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On Classes of Stieltjes Type Operator-Valued Functions with Gaps

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We introduce and investigate classes of operator-valued functions with gaps, which can be realized as fractional linear transformations of operator-valued transfer functions of conservative scattering systems.

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Classes of Stieltjes type operator-valued functions with gaps on the positive semi-axis (i.e., with intervals of holomorphy and definiteness) are considered. We prove criteria that a given function, whose values are operators in a finite-dimensional Hilbert space, belongs to these classes. Moreover, we investigate classes of Stieltjes type operator-valued functions which admit a realization, i.e., which can be represented as fractional linear transformations of operator-valued transfer functions of conservative scattering systems of the form

$$\Theta = (\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-, \mathbb{A}, K, I, E)$$

where $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$, $\operatorname{Im} \mathbb{A} = KK^*$, $\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$, A is a closed Hermitian operator in $\mathfrak{H}$, and T is closed with dense domain of definition in $\mathfrak{H}$.

In the class of realizable Stieltjes type operator-valued functions the following subclasses are investigated:

1. the subclass, where $\overline{\mathfrak{D}(A)} = \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$
2. the subclass, where $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$
3. the subclass, where $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.

We prove analytical criteria for a given operator-valued function to belong to the mentioned subclasses (with gaps). These criteria are analoga, supplements, and refinements of some of the results stated by M.G. Krein and A.A. Nudel'man [7].

§ 1 The classes $S_\pm[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ of operator-valued functions

According to M.G. Kreĭn [8], a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, will be called a *Stieltjes type operator-valued function* if the following conditions hold:

1. $V(z)$ is holomorphic on $\operatorname{Ext}[0,\infty) := \{z : z \notin [0,\infty)\}$
2. $V(z) \geq 0$ for $z < 0$
3. $V(z)$ is an operator-valued R-function, i.e., $\operatorname{Im} V(z)/\operatorname{Im} z \geq 0$.

The class of Stieltjes type operator-valued functions will be denoted by S.

Let $\{(\alpha_j, \beta_j)\}_{j=1}^{m}$ be a system of mutually disjoint intervals on the positive semi-axis.

Definition: By $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ we denote the class of functions $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, such that the following two conditions hold:

1. $V(z) \in S$.
2. $V(z)$ is holomorphic and positive on all intervals (α_j,β_j), i.e., $(V(z)f, f) > 0$ for all $f \in E$, $f \neq 0$, and all $z \in (\alpha_j,\beta_j)$ ($V(z)$ is holomorphic and negative on the intervals (α_j,β_j), respectively, i.e., $(V(z)f, f) < 0$ for all $f \in E$, $f \neq 0$, and all $z \in (\alpha_j,\beta_j)$).

Theorem 1: *A scalar function $V(z)$ belongs to the classes $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S$.

(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$ $\left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S, \text{ respectively}\right)$.

Proof: First we consider the class $S_{+}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$. Let (i) and (ii) be fulfilled. Since (i), a well known theorem (see [7]) gives us

$$V(z) = c \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) f(t)\, dt, \tag{2}$$

where $c > 0$, $f(t)$ is a summable function such thast $0 \le f(t) \le 1$ a.e. and $\int_{-\infty}^{+\infty} (1 + t^2)^{-1} f(t)\, dt < \infty$. Moreover, the representation (2) is unique. It is not hard to see that

$$\frac{\beta_j - z}{\alpha_j - z} = c_j \exp \int_{\alpha_j}^{\beta_j} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) f(t)\, dt. \tag{3}$$

Since (ii), we get

$$\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) = c_1 \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) f_1(t)\, dt, \tag{4}$$

in an analogous way, where $c_1 > 0$ and the function $f_1(t)$ has the same properties as $f(t)$. Using (3) and (4), we obtain

$$V(z) = c_2 \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) f_2(t)\, dt,$$

where

$$f_2(t) = \begin{cases} f_1(t) & \text{for } t \in \mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j,\beta_j) \\ f_1(t) - 1 & \text{for } t \in \bigcup_{j=1}^{m}(\alpha_j,\beta_j) \end{cases}.$$

Because of the uniqueness of the representation (2) it follows

$$V(z) = c \exp \int_{\mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j,\beta_j)} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) f(t)\, dt, \tag{5}$$

where $c > 0$, $0 \le f(t) \le 1$ a.e. on $\mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j,\beta_j)$. By a well known theorem (see [7]), the relation (5) implies that $V(z)$ is holomorphic and positive on all intervals (α_j,β_j).

Now assume that $V(z) \in S_{+}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$. Then $V(z) \in S_{+}[(\alpha_1,\beta_1)]$. We will show that the inclusion $((\beta_1 - z)/(\alpha_1 - z))V(z) \in S$ is true. In fact, setting $\xi = (\beta_1 - z)/(\alpha_1 - z)$ and $V_1(\xi) = V(z)$

we get

$$\frac{\operatorname{Im} V_1(\xi)}{\operatorname{Im} \xi} = \frac{|\alpha_1 - z|^2}{\beta_1 - \alpha_1} \frac{\operatorname{Im} V(z)}{\operatorname{Im} z} \geq 0,$$

hence, $V_1(\xi) \in R$, i.e., $V_1(\xi)$ is an R-function. It is not hard to see that $z \in (\alpha_1, \beta_1)$ implies $\xi \in (-\infty, 0)$, and since $V(z) \in S_+[(\alpha_1, \beta_1)]$, it follows $V_1(\xi) \in S$. Now a theorem of M.G. Krein (see [8]) gives us $\xi V_1(\xi) \in R$. Thus $((\beta_1 - z)/(\alpha_1 - z))V(z) \in R$. Since $((\beta_1 - z)/(\alpha_1 - z))V(z) \geq 0$ if $z \in (-\infty, 0)$, we obtain $((\beta_1 - z)/(\alpha_1 - z))V(z) \in S$.

We will show that the implication

$$\prod_{j=1}^{k} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \ (1 < k < m) \ \Rightarrow \ \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$$

is true. In fact, it is not hard to see that

$$\prod_{j=1}^{k} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_+[(\alpha_{k+1}, \beta_{k+1})].$$

Now by analogous arguments as above we get

$$\frac{\beta_{k+1} - z}{\alpha_{k+1} - z} \prod_{j=1}^{k} \frac{\beta_j - z}{\alpha_j - z} V(z) = \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S.$$

Thus the first part of the theorem is proved.

Now let $V(z) \in S_-[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$. We will show that the ( )-part of (ii) is true. It is not hard to see (cf. [6]) that $V(z) \in R$ if and only if $-V(z)^{-1} \in R$. Thus, the relation $V(z) \in S_-[(\alpha_1, \beta_1)]$ implies $-V(z)^{-1} \in R$ and $-V(z)^{-1} > 0$, $z \in (\alpha_1, \beta_1)$. Setting $\xi = (\beta_1 - z)/(\alpha_1 - z)$ and $V_1(\xi) = -V(z)^{-1}$, we get

$$\frac{\beta_1 - z}{\alpha_1 - z}\left(-\frac{1}{V(z)}\right) \in R, \text{ hence } -\left(\frac{\beta_1 - z}{\alpha_1 - z}\left(-\frac{1}{V(z)}\right)\right)^{-1} = \frac{\alpha_1 - z}{\beta_1 - z} V(z) \in R.$$

Since $\frac{\alpha_1 - z}{\beta_1 - z} V(z) \geq 0$ if $z \in (-\infty, 0)$, it follows $\frac{\alpha_1 - z}{\beta_1 - z} V(z) \in S$. Using an analogous induction method as in the first part of the proof we obtain the ( )-part of (ii).

Now assume (i) and (ii)/( )-part. Consider the function $-V(z)^{-1} \in R$ and use analogous arguments as in the proof of the sufficiency in the first part. This gives us that $-V(z)^{-1}$ is holomorphic and positive on all intervals (α_j, β_j). Thus the theorem is proved ∎

Theorem 2: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the classes $S_\pm[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S$.

(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$ $\left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S, \text{ respectively}\right)$.

Proof: Let $V(z) \in S_+[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$. Considering the scalar function $(V(z)f, f) \in S$ and using Theorem 1 we get

$$\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} (V(z)f, f) = \left(\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)f, f\right) \in S.$$

Hence, the operator-valued function $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)$ belongs to S.

The sufficiency of conditions (i) and (ii) is trivial. The proof for the class $S_{-}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ is analogous. Thus the theorem is proved ∎

Definition: We will say that a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class

$$S_{+}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_{-}[\bigcup_{j=1}^{n}(c_j,d_j)]$$

if the following three conditions hold:

1. $V(z) \in S$.
2. $V(z)$ is holomorphic and positive on the intervals (α_j,β_j) $(j = 1,\dots,m)$.
3. $V(z)$ is holomorphic and negative on the intervals (c_k,d_k) $(k = 1,\dots,n)$.

Theorem 2 immediately implies the following

Theorem 3: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S_{+}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_{-}[\bigcup_{k=1}^{n}(c_k,d_k)]$ if and only if the following two conditions hold*:

1. $V(z) \in S$.
2. $\prod_{j=1}^{m}\frac{\beta_j - z}{\alpha_j - z}\prod_{k=1}^{n}\frac{c_k - z}{d_k - z}V(z) \in S$.

§ 2 Realizable operator-valued functions of the class $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$

Let A be a closed Hermitian operator in a Hilbert space $\mathfrak{H}$, whose defect numbers are finite and coincide. This operator can be considered as acting from $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ into $\mathfrak{H}$. Let A^* be the adjoint operator. Clearly, $\overline{\mathfrak{D}(A^*)} = \mathfrak{H}$ (where the closure is taken in $\mathfrak{H}$). We set $\mathfrak{H}_+ = \mathfrak{D}(A^*)$ and introduce the scalar product $(f,g)_+ = (f,g) + (A^*f,A^*g)$ $(f,g \in \mathfrak{H}_+)$. We consider the rigged Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ (cf. [2]).

We will say that a closed and densely defined operator T in $\mathfrak{H}$ belongs to the *class* Ω_A if the following two conditions

1. $T \supset A$, $T^* \supset A$ (A is closed and Hermitian)
2. $-i$ is a regular point of T

are fulfilled.

A bounded operator $\mathbb{A}: \mathfrak{H}_+ \to \mathfrak{H}_-$ (i.e., $\mathbb{A} \in [\mathfrak{H}_+,\mathfrak{H}_-]$) is called a *biextension* of the Hermitian operator A if $\mathbb{A} \supset A$ and $\mathbb{A}^* \supset A$. Identifying the dual space of $\mathfrak{H}_\pm$ with $\mathfrak{H}_\mp$, we see that $\mathbb{A}^* \in [\mathfrak{H}_+,\mathfrak{H}_-]$. If $\mathbb{A} = \mathbb{A}^*$, then $\mathbb{A}$ is called a *selfadjoint* biextension of A.

By $\hat{A}$ we denote the restriction of $\mathbb{A}$ to $\mathfrak{D}(\hat{A}) = \{f \in \mathfrak{H}_+ : \mathbb{A}f \in \mathfrak{H}\}$. It is called a *quasikernel* of $\mathbb{A}$ (cf. [10,11]). A selfadjoint biextension is called a *strong* biextension if $\hat{A} = \hat{A}^*$ (cf. [10,11]).

Let $T \in \Omega_A$. Then $\mathbb{A} \in [\mathfrak{H}_+,\mathfrak{H}_-]$ is called a $(*)$-extension of T if

$$\mathbb{A} \supset T \supset A,\ \mathbb{A}^* \supset T^* \supset A. \tag{6}$$

Moreover, if $\mathbb{A}_R = (\mathbb{A} + \mathbb{A}^*)/2$ is a strong selfadjoint biextension, then $\mathbb{A}$ is called a *correct* $(*)$-extension of T.

By definition, the class Λ_A denotes the set of all operators $T \in \Omega_A$ such that A coincides with the maximal common Hermitian part of T and T^*.

Definition: The operator colligation

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \tag{7}$$

is called *rigged* if the following four conditions hold:

1. $J = J^* = J^{-1}$ $(\dim E < \infty)$.
2. K is a bounded linear operator from E into $\mathfrak{H}_-$.
3. $\mathbb{A}$ is a correct $(*)$-extension of $T \in \Lambda_A$, and

$$\operatorname{Im} \mathbb{A} = (\mathbb{A} - \mathbb{A}^*)/2i = KJ^*K^*. \tag{8}$$

4. The ranges of K and $\operatorname{Im} \mathbb{A}$ coinside.

The operaror-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{9}$$

is called a *Livsic type characteristic function* of the colligation Θ.

Furthermore, we introduce the function

$$V_\Theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K. \tag{10}$$

It is well known (cf. [3, 9]) that the functions $V_\Theta(z)$ and $W_\Theta(z)$ are associated by the relations

$$V_\Theta(z) = i(W_\Theta(z) + I)^{-1}(W_\Theta(z) - I)J \quad \text{and} \quad W_\Theta(z) = (I + iV_\Theta(z)J)^{-1}(I - iV_\Theta(z)J). \tag{11}$$

We consider the conservative system (cf. [9])

$$\begin{aligned} (\mathbb{A} - zI)x &= KJ\varphi_- \\ \varphi_+ &= \varphi_- - 2iK^*x, \end{aligned} \tag{12}$$

where $x \in \mathfrak{H}_+$, $\varphi_\pm \in E$, φ_- is the so-called *input vector*, φ_+ is the output vector, and x is the inner state. It is not hard to see that the transfer function $\Pi(z)$ of such a system (i.e., $\varphi_+ = \Pi(z)\varphi_-$) coincides with the operator function $W_\Theta(z)$. If $J \neq I$, then the system is called a *crossing* system and if $J = I$, it is called a *scattering* system (cf. [1]). In the following we will write a conservative system Θ in the form of a rigged operator colligation.

Definition: A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, is called *realizable* if it can be represented as

$$V(z) = V_\Theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K = i(W_\Theta(z) + I)^{-1}(W_\Theta(z) - I), \tag{13}$$

where Θ is a conservative scattering system of the form (7).

Theorem 4: *Let $V(z)$ be a realizable function, whose values are operators in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(\mathbb{A}_R - zI)^{-1}K$. Let $A > 0$ and let (α,β) be an arbitrary interval of the positive semi-axis. Then $V(z)$ belongs to the class $S_+[(\alpha,\beta)]$ if and only if the following two conditions hold:*

1. $\mathbb{A}_R \geq 0$.
2. *For an arbitrary set $\{z_i\}_{i=1}^p$ of non-real complex numbers such that $z_i \neq \bar{z}_l$ and for all $\varphi_i \in N_{z_i}$ (where N_{z_i} is the deficiency space of A) it holds*

$$\sum_{i,l=1}^{p} (B(z_i, z_l)\varphi_i, \varphi_l) \geq 0, \tag{15}$$

where

$$B(\lambda,\mu) = \frac{\beta-\alpha}{(\alpha-\lambda)(\alpha-\overline{\mu})}\mathbb{A}_R + \frac{\alpha\beta-\beta(\lambda+\overline{\mu})+\lambda\overline{\mu}}{(\alpha-\lambda)(\alpha-\overline{\mu})}I. \tag{16}$$

Proof: Assume that the conditions (14) and (15) hold. We will show that $V(z) \in S_+[(\alpha,\beta)]$. Since $V(z)$ is realizable, there exists a conservative scattering system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}$$

such that $V(z) = V_\Theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K$. The operator $\mathbb{A}$ is a $(*)$-extension of some densely defined closed operator T, i.e., condition (6) holds, where A is the common maximal Hermitian part of T and T^*. Let N_z be the deficiency space of the Hermitian operator A and let $\{z_i\}_{i=1}^p$ be an arbitrary set of non-real complex numbers such that $z_i \neq \overline{z}_l$. Moreover, let $\varphi_i \in N_{z_i}$. According to [11], there exists a vector $h_i \in E$ such that

$$\varphi_i = (\mathbb{A}_R - zI)^{-1}Kh_i \quad (i = 1, \dots, p). \tag{17}$$

Set $w_i = (\beta - z_i)/(\alpha - z_i)$. We will prove the inequality

$$\sum_{i,l=1}^{p}\left(\frac{w_i V(z_i) - \overline{w}_l V(\overline{z}_l)}{z_i - \overline{z}_l}h_i, h_l\right) \geq 0. \tag{18}$$

In fact,

$$\sum_{i,l=1}^{p}\left(\frac{w_i V(z_i) - \overline{w}_l V(\overline{z}_l)}{z_i - \overline{z}_l}h_i, h_l\right)$$

$$= \sum_{i,l=1}^{p}\left(\frac{w_i(\mathbb{A}_R - z_i I)^{-1} - \overline{w}_l(\mathbb{A}_R - \overline{z}_i I)^{-1}}{z_i - z_l}Kh, Kh_l\right)$$

$$= \sum_{i,l=1}^{p}\left(\frac{(\mathbb{A}_R - \overline{z}_l I)^{-1}(w_i(\mathbb{A}_R - \overline{z}_l I) - \overline{w}_l(\mathbb{A}_R - z_i I))(\mathbb{A}_R - z_i I)^{-1}}{z_i - \overline{z}_l}Kh_i, Kh_l\right)$$

$$= \sum_{i,l=1}^{p}\left(\left(\frac{w_i - \overline{w}_l}{z_i - \overline{z}_l}\mathbb{A}_R + \frac{z_i\overline{w}_l - \overline{z}_l w_i}{z_i - \overline{z}_l}I\right)\varphi_i, \varphi_l\right)$$

$$= \sum_{i,l=1}^{p}\left(\left(\frac{\beta-\alpha}{(\alpha - z_i)(\alpha - \overline{z}_l)}\mathbb{A}_R + \frac{\alpha\beta - \beta(z_i + \overline{z}_l) + z_i\overline{z}_l}{(\alpha - z_i)(\alpha - \overline{z}_l)}I\right)\varphi_i, \varphi_l\right)$$

$$= \sum_{i,l=1}^{p}(B(z_i, z_l)\varphi_i, \varphi_l) \geq 0.$$

Setting $p = 1$, $z_1 = z$, $h_1 = h$, we obtain from (17) and (18)

$$\left(\frac{\frac{\beta - z}{\alpha - z}V(z) - \frac{\beta - \overline{z}}{\alpha - \overline{z}}V(\overline{z})}{z - \overline{z}}h, h\right) \geq 0 \quad \text{and hence} \quad \frac{\operatorname{Im}\left(\frac{\beta - z}{\alpha - z}V(z)\right)}{\operatorname{Im} z} \geq 0,$$

i.e., the operator-valued function $((\beta - z)(\alpha - z))V(z)$ is an operator-valued R-function. Now condition (14) implies that $V(z) \in S$ (cf. [4]). Since for $z < 0$ we have $(\beta - z)(\alpha - z) \geq 0$ it holds $((\beta - z)(\alpha - z))V(z) \in S$. By Theorem 2 we get $V(z) \in S_+[(\alpha,\beta)]$.

Now assume $V(z) \in S_+[(\alpha,\beta)]$. Thus $V(z) \in S$ and hence $A|_R \geq 0$ (cf. [4]). It remains to prove (15). In fact, by Theorem 2 the condition $V(z) \in S_+[(\alpha,\beta)]$ implies that $((\beta - z)(\alpha - z))V(z) \in S$. Thus, the operator-valued function $((\beta - z)(\alpha - z))V(z)$ has a representation of the form

$$\frac{\beta - z}{\alpha - z} V(z) = \gamma + \int_0^\infty \frac{1}{t - z}\, d\sigma(t), \tag{19}$$

where $\gamma \geq 0$, $\sigma(t)$ is a non-decreasing operator-valued function in E such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$. Let $\{z_i\}_{i=1}^p$ be an arbitrary set of non-real complex numbers such that $z_i \neq \overline{z}_l$. Let $\varphi_i \in N_{z_i}$. Then according to [11] there exist vectors $h_i \in E$ such that (17) holds. Setting $w_i = (\beta - z_i)(\alpha - z_i)$ we obtain

$$\sum_{i,l=1}^{p} \left(\frac{w_i V(z_i) - \overline{w}_l V(\overline{z}_l)}{z_i - \overline{z}_l} h_i, h_l \right) = \sum_{i,l=1}^{p} \left(\int_0^\infty \frac{1}{(t - z_i)(t - \overline{z}_l)} d\sigma(t) h_i, h_l \right) \geq 0.$$

It is clear from the proof of the sufficiency part that the inequalities (15) and (18) are equivalent. Thus, the above inequality yields the needed result ∎

Remark: If there is no gap, i.e., if $\alpha = \beta$, the inequality (15) holds trivially and we obtain the results of [4].

Theorem 5: *Let $V(z)$ be a realizable function, whose values are operators in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$. Let (α,β) be an arbitrary interval of the positive semi-axis. Then $V(z)$* belongs to the class $S_-[(\alpha,\beta)]$ *if and only if the following two conditions hold:*

1. $A|_R \geq 0$.
2. *For an arbitrary set $\{z_i\}_{i=1}^p$ of non-real complex numbers such that $z_i \neq z_l$ and for all $\varphi_i \in N_{z_i}$ (where N_{z_i} is the deficiency space of A) it holds*

$$\sum_{i,l=1}^{p} (B(z_i, z_l)\varphi_i, \varphi_l) \geq 0,$$

where

$$B(\lambda,\mu) = \frac{\alpha - \beta}{(\beta - \lambda)(\beta - \overline{\mu})} A|_R + \frac{\alpha\beta - \alpha(\lambda + \overline{\mu}) + \lambda\overline{\mu}}{(\beta - \lambda)(\beta - \overline{\mu})} I. \tag{20}$$

Proof: This theorem can be proved in the same way as Theorem 4 (set $w_i = (\alpha - z_i)(\beta - z_i)$) ∎

Note further that in the case $\alpha = \beta$ (i.e., if there are no gaps) Theorem 5 is an extension of results of [4]. Moreover, it is not hard to see that the above used method allows us to obtain analogous results for the classes

$$S_\pm[\cup_{j=1}^m (\alpha_j, \beta_j)] \quad \text{and} \quad S_+[\cup_{j=1}^m (\alpha_j, \beta_j)] \cap S_-[\cup_{j=1}^n (c_j, d_j)].$$

In fact, we have the following

Theorem 6: *Let $V(z)$ be a realizable function, whose values are operators in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$. Let (α_j,β_j) $(j = 1,\dots,m)$ and (c_k, d_k) $(k = 1,\dots,n)$ be arbitrary mutually disjoint intervals of the positive semi-axis. Then $V(z)$ belongs to the class $S_+[\cup_{j=1}^m (\alpha_j,\beta_j)] \cap S_-[\cup_{k=1}^n (c_k, d_k)]$ if and only if the following two conditions hold:*

1. $A|_R \geq 0$.

2. *For an arbitrary set* $\{z_i\}_{i=1}^{p}$ *of non-real complex numbers such that* $z_i \neq \bar{z}_l$ *and for all* $\varphi_i \in N_{z_i}$ *(where* N_{z_i} *is the deficiency space of* A*) it holds*

$$\sum_{i,l=1}^{p} (B(z_i, z_l)\varphi_i, \varphi_l) \geq 0,$$

where

$$B(\lambda,\mu) = \frac{w(\lambda) - w(\bar{\mu})}{\lambda - \bar{\mu}} A|_R + \frac{\lambda w(\bar{\mu}) - \bar{\mu} w(\lambda)}{\lambda - \bar{\mu}} \tag{21}$$

and

$$w(\lambda) = \prod_{j=1}^{m} \frac{\beta_j - \lambda}{\alpha_j - \lambda} \prod_{k=1}^{n} \frac{c_k - \lambda}{d_k - \lambda}. \tag{22}$$

§ 3 Some subclasses of realizable Stieltjes type operator-valued functions with gaps

By a result of M.G. Krein (see [8]) each Stieltjes type function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, can be represented in the form

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{(t - z)}, \tag{23}$$

where $\gamma \geq 0$, $\sigma(t)$ is a non-decreasing operator-valued function in E such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$. According to [5], we introduce the following notion.

Definition: We will say that a Stieltjes type function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S(R)$, if $\gamma f = 0$ for all f of the subclass

$$E_\infty^\perp = \left\{ f \in E : \int_0^\infty (d\sigma(t)f, f)_E < \infty \right\}. \tag{24}$$

As it was proved in [5], each operator-valued function $V(z) \in S(R)$ can be realized by a conservative scattering system Θ, i.e., it holds (13).

Definition: Following [5], we introduce the following subclasses of $S(R)$:

(i) The class $S^0(R)$ consisting of all $V(z) \in S(R)$ such that

$$\int_0^\infty (d\sigma(t)f, f) = \infty \quad (f \in E,\ f \neq 0). \tag{25}$$

(ii) The class $S^1(R)$ consisting of all $V(z) \in S(R)$ such that $\gamma = 0$ and

$$\int_0^\infty (d\sigma(t)f, f) < \infty \quad (f \in E) \tag{26}$$

in the representation (23).

(iii) The class $S^{01}(R)$ consisting of all $V(z) \in S(R)$ such that $E_\infty^\perp \neq \{0\}$ and $E_\infty^\perp \neq E$.

It is not hard to see that

$$S(R) = S^{0}(R) \cup S^{1}(R) \cup S^{01}(R). \tag{27}$$

Definition: We introduce the following subclasses of $S(R)$, $S^{0}(R)$, $S^{1}(R)$ and $S^{01}(R)$.

(i) The class $S_{\pm}[R, \bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ consisting of all $V(z) \in S(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_j, β_j).

(ii) The class $S_{\pm}^{0}[R, \bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ consisting of all $V(z) \in S^{0}(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_j, β_j).

(iii) The class $S_{\pm}^{1}[R, \bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ consisting of all $V(z) \in S^{1}(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_j, β_j).

(iv) The class $S_{\pm}^{01}[R, \bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ consisting of all $V(z) \in S^{01}(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_j, β_j).

Let Θ be a conservative scattering system of the form (7) such that $V_{\Theta}(z) = V(z)$ and let A and T be the operators of (6). Then (cf. [5])

$$\overline{\mathfrak{D}(A)} = \mathfrak{H}, \quad \mathfrak{D}(T) \neq \mathfrak{D}(T^{*}) \text{ if } V(z) \in S_{+}^{0}[R, \textstyle\bigcup_{j=1}^{m}(\alpha_j, \beta_j)],$$

$$\overline{\mathfrak{D}(A)} \neq \mathfrak{H}, \quad \mathfrak{D}(T) = \mathfrak{D}(T^{*}) \text{ if } V(z) \in S_{\pm}^{1}[R, \textstyle\bigcup_{j=1}^{m}(\alpha_j, \beta_j)],$$

$$\overline{\mathfrak{D}(A)} \neq \mathfrak{H}, \quad \mathfrak{D}(T) \neq \mathfrak{D}(T^{*}) \text{ if } V(z) \in S_{\pm}^{01}[R, \textstyle\bigcup_{j=1}^{m}(\alpha_j, \beta_j)].$$

Theorem 7: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S_{\pm}^{0}[R, (\alpha, \beta)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S^{0}(R)$.

(ii) $\frac{\beta - z}{\alpha - z} V(z) \in S^{0}(R)$ $\left(\frac{\alpha - z}{\beta - z} V(z) \in S^{0}(R), \textit{respectively}\right)$. (28)

Proof: Assume that the conditions (28) hold. Since $S^{0}(R) \in S$, we have $V(z) \in S_{\pm}[R, (\alpha, \beta)]$ by Theorem 2 and, hence, $V(z) \in S_{\pm}^{0}[R, (\alpha, \beta)]$ because $V(z) \in S^{0}(R)$. Conversely, assume that $V(z) \in S_{+}^{0}[R, (\alpha, \beta)]$. Then, clearly, $V(z) \in S^{0}(R)$. It remains to show the first inclusion of (ii). It is well known that

$$\int_{0}^{\infty} (d\sigma(t) f, f) = \lim_{\eta \uparrow \infty} (\eta \operatorname{Im} V(i\eta) f, f), \tag{29}$$

where $\sigma(t)$ is the operator-valued measure of the representation (23) (cf. [6]). The definitive of $S^{0}(R)$ and (29) imply

$$\lim_{\eta \uparrow \infty} (\eta \operatorname{Im} V(i\eta) f, f) = \infty. \tag{30}$$

Since $V(z) = K^{*}(A|_R - zI)^{-1} K$, we obtain

$$\operatorname{Im} V(z) = \operatorname{Im} z K^{*}(A|_R - \bar{z} I)^{-1} (A|_R - zI)^{-1} K. \tag{31}$$

Set

$$f_{\eta} = (A|_R - i\eta I)^{-1} K f. \tag{32}$$

From (30) - (32) it follows

$$\lim_{\eta \uparrow \infty} \eta^{2} (f_{\eta}, f_{\eta}) = \lim_{\eta \uparrow \infty} \eta^{2} ((A|_R - i\eta I)^{-1} K f, (A|_R - i\eta I)^{-1} K f) \tag{33}$$

$$= \lim_{\eta\uparrow\infty} \eta^2(K^*(A|_R + i\eta I)^{-1}(A|_R - i\eta I)^{-1}Kf, f) = \lim_{\eta\uparrow\infty}(\eta \operatorname{Im} V(i\eta)f, f) = \infty.$$

We will show that

$$\lim_{\eta\uparrow\infty}\left(\eta \operatorname{Im}\frac{\beta - i\eta}{\alpha - i\eta}V(i\eta)f, f\right) = \infty. \tag{34}$$

In fact, setting $z_i = z_l = i\eta$ in the inequality (18) and regarding the considerations of the proof of this inequality we obtain

$$\lim_{\eta\uparrow\infty}\left(\eta \operatorname{Im}\frac{\beta - i\eta}{\alpha - i\eta}V(i\eta)f, f\right) = \lim_{\eta\uparrow\infty}\left(\frac{\eta^2(\beta-\alpha)}{\alpha^2+\eta^2}(A|_R f_\eta, f_\eta) + \frac{\eta^2\alpha\beta}{\alpha^2+\eta^2}(f_\eta, f_\eta) + \frac{\eta^4}{\alpha^2+\eta^2}(f_\eta, f_\eta)\right)$$

$$\geq \lim_{\eta\uparrow\infty}\frac{\eta^4}{\alpha^2+\eta^2}(f_\eta, f_\eta) = \lim_{\eta\uparrow\infty}\frac{\eta^2}{\alpha^2+\eta^2}\eta^2(f_\eta, f_\eta) = \infty.$$

We mention that we have also used Theorem 4 ($A|_R \geq 0$) and (33).

Finally, assume that $V(z) \in S^\circ_-[R,(\alpha,\beta)]$. We will show that $((\alpha - z)/(\beta - z))V(z) \in S^\circ(R)$. Setting $w = (\alpha - i\eta)(\beta - i\eta)$ and using (13) we get

$$\begin{aligned}
(\operatorname{Im} wV(i\eta)f, f) &= \frac{1}{2}\left(\left(wK^*(A|_R - i\eta I)^{-1}K - \bar{w}K^*(A|_R + i\eta I)^{-1}K\right)f, f\right)\\
&= \frac{1}{2i}\left(\left(w(A|_R - i\eta I)^{-1} - \bar{w}(A|_R + i\eta I)^{-1}\right)Kf, Kf\right)\\
&= \left(\frac{w - \bar{w}}{2i}A|_R f_\eta, f_\eta\right) + \frac{w - \bar{w}}{2}\eta(f_\eta, f_\eta)\\
&= \frac{\eta(\alpha - \beta)}{|\beta - i\eta|^2}(A|_R f_\eta, f_\eta) + \frac{\eta(\alpha\beta + \eta^2)}{|\beta - i\eta|^2}(f_\eta, f_\eta),
\end{aligned} \tag{35}$$

where f_η has the form (32). But $(A|_R f_\eta, f_\eta) = \left(\frac{\operatorname{Im} i\eta V(i\eta)}{\eta}f, f\right)$. In fact,

$$\begin{aligned}
\left(\frac{\operatorname{Im} i\eta V(i\eta)}{\eta}f, f\right) &= \left(\frac{i\eta V(i\eta) + i\eta V(-i\eta)}{2i\eta}f, f\right)\\
&= \left(\frac{i\eta K^*(A|_R - i\eta I)^{-1}K + i\eta K^*(A|_R + i\eta I)^{-1}K}{2i\eta}f, f\right)\\
&= \left(\frac{(A|_R + i\eta I)^{-1}(i\eta(A|_R + i\eta I) + i\eta(A|_R - i\eta I))(A_R - i\eta I)^{-1}}{2i\eta}Kf, Kf\right)\\
&= (A|_R f_\eta, f_\eta).
\end{aligned}$$

Regarding (23) we obtain

$$\left(\frac{\operatorname{Im} i\eta V(i\eta)}{\eta}f, f\right) = (\gamma f, f) + \int_0^\infty \frac{t}{t^2 + \eta^2}(d\sigma(t)f, f).$$

Using Lebesgue's Dominated Convergence Theorem (cf. [6]), it follows

$$\lim_{\eta\uparrow\infty}(A|_R f_\eta, f_\eta) = \lim_{\eta\uparrow\infty}\left(\frac{\operatorname{Im} i\eta V(i\eta)}{\eta}f, f\right) = (\gamma f, f) < \infty. \tag{36}$$

Now (35) and (36) imply

$$\lim_{\eta\uparrow\infty}\left(\eta \operatorname{Im}\frac{\alpha - i\eta}{\beta - i\eta}V(i\eta)f, f\right) = \lim_{\eta\uparrow\infty}\left(\frac{\eta^2(\alpha-\beta)}{\beta^2+\eta^2}(A|_R f_\eta, f_\eta) + \frac{\eta^2\alpha\beta}{\beta^2+\eta^2}(f_\eta, f_\eta) + \frac{\eta^2}{\beta^2+\eta^2}\eta^2(f_\eta, f_\eta)\right)$$

$$\geq \lim_{\eta\uparrow\infty}\left(\frac{\eta^2(\alpha-\beta)}{\beta^2+\eta^2}(A|_R f_\eta, f_\eta)+\frac{\eta^2}{\beta^2+\eta^2}\eta^2(f_\eta, f_\eta)\right)\geq\infty. \tag{37}$$

Thus the theorem is proved ∎

Theorem 8: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S^1_{\pm}[R,(\alpha,\beta)]$ if and only if the following two conditions hold:*

(i) $V(z)\in S^{-1}(R)$.

(ii) $\frac{\beta-z}{\alpha-z}V(z)\in S^{-1}(R)$ $\left(\frac{\alpha-z}{\beta-z}V(z)\in S^{-1}(R)\right.$, *respectively*$\left.\right)$.

Proof: The sufficiency is obvious (compare the proof of Theorem 7). Now assume that $V(z)\in S^1_+[R,(\alpha,\beta)]$. Clearly, $V(z)\in S^{-1}(R)$. We will show that $((\beta-z)/(\alpha-z))V(z)\in S^{-1}(R)$. Since $V(z)$ is realizable, the relation (13) holds. In this relation, the operator $A|_R$ is a bounded linear operator from $\mathfrak{H}_+$ into $\mathfrak{H}_-$. Let R be the (isometric) Riesz-Berezanskii operator, which arises in a natural way in the therory of nested Hilbert spaces (cf. [2]). The operator R has the properties $(f,g)_-=(Rf,Rg)_+=(Rf,g)=(f,Rg)$ $(f,g\in\mathfrak{H}_-)$. Thus

$$\begin{aligned}|(A|_R f_\eta, f_\eta)| &= |(\mathbb{R}A|_R f_\eta, f_\eta)_+|\leq\|\mathbb{R}A|_R\|\,\|f_\eta\|_+^2=\|\mathbb{R}A|_R\|(\|f_\eta\|^2+\|A^*f_\eta\|^2)\\ &=\|\mathbb{R}A|_R\|(\|f_\eta\|^2+\eta^2\|Pf_\eta\|^2)\leq\|\mathbb{R}A|_R\|(1+\eta^2)\|f_\eta\|^2,\end{aligned} \tag{38}$$

where P is the orthoprojector of $\mathfrak{H}$ onto $\overline{\mathfrak{D}(A)}$ and the operator A is the maximal common Hermitian part of the operators T and T^* that arise realizing the operator-valued function $V(z)$ as a transfer function of the conservative scattering system (7). Furthermore, as it was noted in the proof of Theorem 7, we have

$$\lim_{\eta\uparrow\infty}\left(\eta\,\mathrm{Im}\frac{\beta-i\eta}{\alpha-i\eta}V(i\eta)f,f\right)=\lim_{\eta\uparrow\infty}\left(\frac{\eta^2(\beta-\alpha)}{\alpha^2+\eta^2}(A|_R f_\eta, f_\eta)+\frac{\eta^2\alpha\beta}{\alpha^2+\eta^2}(f_\eta,f_\eta)+\frac{\eta^4}{\alpha^2+\eta^2}(f_\eta,f_\eta)\right)$$

Since $\int_0^\infty(d\sigma(t)f,f)=\lim_{\eta\uparrow\infty}\eta^2(f_\eta,f_\eta)<\infty$, the realization (38) implies $\lim_{\eta\uparrow\infty}\left(\eta\,\mathrm{Im}\frac{\beta-i\eta}{\alpha-i\eta}V(i\eta)f,f\right)<\infty$. Now assume $V(z)\in S^1_-[R,(\alpha,\beta)]$. We will show that $((\alpha-z)/(\beta-z))V(z)\in S^{-1}(R)$. Using (37) we get

$$\lim_{\eta\uparrow\infty}\left(\eta\,\mathrm{Im}\frac{\alpha-i\eta}{\beta-i\eta}V(i\eta)f,f\right)\leq\lim_{\eta\uparrow\infty}\left(\frac{\eta^2\alpha\beta}{\beta^2+\eta^2}(f_\eta,f_\eta)+\frac{\eta^2}{\beta^2+\eta^2}(f_\eta,f_\eta)\right)<\infty.$$

In order to obtain the last estimates we have used the fact that $A|_R\geq 0$ ∎

The following theorem is an immediate consequence of Theorems 7 and (8).

Theorem 9: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S^{01}_{\pm}[R,(\alpha,\beta)]$ if and only if the following two conditions hold:*

(i) $V(z)\in S^{0-1}(R)$.

(ii) $\frac{\beta-z}{\alpha-z}V(z)\in S^{0-1}(R)$ $\left(\frac{\alpha-z}{\beta-z}V(z)\in S^{0-1}(R)\right.$, *respectively*$\left.\right)$.

Combining the results of Theorems 7 - 9 and regarding (27) we obtain the following

Theorem 10: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S_{\pm}[R,(\alpha,\beta)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S(R)$.

(ii) $\frac{\beta - z}{\alpha - z} V(z) \in S(R)$ $\left(\frac{\alpha - z}{\beta - z} V(z) \in S(R)\text{, respectively}\right)$.

Theorem 11: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S^{\circ}_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S^{\circ}(R)$.

(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R)$ $\left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S^{\circ}(R)\text{, respectively}\right)$.

Proof: The sufficiency of the conditions is easy to prove. Since $S^{\circ}(R) \subset S$, we have $V(z) \in S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ because of Theorem 2. But since $V(z) \in S^{\circ}(R)$, we obtain $V(z) \in S^{\circ}_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$.

The necessity is proved with aid of mathematical induction. For $n = 1$ the result was proved in Theorem 7. Now assume that for $m = p$ from $V(z) \in S^{\circ}_{+}[R,\bigcup_{j=1}^{p}(\alpha_j,\beta_j)]$ it follows that

(i) $V(z) \in S^{\circ}(R)$.

(ii) $\prod_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R)$.

We will show that this fact remains true for $n = p + 1$. Assume that $V(z) \in S^{\circ}_{+}[R,\bigcup_{j=1}^{p+1}(\alpha_j,\beta_j)]$. Then, clearly, $V(z) \in S^{\circ}_{+}[R,\bigcup_{j=1}^{p}(\alpha_j,\beta_j)]$ and hence

(i) $V(z) \in S^{\circ}(R)$.

(ii) $\prod_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R)$.

Since $V(z)$ is holomorphic and positive on the interval $(\alpha_{p+1},\beta_{p+1})$, we obtain

$$\prod_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}_{+}[R,(\alpha_{p+1},\beta_{p+1})].$$

Hence by Theorem 7,

$$\frac{\beta_{p+1} - z}{\alpha_{p+1} - z} \prod_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) = \prod_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R).$$

An analogous proof works in the case $V(z) \in S^{\circ}_{-}[R,\bigcup_{j=1}^{n}(\alpha_j,\beta_j)]$ ∎

It is not hard to see that for the classes

$$S^{1}_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)],\ S^{\circ 1}_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \text{ and } S_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$$

analogous results hold. Combining the above stated theorems we get the following

Theorem 12: *A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space E, belongs to the class $S_{+}[R,\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_{-}[\bigcup_{k=1}^{n}(c_k,d_k)]$ if and only if the following two conditions hold:*

(i) $V(z) \in S(R)$.

(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} \prod_{k=1}^{n} \frac{c_k - z}{d_k - z} V(z) \in S(R)$.

Note that analogous results can be formulated for the classes

$S_+^{\circ}[R, \bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_-^{\circ}[\bigcup_{k=1}^{m}(c_k,d_k)]$,

$S_+^{1}[R, \bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_-^{1}[\bigcup_{k=1}^{m}(c_k,d_k)]$,

$S_+^{\circ 1}[R, \bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_-^{\circ 1}[\bigcup_{k=1}^{m}(c_k,d_k)]$.

Definition: Let A be a symmetric operator in a Hilbert space $\mathfrak{H}$. The interval (α,β) is called a *gap* of the operator A if

$$\left\| Af - \frac{\alpha+\beta}{2} f \right\| \geq \frac{\beta-\alpha}{2} \|f\| \text{ for all } f \in \mathfrak{D}(A). \tag{39}$$

Theorem 13: *Let $V(z)$ b a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$, where (6) holds. Let $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $A \geq 0$. Let (α,β) be an arbitrary interval of the positive semi-axis. Then $V(z) \in S_+[(\alpha,\beta)]$ and (α,β) is a gap of the operator A if and only if the following two conditions hold*:

(i) $A|_R \geq 0$.

(ii) $(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi) \geq 0 \ \forall\, \varphi \in \mathfrak{H}_+$. (40)

Proof: Assume that (40) holds. Let $\{z_i\}_{i=1}^{p}$ be an arbitrary set of non-real complex numbers such that $z_i \neq z_l$. Let N_{z_k} be the deficiency space of the operator A and $\varphi_i \in N_{z_i}$. Set $\varphi = \sum_{i=1}^{p}(\alpha - z_i)^{-1}\varphi_i$. Since $A^*\varphi_i = z_i\varphi_i$, we obtain from (40)

$$\begin{aligned}
&(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi) \\
&= \sum_{i,l=1}^{p} \frac{1}{(\alpha - z_i)(\alpha - \overline{z}_l)}(A|_R\varphi_i,\varphi_l) + \frac{\alpha\beta}{\beta-\alpha}\sum_{i,l=1}^{p} \frac{1}{(\alpha - z_i)(\alpha - \overline{z}_l)}(\varphi_i,\varphi_l) \\
&\quad - \frac{\beta}{\beta-\alpha}\sum_{i,l=1}^{p} \frac{1}{(\alpha - z_i)(\alpha - \overline{z}_l)}(A^*\varphi_i,\varphi_l) - \frac{\beta}{\beta-\alpha}\sum_{i,l=1}^{p} \frac{1}{(\alpha - z_i)(\alpha - \overline{z}_l)}(\varphi_i,A^*\varphi_l) \\
&\quad + \frac{1}{\beta-\alpha}\sum_{i,l=1}^{p} \frac{1}{(\alpha - z_i)(\alpha - \overline{z}_l)}(A^*\varphi_i,A^*\varphi_l) \\
&= \frac{1}{\beta-\alpha}\sum_{i,l=1}^{p}\left(\left(\frac{\beta-\alpha}{(\alpha - z_i)(\alpha - \overline{z}_l)}A|_R + \frac{\alpha\beta - \beta(z_i + \overline{z}_l) + z_i\overline{z}_l}{(\alpha - z_i)(\alpha - \overline{z}_l)} I\right)\varphi_i,\varphi_l\right) \\
&= \frac{1}{\beta-\alpha}\sum_{i,l=1}^{p}(B(z_i,z_l)\varphi_i,\varphi_l) \geq 0,
\end{aligned}$$

where $B(\lambda,\mu)$ has the form (16). Thus Theorem 4 yields $V(z) \in S_+[(\alpha,\beta)]$.

Now we will show that (α,β) is a gap of A. In fact, if the vector φ of the inequality (40) belongs to $\mathfrak{D}(A)$, then with regard to the inclusions $A^* \supset A$ and $A|_R \supset A$ we obrain

$$(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi)$$

$$= (A\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) + \frac{2\beta}{\beta-\alpha}(A\varphi,\varphi) + \frac{1}{\beta-\alpha}(A\varphi,A\varphi)$$

$$= \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) + \frac{1}{\beta-\alpha}(A\varphi,A\varphi) - \frac{\alpha+\beta}{\beta-\alpha}(A\varphi,\varphi) \ge 0.$$

This implies

$$(\alpha+\beta)(A\varphi,\varphi) \le \alpha\beta(\varphi,\varphi) + (A\varphi,A\varphi). \tag{41}$$

But conditions (39) and (41) are equivalent. In fact, if (39) holds, we get

$$\left(\left(A - \frac{\alpha+\beta}{2}I\right)\varphi,\left(A - \frac{\alpha+\beta}{2}I\right)\varphi\right) \ge \left(\frac{\beta-\alpha}{2}\right)^2(\varphi,\varphi),$$

hence

$$(A\varphi,A\varphi) - (\alpha+\beta)(A\varphi,\varphi) + \left(\frac{\alpha+\beta}{2}\right)^2(\varphi,\varphi) \ge \left(\frac{\beta-\alpha}{2}\right)^2(\varphi,\varphi)$$

and $(\alpha+\beta)(A\varphi,\varphi) \le \alpha\beta(\varphi,\varphi) + (A\varphi,\varphi)$, i.e., (41). It is not hard to see that the converse conclusion is also true. Thus, the interval (α,β) is a gap of the operator A.

Now let $V(z) \in S_+[(\alpha,\beta)]$ and the interval (α,β) be a gap of the operator A. Then by Theorem 4 it holds (15). As it was proved above, this yields the inequality (40) for all vectors φ of the form

$$\varphi = \sum_{i=1}^{p}(\alpha - z_i)^{-1}\varphi_i, \tag{42}$$

where $\{z_i\}_{i=1}^{p}$ be an arbitrary set of non-real complex numbers such that $z_i \neq z_l$ and φ_i is an arbitrary vector of the deficiency space N_{z_i}. Let $\mathfrak{H}_1 = \bigvee_{z\neq\bar z} N_z$, where the closure is taken with respect to the metric of the space $\mathfrak{H}$. Then, clearly, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where the subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are invariant subspaces of the operator A and the operator $A_2 = A|_{\mathfrak{H}_2}$ is selfadjoint. Thus, $A = A_1 \oplus A_2$, where $A_1 = A|_{\mathfrak{H}_1}$. It is easy to see that $A^* = A_1^* \oplus A_2$. It follows that each vector $\varphi \in \mathfrak{H}_+$ can be representeds in the form $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in \overline{\mathfrak{D}(A_1^*)}$ and $\varphi_2 \in \mathfrak{D}(A_2)$. Since the operators $A|_R$ and A^* are continuous operators from $\mathfrak{H}_+$ into $\mathfrak{H}_-$, we can extend the inequality (40) from all vecotrs of the form (42) to all vectors $\varphi \in \overline{\mathfrak{D}(A_1^*)}$. It is easy to see that an arbitrary selfadjoint extension $\tilde{A}$ of the operator A has the form $\tilde{A} = \tilde{A}_1 \oplus A_2$, wherte $\tilde{A}_1$ is a selfadjoint extension of the operator A_1 in the space $\mathfrak{H}_1$. Since by assumption the interval (α,β) is a gap of the operator A, it is also a gap of the operator A_2. Thus, for the operator A_2 it holds (39) and hence (41), as it was shown above. Setting $\varphi = \varphi_1 + \varphi_2$ in (40) we obtain

$$(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi)$$

$$= (A|_R\varphi_1,\varphi_1) + \frac{\alpha\beta}{\beta-\alpha}(\varphi_1,\varphi_1) - \frac{\beta}{\beta-\alpha}(A^*\varphi_1,\varphi_1) - \frac{\beta}{\beta-\alpha}(\varphi_1,A^*\varphi_1) + \frac{1}{\beta-\alpha}(A^*\varphi_1,A^*\varphi_1)$$

$$+ (A|_R\varphi_2,\varphi_2) + \frac{\alpha\beta}{\beta-\alpha}(\varphi_2,\varphi_2) - \frac{\beta}{\beta-\alpha}(A^*\varphi_2,\varphi_2) - \frac{\beta}{\beta-\alpha}(\varphi_2,A^*\varphi_2) + \frac{1}{\beta-\alpha}(A^*\varphi_2,A^*\varphi_2)$$

$$= \left((A|_R\varphi_1,\varphi_1) + \frac{\alpha\beta}{\beta-\alpha}(\varphi_1,\varphi_1) - \frac{\beta}{\beta-\alpha}(A_1^*\varphi_1,\varphi_1) - \frac{\beta}{\beta-\alpha}(\varphi_1,A_1^*\varphi_1) + \frac{1}{\beta-\alpha}(A_1^*\varphi_1,A_1^*\varphi_1)\right)$$

$$+ \left(\frac{\alpha\beta}{\beta-\alpha}(\varphi_2,\varphi_2) + \frac{1}{\beta-\alpha}(A_2\varphi_2,A_2\varphi_2) - \frac{\alpha+\beta}{\beta-\alpha}(A_2\varphi_2,\varphi_2)\right) \ge 0.$$

We note that the last inequality holds since the terms in the big brackets are non-negative ∎

Theorem 14: *Let $V(z)$ be a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$, where (6) holds. Let $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $A \geq 0$. Let (α,β) be an arbitrary interval of the positive semi-axis. Then $V(z) \in S_-[(\alpha,\beta)]$ and (α,β) is a gap of the operator A if and only if the following two conditions hold:*

(i) $A| \geq 0$.

(ii) $-(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi) \geq 0 \; \forall\varphi \in \mathfrak{H}_+$. (43)

Proof: Assume that (43) holds. Let $\{z_i\}_{i=1}$ be an arbitrary set of non-real complex numbers such that $z_i \neq z_l$. Let N_{z_i} be the feficiency space of the operator A and $\varphi_i \in N_{z_i}$. Setting $\varphi = \sum_{i=1}^{p}(\alpha - z_i)^{-1}\varphi_i$ in (43), we obtain with regard to $A^*\varphi_i = z_i\varphi_i$

$$-(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi)$$

$$= -\sum_{i,l=1}^{p}\frac{1}{(\beta-z_i)(\beta-\overline{z_l})}(A|_R\varphi_i,\varphi_l) + \frac{\alpha\beta}{\beta-\alpha}\sum_{i,l=1}^{p}\frac{1}{(\beta-z_i)(\beta-\overline{z_l})}(\varphi_i,\varphi_l)$$

$$-\frac{\alpha}{\beta-\alpha}\sum_{i,l=1}^{p}\frac{1}{(\beta-z_i)(\beta-\overline{z_l})}(A^*\varphi_i,\varphi_l) - \frac{\alpha}{\beta-\alpha}\sum_{i,l=1}^{p}\frac{1}{(\beta-z_i)(\beta-\overline{z_l})}(\varphi_i,A^*\varphi_l)$$

$$+\frac{1}{\beta-\alpha}\sum_{i,l=1}^{p}\frac{1}{(\beta-z_i)(\beta-\overline{z_l})}(A^*\varphi_i,A^*\varphi_l)$$

$$= \frac{1}{\beta-\alpha}\sum_{i,l=1}^{p}\left(\left(\frac{\alpha-\beta}{(\beta-z_i)(\beta-\overline{z_l})}A|_R + \frac{\alpha\beta-\alpha(z_i+\overline{z_l})+z_i\overline{z_l}}{(\beta-z_i)(\beta-\overline{z_l})}I\right)\varphi_i,\varphi_l\right)$$

$$= \frac{1}{\beta-\alpha}\sum_{i,l=1}^{p}\big(B(z_i,z_l)\varphi_i,\varphi_l\big) \geq 0,$$

where $B(\lambda,\mu)$ has the form (20). This implies the inclusion $V(z) \in S_-[(\alpha,\beta)]$ by Theorem 5.

We will now show that (α,β) is a gap of the operator A. In fact, if the vector φ in the inequality (43) belongs to $\mathfrak{D}(A)$, then with regard to the inclusions $A^* \supset A$ and $A|_R \supset A$ we obtain

$$-(A|_R\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\alpha}{\beta-\alpha}(\varphi,A^*\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi)$$

$$= -(A\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{2\alpha}{\beta-\alpha}(A\varphi,\varphi) + \frac{1}{\beta-\alpha}(A^*\varphi,A^*\varphi)$$

$$= \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) + \frac{1}{\beta-\alpha}(A\varphi,A\varphi) - \frac{\alpha+\beta}{\beta-\alpha}(A\varphi,\varphi) \geq 0.$$

This yields $(\alpha+\beta)(A\varphi,\varphi) \leq \alpha\beta(\varphi,\varphi) + (A\varphi,A\varphi)$. Thus, the relation (41) is true for the operator A. As it was shown above, this implies that the interval (α,β) is a gap of A. The necessity part can be proved in an analogous way as the necessity part of Theorem 13 ∎

As a corollary of Theorems 13 and 14 we obtain the following general

Theorem 15: *Let $V(z)$ be a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$, where (6) holds. Let $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $A \geq 0$. Let (α_j,β_j) $(j = 1,\dots,m)$ and (c_k, d_k) $(k = 1,\dots,n)$ two arbitrary sets of mutually disjoint intervals of the po-*

sitive semi-axis. Then $V(z) \in S_+[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_-[\bigcup_{k=1}^{n}(c_k,d_k)]$ *and all intervals* (α_j,β_j) *and* (c_k,d_k) *are gaps of the operator* $\mathbb{A}$ *if and only if the following tree conditions hold:*

(i) $\mathbb{A}_I \geq 0$.

$$\text{(ii)}\quad (\mathbb{A}_R\varphi,\varphi) + \frac{\alpha_j\beta_j}{\beta_j-\alpha_j}(\varphi,\varphi) - \frac{\beta_j}{\beta_j-\alpha_j}(\mathbb{A}^*\varphi,\varphi) - \frac{\beta_j}{\beta_j-\alpha_j}(\varphi,\mathbb{A}^*\varphi) + \frac{1}{\beta_j-\alpha_j}(\mathbb{A}^*\varphi,\mathbb{A}^*\varphi) \geq 0$$

for each $\varphi \in \mathfrak{H}_+$ *and all* $j = 1,\ldots,m$.

$$\text{(iii)}\quad -(\mathbb{A}_R\varphi,\varphi) + \frac{c_k d_k}{d_k-c_k}(\varphi,\varphi) - \frac{c_k}{d_k-c_k}(\mathbb{A}^*\varphi,\varphi) - \frac{c_k}{d_k-c_k}(\varphi,\mathbb{A}^*\varphi) + \frac{1}{d_k-c_k}(\mathbb{A}^*\varphi,\mathbb{A}^*\varphi) \geq 0$$

for each $\varphi \in \mathfrak{H}_+$ *and all* $k = 1,\ldots,n$.

Remark: Theorems 14 and 15 were obtained in collaboration with E. R. Tsekanovskii.

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