Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11(1992)2, 173 - 182

Asymptotics of the Solution of a Boundary Integral Equation Under a Small Perturbation of a Corner

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The boundary integral equation of the Dirichlet problem is considered in a plane domain with a smooth boundary which is a small perturbation of a contour with an angular point. The asymptotics of the solution are given with respect to a perturbation parameter ϵ. The problem studied in this article serves as an example of the use of a general method which is also applicable to the three-dimensional case, to the Neumann problem and to problems of hydrostatics and elasticity.

Key words: *Boundary integral equations, small perturbations*

AMS subject classification: 35J05, 45M05, 31B10, 35B40

1. Introduction

In the present article we consider the boundary integral equation of the Dirichlet problem in a plane domain if the boundary is smoothed near an angular point. The main terms of the asymptotics of the solution are given with respect to a perturbation parameter ϵ.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with an angular point on its contour $\partial\Omega$. A domain Ω_ϵ with smooth boundary is obtained by a small perturbation of $\partial\Omega$ near the vertex of the angle. Let ψ be a smooth function in $\mathbf{R}^2$. It is the classical approach to solve the Dirichlet problem

$$\triangle u_\epsilon = 0 \quad \text{in } \Omega_\epsilon, \quad u_\epsilon = \psi|_{\partial\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon \tag{1}$$

by expressing the function u_ϵ in the form of a double-layer potential

$$u_\epsilon(x) = \frac{1}{2\pi} \int\limits_{\partial\Omega_\epsilon} \mu_\epsilon(y) \frac{\partial}{\partial\nu} \log|x-y|\, ds_y, \tag{2}$$

where ν denotes the outward unit normal vector to $\partial\Omega_\epsilon$. This leads to the well-known boundary integral equation

$$\frac{1}{2}\mu_\epsilon + T\mu_\epsilon = \psi. \tag{3}$$

The operator T is the direct value of the double-layer potential (2). The density function μ_ϵ is expressed in terms of solutions of auxiliary boundary value problems, using a method which is described in [3]. The following representations are the main results of this article:

$$\mu_\epsilon(x) \sim \begin{cases} \left(\psi(x) - v(x) - \epsilon^{\pi/(2\pi-\alpha)} w_+\left(\frac{x}{\epsilon}\right)\right)\Big|_{\partial\Omega_\epsilon} & \text{if } 0 < \alpha < \pi \\ \left(\psi(x) - v(x) - \epsilon^{\pi/\alpha} w_-\left(\frac{x}{\epsilon}\right)\right)\Big|_{\partial\Omega_\epsilon} & \text{if } \pi \le \alpha < 2\pi, \end{cases}$$

where α denotes the size of the angle. The functions v, w_+ and w_- are solutions of certain exterior Neumann problems. The remainder function can be estimated uniformly in the norm of $C(\partial\Omega_\epsilon)$ by $\mathcal{O}(\epsilon^\gamma)$ with $\gamma > 1$, which is proved in Section 3.

The considered problem serves as an example for the use of a general method which is also applicable to the three-dimensional case, to the Neumann problem and to problems of hydrodynamics and elasticity. Moreover, the method can be used to obtain complete asymptotic series for solutions of boundary integral equations.

2. Formal asymptotics

We suppose that the origin O belongs to the boundary $\partial\Omega$ of the domain Ω and that $\partial\Omega \setminus \{O\}$ is smooth. In a neighbourhood of the origin Ω coincides with the wedge

$$K = \{x = (r, \varphi) \mid r > 0,\ \varphi \in (0, \alpha)\} \quad (0 < \alpha < 2\pi).$$

Let ω be a domain with smooth boundary, which coincides with $K \setminus B_1(0)$ outside the unit circle $B_1(0)$. For the sake of simplicity it is assumed $\Omega \subset K$, $\omega \subset K$ (see Remark 1 below).

Now, domains ω_ϵ and Ω_ϵ are introduced, which depend on a small positive parameter ϵ:

$$\omega_\epsilon = \left\{ x \,\middle|\, \frac{x}{\epsilon} \in \omega \right\}, \quad \Omega_\epsilon = \Omega \cap \omega_\epsilon.$$

Later, several subsets of these domains and their boundaries $\partial\omega_\epsilon$ and $\partial\Omega_\epsilon$ will be considered. For that purpose the following notation is used:

$$\begin{array}{rclrcl} D_\epsilon & = & \Omega \setminus \overline{\Omega}_\epsilon, & \gamma_\epsilon & = & \partial\Omega_\epsilon \setminus \partial\Omega, \\ D_1 & = & K \setminus \overline{\omega}, & \gamma_1 & = & \partial\omega \setminus \partial K, \\ D_2 & = & K \setminus \overline{\Omega}, & \gamma_2 & = & \partial\Omega_\epsilon \setminus \partial\omega_\epsilon. \end{array}$$

Clearly, $D_\epsilon \subset B_\epsilon(0)$.

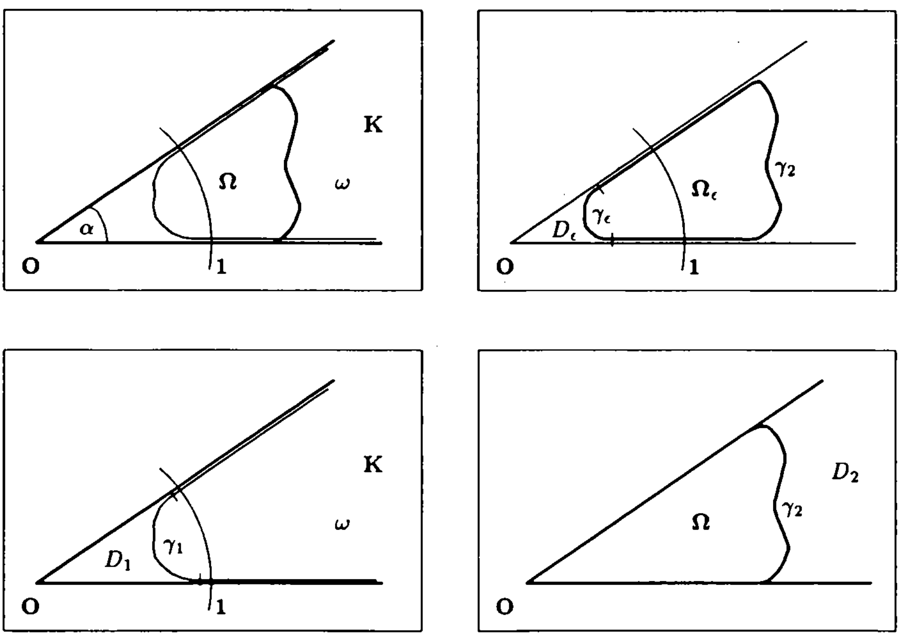

Figure 1: Example with $\epsilon = 0.6$ and $\alpha = 35°$.

Complementary domains with respect to $\mathbf{R}^2$ are denoted by the superscript c. A Neumann problem which is related to the Dirichlet problem (1) is considered:

$$\Delta v_\epsilon = 0 \quad \text{in } \Omega_\epsilon^c, \quad \frac{\partial v_\epsilon}{\partial \nu} = \frac{\partial u_\epsilon}{\partial \nu} \quad \text{on } \partial\Omega_\epsilon. \tag{4}$$

We look for the solution of this and all following Neumann problems in the class of functions with behaviour $o(1)$ at infinity. A solution exists, if a solvability condition is satisfied. For problem (4) the condition

$$\int_{\partial\Omega_\epsilon} \frac{\partial u_\epsilon}{\partial \nu}\, ds = 0$$

is fulfilled because of the harmonicity of u_ϵ in Ω_ϵ and Green's formula.

We follow the approach of [6], which goes back to one of the authors. The representation formulae for harmonic functions,

$$u_\epsilon(x) = \frac{1}{2\pi} \int_{\partial\Omega_\epsilon} \left(u_\epsilon(y) \frac{\partial}{\partial \nu_y} \log|x-y| - \frac{\partial u_\epsilon}{\partial \nu_y}(y) \log|x-y| \right) ds_y$$

for $x \in \Omega_\epsilon$ and

$$v_\epsilon(x) = -\frac{1}{2\pi} \int_{\partial\Omega_\epsilon} \left(v_\epsilon(y) \frac{\partial}{\partial \nu_y} \log|x-y| - \frac{\partial v_\epsilon}{\partial \nu_y}(y) \log|x-y| \right) ds_y$$

for $x \in \Omega_\epsilon^c$, yield on the boundary $\partial\Omega_\epsilon$

$$\frac{1}{2}\psi = T\psi - \frac{1}{2\pi} \int_{\partial\Omega_\epsilon} \frac{\partial u_\epsilon}{\partial \nu} \log|x-y|\, ds_y$$

and

$$\frac{1}{2}v_\epsilon = -Tv_\epsilon + \frac{1}{2\pi} \int_{\partial\Omega_\epsilon} \frac{\partial v_\epsilon}{\partial \nu} \log|x-y|\, ds_y \tag{5}$$

taking into account the jump conditions of the double-layer potential. Equation (5) and the boundary condition of (4) lead to

$$\frac{1}{2}(\psi - v_\epsilon) + T(\psi - v_\epsilon) = \psi \quad \text{on } \partial\Omega_\epsilon,$$

which shows that $(\psi - v_\epsilon)|_{\partial\Omega_\epsilon}$ is a solution of the boundary integral equation (3). Since (3) is uniquely solvable, we can represent its solution by

$$\mu_\epsilon = (\psi - v_\epsilon)|_{\partial\Omega_\epsilon}. \tag{6}$$

In order to find asymptotics of μ_ϵ, the following method (see [3]) is applied to the problems (1) and (4):

An approximation for the solution of a boundary value problem in Ω_ϵ is obtained by solving an analogous problem in the limit domain Ω. This leads to an error concentrated near the origin. The asymptotics of the error function are determined and by the transformation $\xi = \frac{x}{\epsilon}$ the scale is changed. Then, a second auxiliary boundary value problem is solved in the unbounded domain ω. The solution equalizes the main term in the asymptotics of the error function, if it is multiplied by ϵ to a certain power.

Using this method a representation for the solution v_ϵ of problem (4) is obtained:

$$v_\epsilon(x) = v(x) + \epsilon^\tau w\left(\frac{x}{\epsilon}\right) + R(x). \tag{7}$$

The Dirichlet solution of (1) appears in (4) on the right-hand side. Therefore, we have to look for its asymptotics according to the same scheme:

$$u_\epsilon(x) = V(x) + \epsilon^\beta W\Big(\frac{x}{\epsilon}\Big) + R_1(x). \tag{8}$$

We start with the treatment of the Dirichlet problem (1). The Taylor expansion is valid for ψ:

$$\psi(x) = \psi(0) + x\nabla\psi(0) + \mathcal{O}\left(r^2\right).$$

A first approximation for the solution u_ϵ of (1) is obtained by solving the following Dirichlet problem in Ω:

$$\Delta V = 0 \quad \text{in } \Omega, \quad V = \psi|_{\partial\Omega} \quad \text{on } \partial\Omega. \tag{9}$$

For the study of the asymptotic behaviour of V near O, it is suitable to consider 3 cases:

Case 1: $0 < \alpha < \pi$

The asymptotics of V contain terms with integer exponents of r ($r = |x|$) caused by the function ψ and solutions of the homogeneous Dirichlet problem for a wedge:

$$r^{j\lambda} a_j \sin j\lambda\varphi \quad \text{with } \lambda = \pi/\alpha \quad (j = 1, 2, \ldots),$$

where a_j are certain constants. We have

$$V(x) = \psi(0) + x\nabla\psi(0) + f_1(r,\varphi) \tag{10}$$

with $f_1 = \mathcal{O}\left(r^{\min\{\lambda,2\}}\right)$ near O. The remainder function $R_0 = u_\epsilon - V$ is harmonic in Ω_ϵ and has the boundary values

$$R_0 = \begin{cases} (\psi - V)|_{\gamma_\epsilon} & \text{on } \gamma_\epsilon \\ 0 & \text{on } \partial\Omega_\epsilon \setminus \gamma_\epsilon. \end{cases} \tag{11}$$

Hence, the main term in the asymptotics of the right-hand side has the order $r^{\min\{\lambda,2\}}$. This term has to be compensated by the solution W of a Dirichlet problem in ω. As it will turn out later, this function W does not influence the first terms in the asymptotics of v_ϵ in this case, since $\beta > 1$.

In order to solve the exterior Neumann problem (4), we consider the solution of the following problem:

$$\Delta v = 0 \quad \text{in } \Omega^c, \quad \frac{\partial v}{\partial \nu} = \frac{\partial V}{\partial \nu} \quad \text{on } \partial\Omega. \tag{12}$$

The solvability condition $\int\limits_{\partial\Omega} \frac{\partial V}{\partial\nu}\,ds = 0$ is satisfied, since V is harmonic in Ω. The asymptotics of v near O contain terms caused by the right-hand side $\frac{\partial V}{\partial\nu}$ and solutions of the homogeneous exterior Neumann problem for a wedge

$$r^{j\sigma} b_j \cos j\sigma(\varphi - \alpha) \quad \text{with } \sigma = \pi/(2\pi - \alpha) \quad (j = 1, 2, \ldots),$$

where b_j are certain constants. It reads as follows:

$$v(x) = v(0) + r^\sigma b_1 \cos\sigma(\varphi - \alpha) + x\nabla\psi(0) + \mathcal{O}\left(r^{\min\{2\sigma,\lambda\}}\right).$$

The function v is not defined everywhere in Ω_ϵ^c and has to be extended within Ω by v^i. The extension is chosen in this manner:

The conditions

$$v^i = v \quad \text{and} \quad \frac{\partial v^i}{\partial\nu} = \frac{\partial v}{\partial\nu} \tag{13}$$

are satisfied on ∂K and v^i has the prescribed asymptotics near O:

$$v^i(x) = v(0) - r^\sigma b_1 \cos\lambda\varphi + x\nabla\psi(0) + f_2(r,\varphi) \tag{14}$$

with $f_2 = \mathcal{O}\left(r^{\min\{2\sigma,\lambda\}}\right)$, so that (13) is fulfilled by the main terms of the asymptotics. The remainder function $R_2 = v_\epsilon - v$ solves the problem

$$\begin{aligned} \Delta R_2 &= \begin{cases} (\sigma^2-\lambda^2)b_1 r^{\sigma-2}\cos\lambda\varphi - \Delta f_2 & \text{in } D_\epsilon \\ 0 & \text{in } \Omega_\epsilon^c \setminus D_\epsilon, \end{cases} \\ \frac{\partial R_2}{\partial\nu} &= \begin{cases} b_1\frac{\partial}{\partial\nu}(r^\sigma\cos\lambda\varphi) + \frac{\partial}{\partial\nu}(f_1-f_2) + \frac{\partial R_0}{\partial\nu} & \text{on } \gamma_\epsilon \\ 0 & \text{on } \partial\Omega_\epsilon\setminus\gamma_\epsilon. \end{cases} \end{aligned} \tag{15}$$

The main term of the error concentrated near O has to be compensated by a function w_+. By the transformation $\xi = \frac{x}{\epsilon}$ the scale is changed and the problem takes the form

$$\begin{aligned} \Delta w_+ &= \begin{cases} (\sigma^2-\lambda^2)b_1|\xi|^{\sigma-2}\cos\lambda\varphi & \text{in } D_1 \\ 0 & \text{in } \omega^c\setminus D_1, \end{cases} \\ \frac{\partial w_+}{\partial\nu} &= b_1\frac{\partial}{\partial\nu}(|\xi|^\sigma\cos\lambda\varphi) \qquad \text{on } \partial\omega. \end{aligned} \tag{16}$$

The solvability condition is satisfied and equation (7) is valid with $\tau = \sigma$. The function w_+ shows the following asymptotic behaviour at infinity:

$$w_+(\xi) = |\xi|^{-\sigma}c_1\cos\sigma(\varphi-\alpha) + \mathcal{O}\left(|\xi|^{-2\sigma}\right)$$

with a certain constant c_1. Since it is not defined for $x \in D_2$, we extend w_+ by a function w_+^i according to (13). It has the prescribed asymptotics:

$$w_+^i(\xi) = -|\xi|^{-\sigma}c_1\cos\lambda\varphi + \mathcal{O}\left(|\xi|^{-2\sigma}\right)$$

at infinity. From (15) and (16) we derive the problem for the remainder function R in equation (7):

$$\begin{aligned} \Delta R &= \begin{cases} -\Delta f_2 & \text{in } D_\epsilon \\ -\epsilon^\sigma\Delta w_+ & \text{in } D_2 \\ 0 & \text{in } \Omega_\epsilon^c\setminus\{D_\epsilon\cup D_2\}, \end{cases} \\ \frac{\partial R}{\partial\nu} &= \begin{cases} \frac{\partial}{\partial\nu}(f_1-f_2) + \frac{\partial R_0}{\partial\nu} & \text{on } \gamma_\epsilon \\ -\epsilon^\sigma\Delta w_+ + \frac{\partial R_0}{\partial\nu} & \text{on } \gamma_2 \\ \frac{\partial R_0}{\partial\nu} & \text{on } \partial\Omega_\epsilon\setminus\{\gamma_\epsilon\cup\gamma_2\}. \end{cases} \end{aligned}$$

In Section 3 it will be useful to have this problem in variational formulation. Throughout this paper test functions and their restrictions on the boundary are denoted by Φ.
Since $\frac{\partial}{\partial\nu}(f_1 - f_2) = 0$ on $\partial D_\epsilon\setminus\gamma_\epsilon$ and $\frac{\partial w_+}{\partial\nu} = 0$ on $\partial D_2\setminus\gamma_2$, Green's formula implies

$$\int_{\Omega_\epsilon^c}\nabla R\,\nabla\Phi\,dx = \int_{D_\epsilon}\nabla(f_1-f_2)\,\nabla\Phi\,dx - \epsilon^\sigma\int_{D_2}\nabla w_+\nabla\Phi\,dx - \int_{\partial\Omega_\epsilon}\frac{\partial R_0}{\partial\nu}\Phi\,ds, \tag{17}$$

with f_1 and f_2 defined in (10) and (14), respectively. The solvability condition is satisfied, since R_0 is harmonic in Ω_ϵ.

Case 2: $\pi < \alpha < 2\pi$

Since $1/2 < \lambda < 1$, the asymptotics of V read as follows:

$$V(x) = \psi(0) + r^{\lambda} a_1 \sin \lambda\varphi + x\nabla\psi(0) + f_1(r,\varphi) \tag{18}$$

with $f_1 = \mathcal{O}\left(r^{2\lambda}\right)$.

The second limit problem is set to compensate the term $r^{\lambda} a_1 \sin \lambda\varphi$ on γ_ϵ:

$$\begin{aligned} \Delta W(\xi) &= 0 && \text{in } \omega, \\ W(\xi) &= \begin{cases} -|\xi|^{\lambda} a_1 \sin \lambda\varphi & \text{on } \gamma_1 \\ 0 & \text{on } \partial\omega \setminus \gamma_1 \end{cases} \end{aligned} \tag{19}$$

and equation (8) is valid with $\beta = \lambda$. The function W shows the asymptotic behaviour $\mathcal{O}\left(|\xi|^{-\lambda}\right)$ at infinity. The solution v of (12) has the asymptotics

$$v(x) = v(0) + r^{\lambda} a_1 \frac{1}{\cos \lambda\pi} \sin \lambda(\varphi - \pi) + x\nabla\psi(0) + \mathcal{O}\left(r^{\min\{2\lambda,\sigma\}}\right)$$

near O and is extended according to (13) with the prescribed asymptotics

$$v^{i}(x) = v(0) + r^{\lambda} a_1 (\tan \lambda\pi + \sin \lambda\varphi) + x\nabla\psi(0) + f_2(r,\varphi) \tag{20}$$

with $f_2 = \mathcal{O}\left(r^{\min\{2\lambda,\sigma\}}\right)$. Compared with Case 1, the function w_- has to compensate the additional term $\frac{\partial W}{\partial \nu}$ on $\partial\omega$. The problem takes the form

$$\begin{aligned} \Delta w_- &= \begin{cases} -\lambda^2 a_1 \tan \lambda\pi |\xi|^{\lambda-2} & \text{in } D_1 \\ 0 & \text{in } \omega^c \setminus D_1, \end{cases} \\ \frac{\partial w_-}{\partial \nu} &= -a_1 \tan \lambda\pi \frac{\partial}{\partial \nu}(|\xi|^{\lambda}) + \frac{\partial W}{\partial \nu} \quad \text{on } \partial\omega. \end{aligned} \tag{21}$$

The solvability condition is satisfied, since W is harmonic in Ω_ϵ. Equation (7) is valid with $\tau = \lambda$. The function w_- shows the asymptotic behaviour $\mathcal{O}\left(|\xi|^{-\lambda}\right)$ at infinity, because it satisfies the condition $\frac{\partial w_-}{\partial \nu} = \frac{\partial W}{\partial \nu}$ on $\partial\omega$ and we have $\lambda < \sigma$. Hence, the extension w_-^{i} has the same behaviour at infinity.

We derive the problem for the remainder function R:

$$\int_{\Omega_\epsilon^c} \nabla R \, \nabla \Phi \, dx = \int_{D_\epsilon} \nabla(f_1 - f_2) \nabla\Phi \, dx - \epsilon^{\lambda} \int_{D_2} \nabla(w_- - W) \nabla\Phi \, dx - \int_{\partial\Omega_\epsilon} \frac{\partial R_1}{\partial \nu} \Phi \, ds \tag{22}$$

with f_1 and f_2 defined in (18) and (20), respectively.

Case 3: $\alpha = \pi$

In this special case the asymptotics of V read as follows:

$$V(x) = \psi(0) + r(\psi_{x_1}(0) \cos\varphi + a_1 \sin\varphi) + f_1(r,\varphi) \tag{23}$$

with $f_1 = \mathcal{O}\left(r^2\right)$. The index x_1 denotes the partial derivative in x_1-direction. The problem for W is set:

$$\begin{aligned} \Delta W(\xi) &= 0 && \text{in } \omega, \\ W(\xi) &= \begin{cases} |\xi|(\psi_{x_2}(0) - a_1) \sin\varphi & \text{on } \gamma_1 \\ 0 & \text{on } \partial\omega \setminus \gamma_1. \end{cases} \end{aligned} \tag{24}$$

Equation (8) is valid with $\beta = 1$. The function W shows the asymptotic behaviour $\mathcal{O}\left(|\xi|^{-1}\right)$ at infinity. The solution v of (12) has the asymptotics

$$v(x) = v(0) + r(b_1 \cos\varphi + a_1 \sin\varphi) + f_2(r,\varphi) \tag{25}$$

with $f_2 = \mathcal{O}\left(r^2\right)$. The same asymptotics are prescribed for the extension $v^{\mathfrak{i}}$. The function w_- solves the problem

$$\begin{aligned} \Delta w_- &= 0 && \text{in } \omega^c, \\ \frac{\partial w_-}{\partial \nu} &= (\psi_{x_1} - b_1)\frac{\partial}{\partial \nu}(|\xi|\cos\varphi) + \frac{\partial W}{\partial \nu} && \text{on } \partial\omega. \end{aligned} \tag{26}$$

Equation (7) is valid with $\tau = 1$. The function w_- as well as its extension $w_-^{\mathfrak{i}}$ show the asymptotic behaviour $\mathcal{O}\left(|\xi|^{-1}\right)$ at infinity. The remainder function solves problem (22) with f_1 and f_2 defined in (23) and (25), respectively.

Remark 1: Since the perturbation was inside the wedge, we had to construct an extension of v inside Ω_ϵ. If the perturbation lies outside the wedge, an extension has to be constructed for the function V outside Ω_ϵ. In the case that the boundary is perturbed partly inside and partly outside the wedge, the Dirichlet solution and the Neumann solution has to be extended within $\Omega_\epsilon \setminus \Omega$ and $\Omega \setminus \Omega_\epsilon$, respectively.

We summarize the results of Section 2, taking into consideration equations (6) and (7):

Lemma 1: *The solution μ_ϵ of the boundary integral equation (3) for the Dirichlet problem (1) has the following asymptotics:*

$$\mu_\epsilon(x) = \begin{cases} \left(\psi(x) - v(x) - \epsilon^{\pi/(2\pi-\alpha)} w_+\left(\frac{x}{\epsilon}\right) - R(x)\right)\Big|_{\partial\Omega_\epsilon} & \text{if } 0 < \alpha < \pi \\ \left(\psi(x) - v(x) - \epsilon^{\pi/\alpha} w_-\left(\frac{x}{\epsilon}\right) - R(x)\right)\Big|_{\partial\Omega_\epsilon} & \text{if } \pi \le \alpha < 2\pi, \end{cases}$$

where v is the solution of (12), w_+ and w_- are the solutions of (16), (21) and (26), respectively.

Remark 2: The expression $\psi(x) - v(x)$ on the right-hand side in the asymptotics of Lemma 1 can be understood as an extension of the solution μ of the boundary integral equation for the Dirichlet problem (9) in Ω (cp. equation (6)).

3. Estimates for the remainder function

In order to demonstrate the quality of the asymptotics of Lemma 1, we want to estimate the remainder function R. For that purpose a new origin is chosen, which lies inside Ω_ϵ. The distance to this origin is denoted by ρ. The weighted Sobolev space $H_{1,\delta}(\Omega_\epsilon^c)$ is defined as the space of functions with the finite norm

$$\|R\|_{1,\delta} = \left(\int_{\Omega_\epsilon^c} \left(\frac{|R|^2}{\rho^2} + |\nabla R|^2\right)\rho^{2\delta} dx\right)^{\frac{1}{2}} \quad \left(0 < \delta < \frac{1}{2}\right),$$

where the derivatives are understood in the generalized sense. A problem of the form

$$\int_{\Omega_\epsilon^c} \nabla R \, \nabla \Phi \, dx = \int_{\Omega_\epsilon^c} \vec{g} \, \nabla \Phi \, dx + \int_{\partial\Omega_\epsilon} h \Phi \, ds \tag{27}$$

with $\int_{\Omega_\epsilon^c} |\vec{g}|^2 \rho^{2\delta} dx < \infty$ and $h \in H^{-\frac{1}{2}}(\partial\Omega_\epsilon)$ has a unique solution in $H_{1,\delta}(\Omega_\epsilon^c)$, if the solvability condition $\int\limits_{\partial\Omega_\epsilon} h\,ds = 0$ is satisfied. The estimate below is valid for the solution R:

$$\|R\|_{1,\delta} \le C\Big(\int\limits_{\Omega_\epsilon^c} |\vec{g}|^2\rho^{2\delta}dx + \|h\|^2_{-\frac{1}{2}}\Big)^{\frac{1}{2}}. \tag{28}$$

Here and in all following estimates the letter C stands for a positive constant which does not depend on ϵ. This constant may be different in different inequalities. In a sequence of estimates indices will be used.

The unique solvability of (27) can be proved in the spirit of the papers by Kondratjev [1] and Maz'ya and Plamenevsky [4].

Lemma 2: *The remainder function $R|_{\partial\Omega_\epsilon}$ of Lemma 1 can be estimated in the norm of the trace space $H^{\frac{1}{2}}(\partial\Omega_\epsilon)$. R satisfies the inequalities*

$$\|R\|_{\frac{1}{2}} \le \begin{cases} C\epsilon^{\min\{\frac{\pi}{\alpha},\, \frac{2\pi}{2\pi-\alpha}\}} & \text{if } 0<\alpha<\pi \\ C\epsilon^{\min\{\frac{2\pi}{\alpha},\, \frac{\pi}{2\pi-\alpha}\}} & \text{if } \pi<\alpha<2\pi \\ C\epsilon^2 & \text{if } \alpha=\pi. \end{cases}$$

Proof: The cases are distinguished according to Section 2:

1) Problem (17) has the form (27). The three terms on the right-hand side are estimated in the following corresponding norms.

(i)
$$\begin{aligned}\Big(\int\limits_{D_\epsilon} |\nabla(f_1-f_2)|^2\rho^{2\delta}dx\Big)^{\frac{1}{2}} &\le C_1\Big(\int\limits_{D_\epsilon}(r^{\min\{\lambda,2\sigma\}-1})^2dx\Big)^{\frac{1}{2}} \\ &\overset{r=\epsilon t}{=} C_1\Big(\int\limits_{D_1}(\epsilon t)^{2\min\{\lambda,2\sigma\}-2}\epsilon^2dx_t\Big)^{\frac{1}{2}} \le C_2\epsilon^{\min\{\lambda,2\sigma\}}.\end{aligned}$$

(ii)
$$\epsilon^\sigma\Big(\int\limits_{D_2}|\nabla w_+|^2\rho^{2\delta}dx\Big)^{\frac{1}{2}} \le C_1\epsilon^{2\sigma}\Big(\int\limits_{D_2} r^{-2\sigma-2}\rho^{2\delta}dx\Big)^{\frac{1}{2}} \le C_2\epsilon^{2\sigma},$$

since $\delta<\sigma$.

(iii)
$$\Big\|\frac{\partial R_0}{\partial\nu}\Big\|_{-\frac{1}{2}} = \sup_{\|\Phi\|_{\frac{1}{2}}\le 1}\Big|\int\limits_{\partial\Omega_\epsilon}\frac{\partial R_0}{\partial\nu}\Phi\,ds\Big| = \sup\Big|\int\limits_{\Omega_\epsilon}\nabla R_0\nabla\Phi\,dx\Big| \le C_1\|\nabla R_0\|_{L_2(\Omega_\epsilon)}, \tag{29}$$

because of the duality between $H^{-\frac{1}{2}}(\partial\Omega_\epsilon)$ and $H^{\frac{1}{2}}(\partial\Omega_\epsilon)$ and the existence of a continuous extension operator from $H^{\frac{1}{2}}(\partial\Omega_\epsilon)$ onto $H^1(\Omega_\epsilon)$. Since R_0 is the solution of the Dirichlet problem (11), its gradient can be estimated by the gradient of an extension of the right-hand side of the boundary condition. We choose as extension the function $(\psi - V)\eta$, where $\eta\in C_0^\infty(\mathbf{R}^2)$ is a cut-off function with $\eta\equiv 1$ in $B_\epsilon(O)$ and $\eta\equiv 0$ in $\mathbf{R}^2\setminus B_{2\epsilon}(O)$:

$$\begin{aligned}\|\nabla R_0\|_{L_2(\Omega_\epsilon)} &\le C_2\Big(\int\limits_{B_{2\epsilon}(O)} r^{2\min\{\lambda,2\}-2}dx\Big)^{\frac{1}{2}} \overset{r=\epsilon t}{=} C_2\Big(\int\limits_{B_2(O)}(\epsilon t)^{2\min\{\lambda,2\}-2}\epsilon^2dx_t\Big)^{\frac{1}{2}} \\ &\le C_3\epsilon^{\min\{\lambda,2\}}.\end{aligned} \tag{30}$$

The estimates of (i),(ii) and (iii) and inequality (28) imply $\|R\|_{1,\delta}\le C\epsilon^{\min\{\lambda,2\sigma\}}$ and the proposition of the lemma follows from the trace lemma.

2) and 3) Problem (22) has the form (27). The first two terms on the right-hand side can be treated as in Case 1, taking into consideration the different behaviour of f_1, f_2, w_- and W. The function R_1 in the third term solves the Dirichlet problem

$$\triangle R_1 = 0 \quad \text{in } \Omega_\epsilon, \qquad R_1(x) = \psi(x) - V(x) - \epsilon^\lambda W\left(\frac{x}{\epsilon}\right) \quad \text{on } \partial\Omega_\epsilon.$$

If we write R_1 as a sum of two harmonic functions $R_1 = R_{1,1} + R_{1,2}$ with $\operatorname{supp} R_{1,1}|_{\partial\Omega_\epsilon} \subset \gamma_\epsilon$ and $\operatorname{supp} R_{1,2}|_{\partial\Omega_\epsilon} \subset \gamma_2$, the function $R_{1,1}$ can be estimated according to (30) and the estimate for $R_{1,2}$ reads as follows:

$$\|\nabla R_{1,2}\|_{L_2(\Omega_\epsilon)} \le C_2\epsilon^\lambda\Big(\int_{\Omega_\epsilon} |\nabla(W(1-\eta))|^2\,dx\Big)^{\frac{1}{2}} \le C_3\epsilon^{2\lambda},$$

where η is a cut-off function with support in a circle which does not contain a point of γ_2. Analogously to (29) we obtain $\|\partial R_1/\partial\nu\|_{-\frac{1}{2}} \le C\epsilon^{2\lambda}$ and the proposition of the lemma follows ∎

It is desirable to get stronger estimates for the remainder function $R|_{\partial\Omega_\epsilon}$. Let Q_1 and Q_2 be domains obtained by intersection of Ω_ϵ^c and circles containing Ω_ϵ and let $Q_1 \subset Q_2$. Following the paper of Meyers [5], the gradient of R can be estimated in an L_p-norm with $2 < p < 2+\kappa$, where $\kappa > 0$ does not depend on ϵ:

$$\|\nabla R\|_{L_p(Q_1)} \le C\Big(\|\vec{g}\|_{L_p(Q_2)} + \|h\|_{W_p^{-\frac{1}{p}}(\partial\Omega_\epsilon)} + \|R\|_{L_2(Q_2)}\Big).$$

Sobolev's imbedding theorem implies

$$\begin{aligned} \|R\|_{C^{0,\delta}(\overline{Q}_1)} &\le C_1\|R\|_{W_p^1(Q_1)} \\ &\le C_2\Big(\|\vec{g}\|_{L_p(Q_2)} + \|h\|_{W_p^{-\frac{1}{p}}(\partial\Omega_\epsilon)} + \|R\|_{L_2(Q_2)}\Big) \end{aligned} \tag{31}$$

with $\delta < \dfrac{\kappa}{2+\kappa}$.

Lemma 3: *The remainder function $R|_{\partial\Omega_\epsilon}$ can be estimated in the norm of the space $C(\partial\Omega_\epsilon)$:*

$$\|R\|_{C(\partial\Omega_\epsilon)} = \begin{cases} \mathcal{O}\left(\epsilon^{\min\{\frac{\pi}{\alpha},\frac{2\pi}{2\pi-\alpha}\}-\delta}\right) & \text{if } 0 < \alpha < \pi \\ \mathcal{O}\left(\epsilon^{\min\{\frac{2\pi}{\alpha},\frac{\pi}{2\pi-\alpha}\}-\delta}\right) & \text{if } \pi < \alpha < 2\pi \\ \mathcal{O}\left(\epsilon^{2-\delta}\right) & \text{if } \alpha = \pi \end{cases}$$

with arbitrary small $\delta > 0$.

Proof: We estimate the terms of the right-hand side in (31). For the last summand the results of Lemma 2 can be applied. The estimates of the terms $\|\vec{g}\|_{L_p(Q_2)}$ and $\|h\|_{W_p^{-1/p}(\partial\Omega_\epsilon)}$ are carried out as in the proof of Lemma 2. In Case 1 we obtain

(i) $$\Big(\int_{D_\epsilon} |\nabla(f_1 - f_2)|^p dx\Big)^{\frac{1}{p}} \le C_1\Big(\int_{D_1} (\epsilon t)^{p\min\{\lambda,2\sigma\}-p}\epsilon^2 dx_t\Big)^{\frac{1}{p}} \le C_2\epsilon^{\min\{\lambda,2\sigma\}-\delta}$$

with $\delta < \frac{\kappa}{2+\kappa}$.

(ii) $$\epsilon^{\sigma}\Big(\int_{D_2\cap Q_2} |\nabla w_+|^p dx\Big)^{\frac{1}{p}} \le C_1\epsilon^{2\sigma}.$$

(iii) $$\Big\|\frac{\partial R_0}{\partial\nu}\Big\|_{W_p^{-\frac{1}{p}}(\partial\Omega_\epsilon)} = \sup_{\|\Phi\|_{W_{p'}^{\frac{1}{p}}(\partial\Omega_\epsilon)}\le 1}\Big|\int_{\partial\Omega_\epsilon}\frac{\partial R_0}{\partial\nu}\Phi\, ds\Big| = \sup\Big|\int_{\Omega_\epsilon}\nabla R_0\nabla\Phi\, dx\Big|$$
$$\le C_1\|\nabla R_0\|_{L_p(\Omega_\epsilon)}$$

and

$$\|\nabla R_0\|_{L_p(\Omega_\epsilon)} \le C_2\Big(\int_{B_2(O)} (\epsilon t)^{p\min\{\lambda,2\}-p}\epsilon^2 dx_t\Big)^{\frac{1}{p}} \le C_3\epsilon^{\min\{\lambda,2\}-\delta}.$$

The other cases can be treated in the same way. The proposition of the lemma follows from (31), since the domain Ω_ϵ is bounded ∎

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Received 16.05.1991; in revised form 17.09.1991

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