous type, endowed with a quasi-distance d and a measure μ. In this section we shall deal with operators T on $L_c^\infty(X,E)$ which have a representation of the type

$$Tf(x) = \int_X K(x,y)f(y)\,d\mu(y) \text{ for } x \notin \operatorname{supp} f \tag{3.1}$$

with a kernel $K \in L_{loc}^1(X \times X \setminus \Delta, L(E,F))$, where Δ stands for the diagonal set in $X \times X$.

Definition 3.1: We say that the kernel $K \in L_{loc}^1(X \times X \setminus \Delta, L(E,F))$ satisfies condition (C_γ), $1 \le \gamma < \infty$, if for all $y \in X$ and fixed $x_0 \in X$ we have

$$\int_{d(x,x_0) > a d(y,x_0)} \|K(x,y) - K(x,x_0)\|_{L(E,F)}^\gamma \, d\mu(x) \le C, \; a > 1. \tag{3.2}$$

We say that this kernel satisfies condition (C_γ') if $K'(x,y) = K(y,x)$ satisfies (C_γ).

Theorem 3.1: *Let the operator* T *be given as in* (3.1) and let there are numbers $\gamma, r_0, s_0 \in \mathbb{R}$ *satisfying* $1 \le \gamma < \infty$, $1 < r_0 \le \infty$ *and* $1/r_0 - 1/s_0 = 1/\gamma - 1$, *such that*

$$\|Tf\|_{L_F^{r_0}} \le C\|f\|_{L_E^{s_0}}. \tag{3.3}$$

(i) *If the kernel* $K \in L^1_{loc}(X \times X \setminus \Delta, L(E,F))$ *satisfies condition* (C_γ), *then the operator* T *of* (3.1) *is of weak type* $(1,\gamma)$, *i.e.,*

$$\mu(\{x\colon \|Tf(x)\|_F > t\}) \le Ct^{-\gamma}\|f\|^{\gamma}_{L^1_E}.$$

(ii) *If the kernel* $K \in L^1_{loc}(X \times X \setminus \Delta, L(E,F))$ *satisfies condition* (C'_γ), *then the operator* T *of* (3.1) *is of type* $(L_E^{\gamma'}, BMO_F)$, *i.e.,*

$$\|Tf\|_{BMO_F} \le C\|f\|_{L_E^{\gamma'}},$$

where $1/\gamma + 1/\gamma' = 1$. *Moreover, after an extension,* T *is of strong type* (p,q), *i.e.,*

$$\|Tf\|_{L^q_F} \le C\|f\|_{L^p_E},$$

where

$$1/p - 1/q = 1 - 1/\gamma, \text{ with } \begin{cases} 1 < p \le s_0 \text{ and } \gamma < q \le r_0 \text{ if } K \text{ satisfies } (C_\gamma) \\ s_0 < p \le \gamma' \text{ and } r_0 < q < \infty \text{ if } K \text{ satisfies } (C'_\gamma). \end{cases}$$

Proof: *Step* 1. Let $f \in L^\infty_c(X,E)$ and λ a positive constant such that we shall make more precise later (if $\mu(X) < \infty$, we restrict λ to be greater than $\|f\|_{L^1_E}/\mu(X)^{1/\gamma}$). Then there are a function $g \in L^1_E \cap L^\infty_E$, a sequence $\{b_j\} \subset L^1(X,E)$ and balls $B(x_j,R_j)$ such that

$$f = g + \textstyle\sum_j b_j$$

with

$$\|g(x)\|_E \le C\lambda \text{ for a.a. } x \in X, \quad \|g\|_{L^1_E} \le C\|f\|_{L^1_E} \tag{3.4}$$

$$\operatorname{supp} b_j \subset B(x_j,R_j) \text{ and } \int b_j(x)\,d\mu(x) = 0 \text{ for all } j, \ \textstyle\sum_j \|b_j\|_{L^1_E} \le C\|f\|_{L^1_E}.$$

The proofs of these properties follow the same lines as in the scalar case (see [5]).

Step 2. If the kernel K satisfies (C_γ), then the operator T is of weak type $(1,\gamma)$. In fact, given $f \in L^\infty_c(X,E)$ we shall control the measure of the set $\{x \in X\colon \|Tf(x)\|_F > t\}$. Observe that in the case $\mu(X) < \infty$ it is enough to consider $t > \|f\|_{L^1_E}/\mu(X)^{1/\gamma}$. Since

$$\mu(\{x \in X\colon \|Tf(x)\|_F > t\}) \le \mu(\{x \in X\colon \|Tg(x)\|_F > t/2\}) + \mu(\{x \in X\colon \|Tb(x)\|_F > t/2\}),$$

it is enough to estimate the two terms on the right-hand side of the above inequality. By (3.4) it follows that $\|g\|_{L_E^{s_0}} \le C\lambda^{1-1/s_0}\|f\|^{1/s_0}_{L^1_E}$. Thus by (3.3) we obtain

$$\mu(\{x \in X\colon \|Tg(x)\|_F > t/2\}) \le 2^{r_0}t^{-r_0}\|g\|^{r_0}_{L_E^{s_0}} \le Ct^{-r_0}\lambda^{r_0/\gamma - 1}\|f\|^{r_0/s_0}_{L^1_E}.$$

Now, let $B_j = B(x_j,R_j)$ be the balls of the Calderón-Zygmund decomposition of $f = g + b$, where $b = \sum_j b_j$, and let $\widetilde{B} = B(x_j, 2aR_j)$, where a is the constant of Definition 3.1. First, we observe that

$$\left(\int_{(\cup_j \widetilde{B}_j)^c} \|Tb(x)\|_F^\gamma\, d\mu(x)\right)^{1/\gamma} \le C\|f\|_{L^1_E}. \tag{3.5}$$

In fact, for each j we have $Tb_j(x) = \int_{B_j} K(x,y)b_j(y)\,d\mu(y)$ and hence, taking into account that

$\int b_j(y)\,d\mu(y) = 0$, we get

$$Tb_j(x) = \int_{B_j} (K(x,y) - K(x,x_j))\, b_j(y)\, d\mu(y).$$

It follows by Minkowski's inequality and Fubini's theorem that

$$\left(\int_{(\tilde{B}_j)^c} \|Tb_j(x)\|_F^\gamma\, d\mu(x)\right)^{1/\gamma} \le \int_{B_j}\left(\int_{(\tilde{B}_j)^c} \|K(x,y) - K(x,x_j)\|_{L(E,F)}^\gamma \|b_j(y)\|_E^\gamma\, d\mu(x)\right)^{1/\gamma} d\mu(y).$$

Now, since $x \in (\tilde{B}_j)^c$ and $y \in B_j$, we have $d(x,x_j) > a\,d(y,x_j)$, and by condition (3.2) it results that

$$\left(\int_{(\tilde{B}_j)^c} \|Tb_j(x)\|_F^\gamma\, d\mu(x)\right)^{1/\gamma} \le C\int_{B_j} \|b_j(y)\|_E\, d\mu(y).$$

Then

$$\int_{(\cup_j \tilde{B}_j)^c} \|Tb(x)\|_F^\gamma\, d\mu(x) \le C\sum_j \int_{(\cup_j \tilde{B}_j)^c} \|Tb_j(x)\|_F^\gamma\, d\mu(x) \le C\sum_j \int_{(\tilde{B}_j)^c} \|Tb_j(x)\|_F^\gamma\, d\mu(x)$$

$$\le C\left(\sum_j \int_{B_j} \|b_j(y)\|_E\, d\mu(y)\right)^\gamma \le C\left(\sum_j \|b_j\|_{L_E^1}\right)^\gamma \le C\|f\|_{L_E^1}^\gamma$$

and (3.5) follows.

Now, we are in conditions to obtain

$$\mu\left(\left\{x \in X:\ \|Tb(x)\|_F > t/2\right\}\right) \le \mu\left(\bigcup_j \tilde{B}_j\right) + 2^\gamma \int_{\{x \in (\cup_j \tilde{B}_j)^c:\ \|Tb(x)\|_F > t/2\}} t^{-\gamma} \|Tb(x)\|_F^\gamma\, d\mu(x)$$

$$\le C_1 \lambda^{-1} \|f\|_{L_E^1} + C_2 t^{-\gamma} \|f\|_{L_E^1}^\gamma,$$

where the first term of the last inequality is a consequence of the Calderón-Zygmund decomposition theorem associated with λ. Thus by (3.5) we get

$$\mu\left(\left\{x \in X:\ \|Tf(x)\|_F > t\right\}\right) \le C\left(\lambda^{-1}\|f\|_{L_E^1} + t^{-\gamma}\|f\|_{L_E^1}^\gamma + t^{-r_0}\lambda^{r_0/\gamma - 1}\|f\|_{L_E^1}^{r_0/s_0}\right).$$

Choosing $\lambda > 0$ so that $\lambda^{-1} = t^{-\gamma}\|f\|_{L_E^1}^{\gamma-1}$(observe that when $\mu(X) < \infty$ such λ always exists because in this case it is enough to consider $t > \|f\|_{L_E^1}/\mu(X)^{1/\gamma}$) we obtain

$$\mu\left(\left\{x \in X:\ \|Tf(x)\|_F > t\right\}\right) \le C t^{-\gamma}\|f\|_{L_E^1}^\gamma$$

which proves (i) when $r_0 \neq \infty$.

In the case $r_0 = \infty$ we proceed as in the previous case. We have

$$\|Tg\|_{L_F^\infty} \le C\left(\int \|g(y)\|_E^{s_0}\, d\mu(y)\right)^{1/s_0} = C\left(\int \|g(y)\|_E^{s_0-1}\|g(y)\|_E\, d\mu(y)\right)^{1/s_0}.$$

By (3.4) it follows that

$$\|Tg\|_{L_F^\infty} \le C\gamma^{1-1/s_0}\|f\|_{L_E^1}^{1/s_0} = C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}. \tag{3.6}$$

Thus

$$\mu\left(\left\{x \in X:\ \|Tf(x)\|_F > 2C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}\right\}\right)$$

$$\le \mu\left(\left\{x \in X:\ \|Tg(x)\|_F > C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}\right\}\right) + \mu\left(\left\{x \in X:\ \|Tb(x)\|_F > C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}\right\}\right)$$

$$= \mu\left(\left\{x \in X: \|Tb(x)\|_F > C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}\right\}\right)$$

$$\leq C_1\lambda^{-1}\|f\|_{L_E^1} + C_2\lambda^{-1}\|f\|_{L_E^1} = C\lambda^{-1}\|f\|_{L_E^1} = C_3\left(C\lambda^{1/\gamma}\|f\|_{L_E^1}^{1/\gamma'}\right)^{-\gamma}\|f\|_{L_E^1}^{\gamma}$$

which proves (i) when $r_0 = \infty$.

Step 3. Let $x_0 \in X$ be fixed and consider the balls $B = B(x_0, R)$ and $\widetilde{B} = B(x_0, 2aR)$. Let $f = \widetilde{g} + \widetilde{b}$ where $\widetilde{g} = f\chi_{\widetilde{B}}$ and $\widetilde{b} = f - \widetilde{g}$. We shall prove that T is of type $\left(L_E^{\gamma'}, BMO_F^{r_0}\right)$. We have $Tf = T\widetilde{g} + T\widetilde{b}$ and consequently

$$\int_B \|Tf(x) - a_B\|_F^{r_0}\,d\mu(x) \leq \int_B \|T\widetilde{g}(x)\|_F^{r_0}\,d\mu(x) + \int_B \|T\widetilde{b}(x) - a_B\|_F^{r_0}\,d\mu(x).$$

Since T is bounded from L^{s_0} into L^{r_0}, we obtain

$$\left(\frac{1}{\mu(B)}\int_B \|T\widetilde{g}(x)\|_F^{r_0}\,d\mu(x)\right)^{1/r_0} \leq \frac{C}{\mu(B)^{1/r_0}}\left(\int_B \|\widetilde{g}(x)\|_E^{s_0}\,d\mu(x)\right)^{1/s_0}.$$

Applying Hölder inequality, it results that the last term of the above inequality is majorized by

$$C\left(\int_B \|\widetilde{g}(x)\|_E^{\gamma'}\,d\mu(x)\right)^{1/\gamma'}\mu(B)^{1/s_0 - 1/\gamma' - 1/r_0}.$$

Since, by hypothesis, $1/s_0 - 1/r_0 = 1/\gamma'$, it follows that

$$\left(\frac{1}{\mu(B)}\int_B \|T\widetilde{g}(x)\|_F^{r_0}\,d\mu(x)\right)^{1/r_0} \leq C\|g\|_{L_E^{\gamma'}}. \tag{3.7}$$

Now, choosing $a_B = \int_X K(x_0, y)\widetilde{b}(y)\,d\mu(y)$, we have

$$T\widetilde{b}(x) - a_B = \int_{(\widetilde{B})^c}(K(x,y) - K(x_0,y))\widetilde{b}(y)\,d\mu(y)$$

and hence, by Hölder inequality, we obtain

$$\left(\frac{1}{\mu(B)}\int_B \|T\widetilde{b}(x) - a_B\|_F^{r_0}\,d\mu(x)\right)^{1/r_0}$$

$$\leq \frac{1}{\mu(B)^{1/r_0}}\left\{\int_B\left\{\left(\int_{(\widetilde{B})^c}\|K(x,y) - K(x_0,y)\|_{L(E,F)}^{\gamma}\,d\mu(y)\right)^{r_0/\gamma}\right.\right.$$

$$\left.\left.\times\left(\int\|\widetilde{b}(y)\|_E^{\gamma'}\,d\mu(y)\right)^{r_0/\gamma'}\right\}d\mu(x)\right\}^{1/r_0}.$$

Now, since $y \in (\widetilde{B})^c$ and $x \notin \operatorname{supp}\widetilde{b}$, it follows that $d(x, x_0) > a\,d(y, x_0)$. Then by condition (C'_γ) we have

$$\left(\frac{1}{\mu(B)}\int_B \|T\widetilde{b}(x) - a_B\|_F^{r_0}\,d\mu(x)\right)^{1/r_0} \leq C\|\widetilde{b}\|_{L_E^{\gamma'}}.$$

From (3.6) and (3.7) it results that

$$\left(\frac{1}{\mu(B)}\int_B \|Tg(x) - a_B\|_F^{r_0}\,d\mu(x)\right)^{1/r_0} \leq C\|f\|_{L_E^{\gamma'}},$$

i.e., T is of type $\left(L_E^{\gamma'}, BMO_F^{r_0}\right)$. Taking into account that $BMO_F^{r_0} = BMO_F$, we obtain (ii).

The remainder of the theorem follows by the Marcinkiewicz interpolation theorem (C_γ - case) and Theorem 2.3 (C'_γ - case) ∎

Remark: If we take $X = \mathbb{R}^n$, $d(x,y) = |x-y|^n$, μ the Lebesgue measure and $K(x,y) = K(y,x)$, from Theorem 3.1 we obtain a generalization of Triebel's theorem (see [4]). The hypothesis of reflexivity on the Banach spaces assumed by Triebel is unnecessary. In particular Theorem 3.1 can be applied to singular integral operators induced by kernels of type $K(x,y) = |x-y|^{-n+\beta}$, $\beta > 0$, i.e., the so-called singular integral operators of potential type.

Corollary 1: *Under the hypotheses of Theorem* 3.1 *we have*

$$\Big\|\Big(\sum_j \|Tf_j\|_F^r\Big)^{1/r}\Big\|_{L^q} \le C\Big\|\Big(\sum_j \|f_j\|_E^r\Big)^{1/r}\Big\|_{L^p} \tag{3.8}$$

for

$$1/p - 1/q = 1 - 1/\gamma, \text{ with } \begin{cases} 1 < p \le s_0 \text{ and } \gamma < q, r \le r_0 \text{ if } K \text{ satisfies } (C_\gamma) \\ s_0 < p \le \gamma' \text{ and } r_0 < q, r < \infty \text{ if } K \text{ satisfies } (C'_\gamma). \end{cases}$$

Proof: We shall apply Theorem 3.1 twice. We have that T is bounded from $L^s(X,E)$ into $L^r(X,F)$ for r and s such that

$$1/s - 1/r = 1 - 1/\gamma, \text{ with } \begin{cases} 1 < s \le s_0 \text{ and } \gamma < r < r_0 \text{ if } K \text{ satisfies } (C_\gamma) \\ s_0 < s < \gamma' \text{ and } r_0 < r < \infty \text{ if } K \text{ satisfies } (C'_\gamma). \end{cases}$$

Let us consider the operator $\widetilde{T}$ on $L_c^\infty(X, l^r(E))$ given by $\widetilde{T}\{f_j\} = \{Tf_j\}$, and the kernel $\widetilde{K}(x,y)$ given by $\widetilde{K}(x,y)\{\lambda_j\} = \{K(x,y)\lambda_j\}$. Since

$$\|\widetilde{K}(x,y)\|_{L(l^r(E), l^r(F))} \le \|K(x,y)\|_{L(E,F)}$$

we see that $\widetilde{K}$ satisfies (C_γ) and (C'_γ) whenever K satisfies (C_γ) and (C'_γ), respectively. Moreover,

$$\widetilde{T}\{f_j\}(x) = \int_X \widetilde{K}(x,y)\{f_j(y)\}\, d\mu(y).$$

By considering the counting measure ν in $\mathbb{N}$, Fubini's theorem first and then Theorem 3.1 yields that

$$\big\|\widetilde{T}\{f_j\}\big\|_{L^r_{l^r(F)}} = \big\|\{Tf_j\}_j\big\|_{L^r_{l^r(F)}} = \big\|\{\|Tf_j\|_{L^r_F}\}_j\big\|_{l^r} \le C\big\|\{\|f_j\|_{L^s_E}\}_j\big\|_{l^r}. \tag{3.9}$$

Since $1/s - 1/r = 1 - 1/\gamma > 0$, we have $r > s$ or $r/s > 1$. Hence Minkowski's inequality assures that

$$\big\|\{\|f_j\|_{L^s_E}\}_j\big\|_{L^r} = \Big(\int_{\mathbb{N}}\Big(\int_X \|f_j\|_E^s\, d\mu\Big)^{r/s} d\nu\Big)^{1/r} \le \Big(\int_X\Big(\int_{\mathbb{N}} \|f_j\|_E^r\, d\nu\Big)^{s/r} d\mu\Big)^{1/s}.$$

From (3.9) we get $\big\|\widetilde{T}\{f_j\}\big\|_{L^r_{l^r(F)}} \le C\big\|\{f_j\}\big\|_{L^s_{l^r(E)}}$, and hence $\widetilde{T}$ is bounded from $L^s(X, l^r(E))$ into $L^r(X, l^r(F))$. Consequently, we are in the conditions of Theorem 3.1 with E and F replaced by $l^r(E)$ and $l^r(F)$, respectively. Hence, the assertion follows ∎

Corollary 2: *Let* $\{T_j\}$ *be a sequence of singular integral operators, induced by a sequence of kernels* $\{K_j\}$ *uniformly bounded from* $L^{s_0}(X,E)$ *into* $L^{r_0}(X,F)$ *for some* $s_0, r_0 \in \mathbb{R}$ *such that* $1/s_0 - 1/r_0 = 1 - 1/\gamma$, $1 \le \gamma < \infty$. *Suppose further that the sequence* $\{K_j\}$ *of associated kernels satisfies*

$$\int_{d(x,x_0) > 2d(y,x_0)} \sup_j \|K_j(x,y) - K_j(x,x_0)\|_{L(E,F)}^\gamma\, d\mu(x) \le C \tag{3.10}$$

$$\int_{d(y,y_0)>2d(x,y_0)} \sup_j \|K_j(x,y) - K_j(y_0,y)\|^{\gamma}_{L(E,F)}\, d\mu(y) \le C. \tag{3.11}$$

Then we have

$$\Big\|\Big(\sum_j \|T_j f_j\|_F^r\Big)^{1/r}\Big\|_{L^q} \le C\Big\|\Big(\sum_j \|f_j\|_E^r\Big)^{1/r}\Big\|_{L^p}$$

for

$$1/p - 1/q = 1 - 1/\gamma, \text{ with } \begin{cases} 1 < p \le s_0 \text{ and } \gamma < q, r \le r_0 \text{ if } K_j \text{ satisfies } (3.10) \\ s_0 < p \le \gamma' \text{ and } r_0 < q, r < \infty \text{ if } K_j \text{ satisfies } (3.11). \end{cases}$$

Proof: Firstly, we observe that the operators T_j are uniformly bounded from $L^s(X,E)$ into $L^r(X,F)$ with s and r given in Corollary 1. Let us consider the operator T on $L^\infty_c(X, l^r(E))$ given by $T\{f_j\} = \{T_j f_j\}$ and the kernel $K \in L(l^r(E), l^r(F))$ given by $K(x,y)\{b_j\} = \{K_j(x,y)b_j\}$. Since

$$\|K(x,y)\|_{L(l^r(E), l^r(F))} = \sup_j \|K_j(x,y)\|_{L(E,F)}$$

we see that the kernel K satisfies (C_γ) and (C'_γ) with $a = 2$. Moreover,

$$T\{f_j\}(x) = \int_X K(x,y)\{f_j\}(y)\, d\mu(y), \quad x \notin \operatorname{supp} f_j.$$

Arguing as in Corollary 1, we have $\big\|T\{f_j\}\big\|_{L^r_{l^r(F)}} \le C\big\|\{f_j\}\big\|_{L^s_{l^r(E)}}$, that is, T is bounded from the space $L^s(X, l^r(E))$ into $L^r(X, l^r(F))$. Thus, Theorem 3.1 applies giving the desired result ∎

Some related results, for $X = \mathbb{R}^n$, can be found in F.J. Ruiz and J.L. Torrea [14].

4. Maximal operators of F. Zó's type

Let $\{\varphi_\nu\}_{\nu>0}$ be a family of scalar-valued functions on $X \times X$ which satisfies for some $\gamma \in \mathbb{R}$, $1 \le \gamma < \infty$, the conditions

$$\int_{d(x,y')>2d(y',y)} \sup_{\nu>0} |\varphi_\nu(x,y') - \varphi_\nu(x,y'')|^{\gamma}\, d\mu(x) \le C \tag{4.1}$$

$$\int_{d(x'',y)>2d(x',x'')} \sup_{\nu>0} |\varphi_\nu(x',y) - \varphi_\nu(x'',y)|^{\gamma}\, d\mu(y) \le C \tag{4.2}$$

$$\int_X |\varphi_\nu(x,y)|^{\gamma}\, d\mu(y) \le C \tag{4.3}$$

for all $x \in X$ and $\nu > 0$. We define the operator Φ_ν by

$$\Phi_\nu f(x) = \int_X \varphi_\nu(x,y) f(y)\, d\mu(y) \tag{4.4}$$

and observe that φ_ν satisfies (C_γ) and (C'_γ) from (4.1) and (4.2), respectively.

Theorem 4.1: *Let $\{\Phi_\nu\}_{\nu>0}$ be a family of operators given by (4.4) which satisfy (4.1) - (4.3) for a fixed $\gamma \ge 1$. If $\Phi^* f$ is the maximal operator given by*

$$\Phi^* f(x) = \sup_{\nu>0} |\Phi_\nu f(x)|$$

we have, for $1/p - 1/q = 1 - 1/\gamma$ and $\gamma < r, q \le \infty$,

(i) $\|\Phi^* f\|_{L^q} \le C\|f\|_{L^p}$ *and* (ii) $\|\{\Phi^* f_j\}_j\|_{L^q_{l^r}} \le C\|\{f_j\}_j\|_{L^p_{l^r}}$.

Proof: *Step* 1. Let $\mathbb{Q}_+$ be the set of positive rational numbers increasingly ordered. We have $\Phi^* f(x) = \sup_{\nu \in \mathbb{Q}_+} |\Phi_\nu f(x)|$. Also, if we set $\mathbb{Q}_n = \mathbb{Q}_+ \cap [0, \nu_n]$, we see that

$$\Phi_n^* f(x) = \sup_{\nu \in \mathbb{Q}_n} |\Phi_\nu f(x)| \text{ increases to } \Phi^* f(x) \text{ as } n \to \infty.$$

Next, let us denote by $l_n^\infty = l^\infty(\mathbb{Q}_n)$ the set of all n-tuples $(a_{\nu_1}, \dots, a_{\nu_n}) \in \mathbb{R}^n$. We can look at $\Phi_n^* f$ as an l_n^∞-valued operator. In fact, if we set $T_n f = (\Phi_{\nu_1} f, \dots, \Phi_{\nu_n} f)$ we have $\|T_n f\|_{l_n^\infty} = \Phi_n^* f$, and consequently $\|T_n f\|_{L^\infty(l_n^\infty)} \le C \|f\|_{L^{\gamma'}}$, where $1/\gamma + 1/\gamma' = 1$.

Step 2. Let us consider the kernel

$$K_n : \mathbb{C} \to l_n^\infty \text{ defined by } K_n(x,y)\lambda = (\varphi_{\nu_1}(x,y)\lambda, \dots, \varphi_{\nu_n}(x,y)\lambda).$$

We see that $K_n \in L(\mathbb{C}, l_n^\infty)$ is locally integrable and satisfies (C_γ) and (C_γ') with $a = 2$. Moreover we have

$$T_n f(x) = \int_X K_n(x,y) f(y)\, d\mu(y).$$

Therefore, all the hypotheses of Theorem 3.1 are satisfied and consequently, T_n is a bounded operator from $L^p(X, \mathbb{R})$ into $L^q(X, l_n^\infty)$, i.e. $\|\Phi_n^* f\|_{L^q} \le C \|f\|_{L^p}$ for $p, q \in \mathbb{R}$ satisfying $1/p - 1/q = 1 - 1/\gamma$ and $\gamma < q \le \infty$. Now an application of the monotone convergence theorem yields the desired inequality (i). The sequential inequality (ii) follows from Corollary 1 of Theorem 3.1 ∎

5. A maximal inequality of Hardy-Littlewood type

In this section we will derive fractionary maximal inequalities of Hardy-Littlewood and Fefferman-Stein type, but also a version of an L^p-L^q inequality for a function of Marcinkiewicz type $J_{r,\varepsilon}$.

Proposition 5.1: *Let* (X, d, μ) *be a* ρ*-normal homogeneous space,* φ *a* C^∞*-function on* $[0, \infty)$ *such that* $0 \le \varphi(t) \le 1$, $\varphi(t) = 1$ *for* $0 \le t \le 1/2$ *and* $\varphi(t) = 0$ *for* $t > 1$, *and let*,

$$\varphi_\nu(x,y) = \rho(\nu)^{1/\gamma} \varphi(\nu^{-1} d(x,y)),\ 1 \le \gamma < \infty,\ \text{for } 0 < \nu < \infty.$$

Then there is a positive constant C such that

$$\int_X |\varphi_\nu(x,y)|^\gamma d\mu(y) \le C$$

for all $x \in X$. *Moreover, if* $d(y, x'') > 2\, d(x', x'')$, *then*

$$|\varphi_\nu(x', y) - \varphi_\nu(x'', y)| \le C \frac{d(x', x'')^\alpha}{d(y, x'')^\alpha \mu(B(x'', d(y, x'')))^{1/\gamma}}$$

for some α *with* $0 < \alpha < 1$.

Proof: *Step* 1. *Since* $\varphi_\nu(x,y) = 0$ if $d(x,y) > \nu$, we have

$$\int_X |\varphi_\nu(x,y)|^\gamma d\mu(y) = \int_{d(x,y) \le \nu} |\varphi_\nu(x,y)|^\gamma d\mu(y)$$

$$= \int_{d(x,y) \le \nu} \rho(\nu) \Big|\varphi\Big(\frac{d(x,y)}{\nu}\Big)\Big|^\gamma d\mu(y) \le C \rho(\nu) \int_{d(x,y) \le \nu} d\mu(y) = C.$$

Step 2. There is a constant C such that the quasi-distance d satisfies $|d(x',y) - d(x'',y)| \le C\nu^{1-\alpha}d(x',x'')^{\alpha}$ for some α, $0 < \alpha < 1$, whenever $\max(d(x',y), d(x'',y)) < \nu$ (see R.A. Macias and C. Segovia [11: Theorem 2]). Hence

$$|\varphi_\nu(x',y) - \varphi_\nu(x'',y)| = \rho(\nu)^{1/\gamma}\left|\varphi\left(\frac{d(x',y)}{\nu}\right) - \varphi\left(\frac{d(x'',y)}{\nu}\right)\right| \le C\rho(\nu)^{1/\gamma}\nu^{-1}|d(x',y) - d(x'',y)|$$

$$\le C\rho(\nu)^{1/\gamma}\nu^{-1}d(x',x'')^{\alpha} \le C\frac{d(x',x'')^{\alpha}}{d(x'',y)^{\alpha}\mu(B(x'',y)))^{1/\gamma}}.$$

On the other hand, if $\min(d(x',y), d(x'',y)) > \nu$, then $|\varphi_\nu(x',y) - \varphi_\nu(x'',y)| = 0$. If, say, $d(x',y) > 2k\nu$ and $d(x'',y) \le \nu$, then

$$d(x',y) \le k(d(x',x'') + d(x'',y)) \le k(d(x',x'') + \nu) \quad \text{and} \quad d(x',x'') \ge k^{-1}d(x',y) - \nu > \nu$$

and thus

$$|\varphi_\nu(x',y) - \varphi_\nu(x'',y)| = |\varphi_\nu(x'',y)| \le \rho(\nu)^{1/\gamma}\frac{d(x',x'')^{\alpha}}{\nu^{\alpha}} \le \frac{d(x',x'')^{\alpha}}{d(x'',y)^{\alpha}\mu(B(x'',d(x'',y)))^{1/\gamma}}.$$

This complete the proof ∎

Corollary: *Let φ be given as in Proposition* 5.1. *Then the family $\{\varphi_\nu\}_{\nu>0}$ satisfies the inequalities* (4.1)-(4.3).

Theorem 5.2: *For $f \in L^1_{loc}(X,\mathbb{R})$ and $0 < \varepsilon \le 1$, let $M_\varepsilon f$ be the fractionary maximal operator given by*

$$M_\varepsilon f(x) = \sup_{x\in B}\frac{1}{\mu(B)^{\varepsilon}}\int_B |f(y)|\,d\mu(y).$$

Then, if $1/p - 1/q = 1 - \varepsilon$ and $1/\varepsilon < q \le \infty$, we have

$$\|M_\varepsilon f\|_{L^q} \le C\|f\|_{L^p}. \tag{5.1}$$

Moreover, for r satisfying $\gamma < r \le \infty$ we have

$$\|\{M_\varepsilon f_j\}\|_{L^q_{l^r}} \le C\|\{f_j\}\|_{L^p_{l^r}}. \tag{5.2}$$

Proof: Let φ be a C^∞-function as in Proposition 5.1. Then

$$\rho(\nu)^{1/\gamma}\chi_{B(x,\nu)}(y) = \rho(\nu)^{1/\gamma}\chi_{[0,1]}(\nu^{-1}d(x,y)) \le \varphi_\nu(x,y)$$

where γ is also given as in Proposition 5.1. Since $0 < C < \rho(\nu)\mu(B(x,\nu))$, we have

$$M_\varepsilon f(x) \le C\sup_\nu\int\varphi_\nu(x,y)|f(y)|\,d\mu(y)$$

with $\varepsilon = 1/\gamma$. Now, we see that (5.1) follows at once from (4.5). From (4.6) we obtain (5.2) ∎

Corollary: *Let X be a normalized homogeneous space, $\{B_j\}_{j\in\mathbb{N}}$ a sequence of disjoint balls in X with B_j centerd at x_j, and*

$$J_{r,\varepsilon}(x) = \left(\sum_{j=1}^{\infty}\left(\frac{\mu(B_j)}{d(x,x_j)^{\varepsilon} + \mu(B_j)^{\varepsilon}}\right)^r\right)^{1/r}$$

a function of Marcinkiewicz type, where $0 < \varepsilon \le 1$ and $1/\varepsilon < r \le \infty$. Then there exists a finite *constant C such that*

$$\|J_{r,\varepsilon}\|_{L^q} \le C\left(\sum_{j=1}^{\infty}\mu(B_j)^p\right)^{1/p} \text{ with } 1/p - 1/q = 1 - \varepsilon.$$

Proof: We have

$$M_\varepsilon\chi_{B_j}(x) \ge \mu(B)^{-\varepsilon}\int_B \chi_{B_j}(y)\,d\mu(y) \text{ for some ball } B = B(x,R),$$

with $R = 2k\,d(x,x_j)$, where k is the constant of homogeneity of X. Therefore,

$$M_\varepsilon\chi_{B_j}(x) \ge C\mu(B_j)/\left(d(x,x_j)^\varepsilon + \mu(B_j)^\varepsilon\right)$$

since $B_j \subset B$ and X is a normalized homogeneous space. The result now follows from Theorem 5.2, taking $f_j = \chi_{B_j}$, $j \in \mathbf{N}$ ∎

We remark that, if we take the normalized homogeneous space $(\mathbb{R}^n, d, \mu)$, where $d(x,y) = |x-y|^n$ and μ is the Lebesgue measure, we have for $\varepsilon = 1 - \gamma/n$ a result due to H.P. Heinig and R. Johnson (see [7]) for a Marcinkiewicz-type function.

6. A theorem of Littlewood-Paley type

Taking $s_0 = r_0 = 2$ and $\gamma = 1$ in Theorem 3.1 we obtain the following result of Littlewood-Paley type.

Theorem 6.1: *Let X be a ρ-normal homogeneous space and $\{\psi_j\}_{j\in\mathbf{Z}}$ a sequence of functions on $X \times X$ such that*

$$\int_X \psi_j(x,y)\,d\mu(x) = \int_X \psi_j(x,y)\,d\mu(y) = 0$$

$$|\psi_j(x,y)| \le \rho(2^{-j}) \text{ and } \psi_j(x,y) = 0 \text{ whenever } d(x,y) \ge 2^j$$

$$|\psi_j(x,y') - \psi_j(x,y'')| \le \rho(2^{-j})\left(2^{-j}d(y',y'')\right)^\alpha,\ \alpha > 0$$

$$|\psi_j(x',y) - \psi_j(x'',y)| \le \rho(2^{-j})\left(2^{-j}d(x',x'')\right)^\beta,\ \beta > 0,$$

and let us set

$$f_j(x) = \int_X \psi_j(x,y)f(y)\,d\mu(y) \quad \text{and} \quad g(f)(x) = \left(\sum_{j=-\infty}^{\infty}|f_j(x)|^2\right)^{1/2}.$$

Then, for all p with $1 < p < \infty$, we have

$$\|g(f)\|_{L^p} \le C\|f\|_{L^p},\quad f \in L^p(X,\mathbf{R}).$$

Proof: *Step* 1. For each positive integer N, let us consider the mapping

$\tilde{g}_N(f)\colon X \to l^2(\mathbf{Z})$ given by $\tilde{g}_N(f)(x) = \{f_j(x)\chi_N(j)\}_{j\in\mathbf{Z}}$

where χ_N is the characteristic function of $[-N,N]$. Then

$$\|\tilde{g}_N(f)(x)\|_{l^2} = \left(\sum_{j=-N}^{N}|f_j(x)|^2\right)^{1/2}.$$

On the other hand, let us consider the kernel

$$K_N \text{ given by } K_N(x,y)\lambda = \{\psi_j(x,y)\chi_N(j)\lambda\}_{j\in\mathbb{Z}}.$$

We have $K_N \in L(\mathbb{C}, l^2) \approx l^2$ and

$$\left\{\int_X \psi_j(x,y)\chi_N(j)f(y)\,d\mu(y)\right\} = \int_X \{\psi_j(x,y)\chi_N(j)\}\,d\mu(y) = \int_X K_N(x,y)f(y)\,d\mu(y).$$

Moreover, if A is a compact subset of $X\times X$, we also have

$$\iint_A \|K_N(x,y)\|_{L(\mathbb{C},l^2)}\,d\mu\otimes\mu(x,y) \le \mu\otimes\mu(A)\Big(\sum_{j=-N}^{N}|\rho(2^{-j})^2\Big)^{1/2},$$

i.e., $K_N \in L^1_{loc}(X\times X, L(\mathbb{C}, l^2))$.

Step 2. In [1: pp. 119-120] it was proved that the operator

$$f \to \sum_{j=-N}^{N} f_j = \sum_{j=-N}^{N}\int_X \psi_j(x,y)f(y)\,d\mu(y)$$

is bounded in $L^2(X)$. Hence

$$\|\tilde{g}_N(f)\|_{L^2(X,l^2)} = \Big\|\Big(\sum_{j=-N}^{N}|f_j|^2\Big)^{1/2}\Big\|_{L^2} \le \Big\|\sum_{j=-N}^{N}\int_X \psi_j(x,y)|f(y)|\,d\mu(y)\Big\|_{L^2} \le C\|f\|_{L^2}.$$

Step 3. From Step 1 and Step e we see that $\tilde{g}_N$ is a singular integral with an operator-valued kernel K_N. To see that $\tilde{g}_N$ maps $L^p(X,\mathbb{R})$ into $L^p(X,l^2)$, $1 < p < \infty$, we have only to check the conditions (C_1) and (C'_1) for the kernel K_N. But from [1: p. 125] we have

$$\int_{d(x,y')>2kd(y',y'')} |\psi_j(x,y') - \psi_j(x,y'')|\,d\mu(x) \le C\min\{(2^{-j}d(y',y''))^{\alpha}, (2^{-j}d(y',y''))^{-\alpha}\}$$
$$\int_{d(x',y)>2kd(x',x'')} |\psi_j(x',y) - \psi_j(x'',y)|\,d\mu(y) \le C\min\{(2^{-j}d(x',x''))^{\beta}, (2^{-j}d(x',x''))^{-\beta}\}$$

where k is the constant of homogeneity of X. Consequently we have

$$\int_{d(x,y')>2kd(y',y'')} \|K_N(x,y') - K_N(x,y'')\|_{L(\mathbb{C},l^2)}\,d\mu(x)$$
$$= \int_{d(x,y')>2kd(y',y'')}\Big(\sum_{j=-N}^{N}|\psi_j(x,y') - \psi_j(x,y'')|^2\Big)^{1/2} d\mu(x)$$
$$\le \sum_{j=-N}^{N}\int_{d(x,y')>2kd(y',y'')} |\psi_j(x,y') - \psi_j(x,y'')|\,d\mu(x)$$
$$\le C\sum_{j=-N}^{N}\min\{(2^{-j}d(y',y''))^{\alpha}, (2^{-j}d(y',y''))^{-\alpha}\} \le C$$

with C independent of N and $d(x',x'')$, i.e., K_N satisfies (C_1). Analogously, we see that K_N is subject to (C'_1).

Step 4. From Theorem 3.1, with $s_0 = r_0 = 2$, $a = 2k$ and $\gamma = 1$, we get

$$\|\tilde{g}_N(f)\|_{L^p(X,l^2)} = \Big\|\Big(\sum_{j=-N}^{N}|f_j|^2\Big)^{1/2}\Big\|_{L^p} \le C\|f\|_{L^p}.$$

The monotone convergence theorem yields now the desired result ∎

Finally, we observe that Theorem 6.1 generalizes slightly Theorem 5.2.1 of J. Aguirre [1], but the proof given is based on the theory of sungular integral operators with operator-valued kernels.

REFERENCES

[1] AGUIRRE, J.: *Multilinear Pseudo-Differential Operators and Paraproducts.* Thesis. St. Louis (Mo): Washington Univ. 1981.

[2] BENEDEK, A., CALDERÓN, A. P., and R. PANZONE: *Convoluton operators on Banach space valued functions.* Proc. Nat. Acad. Sci. USA **48** (1962), 356 - 365.

[3] CALDERÓN, A. P.: *Inequalities for the maximal function relative to a metric.* Studia Math. **57** (1976), 297 - 306.

[4] COIFMAN, R. R., and G. WEISS: *Analyse Harmonique Non-Commutative sur Certains Spaces Homogenes.* Lect. Notes Math. 242 (1971), 1 158.

[5] COIFMAN, R. R., and G. WEISS: *Extensions of Hardy spaces and their uses in analysis.* Bull. Amer. Math. Soc. **83** (1977), 569 - 645.

[6] GARCIA–CUERVA, J., and J. L. RUBIO DE FRANCIA: *Weighted Norm Inequalities and Related Topics.* Amsterdam: North-Holland Publ. Comp. 1985.

[7] HEINIG, H. P., and R. JOHNSON: *Weighted norm inequalities for L^r-valued integral operators and applications.* Math. Nachr. **107** (1982), 161 - 174.

[8] HÖRMANDER, L.: *Estimates for translation invariant operators in L^p spaces.* Acta Math. **104** (1960), 93 - 139.

[9] KORANYI, A., and S. VAGI: *Singular integrals on homogeneous spaces and some problems of classical analysis.* Ann. Scuola Norm. Sup. Pisa **25** (1971), 575 - 647.

[10] MACIAS, R. A.: *Interpolation Theorem on Generalized Hardy Spaces.* Ph. D. Thesis. St. Louis (Mo.): Washington Univ. 1975.

[11] MACIAS, R. A., and C. SEGOVIA: *Lipschitz functions on spaces of homogeneous type.* Adv. Math. **33** (1979), 257 - 270.

[12] RUBIO DE FRANCIA, J. L., RUIZ, F. J., and J. L. TORREA: *Calderón-Zygmund theory for operator-valued kernels.* Adv. Math. **62** (1982), 7 - 48.

[13] RUIZ, F. J., and J. L. TORREA: *Vector-valued Calderón-Zygmund theory and Carleson measures on spaces of homogeneous nature.* Studia Math. **88** (1988), 221 - 243.

[14] RUIZ, F. J., and J. L. TORREA: *Weighted and vector-valued inequalities for potential operators.* Trans. Am. Math. Soc. **295** (1986), 213 - 232.

[15] SCHWARTZ, J.: *A remark on inequalities of Calderón-Zygmund type for vector-valued functions.* Comm. Pure Appl. Math. **14** (1961), 785 - 799.

[16] TRIEBEL, H.: *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation.* Ark. Math. **11** (1973), 13 - 64.

Received 30.05.1991

Prof. Dr. Benjamin Bordin and Prof. Dr. Dicesar Lass Fernandez
Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
Caixa Postal 6065
13081 - Campinas SP, Brasil