On Classes of Stieltjes Type Operator-Valued Functions with Gaps

V. E. TSEKANOVSKII

We introduce and investigate classes of operator-valued functions with gaps, which can be realized as fractional linear transformations of operator-valued transfer functions of conservative scattering systems.

Key words *Operator-valued functions, accretive extensions, conservative systems* AMS subject classification: 47 B 44

Classes of Stieltjes type operator-valued functions with gaps on the positive semi-axis (i.e., with intervals of holomorphy and definiteness) are considered. We prove criteria that a given function, whose values are operators in a finite-dimensional Hilbert space, belongs to these classes. Moreover, we investigate classes of Stieltjes type operator-valued functions which admit a realization, i.e., which can be represented as fractional linear transformations of operator-valued transfer functions of conservative scattering systems of the form Classes of Stieltjes type operator-valued functions with gaps on the positive semi-axis (i.e.,
with intervals of holomorphy and definiteness) are considered. We prove criteria that a given
function, whose values are opera

 $\Theta = (\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-, A, K, I, E)$

in \mathfrak{H} , and *T* is closed with dense domain of definition in \mathfrak{H} . 2. the subclass, where * Si,, (*T)* * (*T)* $\Theta = (\mathfrak{H}_2 \subset \mathfrak{H} \subset \mathfrak{H}_2, A, K, I, E)$
 $\text{Re } A \in [\mathfrak{H}_2, \mathfrak{H}_2]$, $\text{Im } A = KK^*, A \supset T \supset A, A^* \supset T$, and T is closed with dense domain of definition is
 $\text{Im } \text{the class of realizable Stieltjes type operator-
investigated:$

1. the subclass, where $\overline{\mathfrak{D}(A$

In the class of realizable Stieltjes type operator-valued functions the following subclasses are investigated:

1. the subclass, where $\overline{\mathfrak{D}(A)} = \mathfrak{H}, \mathfrak{D}(T) \dagger \mathfrak{D}(T^*)$
2. the subclass, where $\overline{\mathfrak{D}(A)} \dagger \mathfrak{H}, \mathfrak{D}(T) \dagger \mathfrak{D}(T^*)$

We prove analytical criteria for a given operator-valued function to belong to the mentioned subclasses (with gaps). These criteria are analoga, supplements, and refinements of some of the results stated by M.G. Krein and A.A. Nudel'man [7].

§ 1 The classes $\mathbf{S}_{\pm}[\bigcup_{j=1}^m (\alpha_j, \beta_j)]$ of operator-valued functions

According to M.G. Krein [8], a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space *E, will* be called a *Stieltjes type operator-valued function* if the following conditions hold: The classes $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_{j}, \beta_{j})]$ of operator-valued functions

ording to M.G. Krein [8], a function $V(z)$, whose values are operators in a finite-di-

Hilbert space E, will be called a *Stieltjes type operator-va*

1. $V(z)$ is holomorphic on $Ext[0,\infty) = \{z: z \in [0,\infty)\}$

 $2. V(z) \ge 0$ for $z < 0$

3. $V(z)$ is an operator-valued R-function, i.e., $Im V(z)/Im z \ge 0$.

The class of Stieltjes type operator-valued functions will be denoted by S.

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 Definition: By $S_x \left[\bigcup_{j=1}^{m} (\alpha_j, \beta_j) \right]$ we denote the class of functions $V(z)$, whose values are

ators in a finite-dimensional Hilbert space E, such that the following two conditions hold:

1. operators in a finite-dimensional Hubert space *E,* such that the following two conditions hold: 1. $V(z) \in S$.

2. $V(z)$ is holomorphic and positive on all intervals (α_j, β_j) , i.e., $(V(z)f, f) > 0$ for all $f \in E$, $f \neq 0$, and all $z \in (\alpha_j, \beta_j)$ ($V(z)$ is holomorphic and negative on the intervals (α_j, β_j) , respectively, i.e., $(V(z)f, f) < 0$ for all $f \in E$, $f \neq 0$, and all $z \in (\alpha_i, \beta_i)$. **Definition:** By $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_{j}, \beta_{j})]$ we denote the class of functions $V(z)$, whose values are rators in a finite-dimensional Hilbert space *E*, such that the following two conditions hold:

1. $V(z) \in S$.

2. $V(z)$: By $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_{j}, \beta_{j})]$
finite-dimensional Hil
i.
bolomorphic and posit
 $\epsilon(\alpha_{j}, \beta_{j}) (V(z)$ is holo
 ϵ 0 for all $f \in E, f \neq 0$,
A scalar function V
(conditions hold:
S.
 $\frac{z}{z}V(z) \in S \left(\prod_{j=1}^{m} \frac{\alpha$ **Proof:** First we consider the class $S_{\text{L}}[\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})]$, Let (i) and (ii) be fulfilled. Since (i), a
 Proof: First we consider the class $S_{\text{L}}[\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})]$, Let (ii) and only if the
 Proof: F

following two conditions hold: **Theorem 1:** A scalar function $V(z)$ belongs to the classes $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ if and only if the
 Sowing two conditions hold:

(i) $V(z) \in S$.

(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$ $\left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \$

(i) $V(z) ∈ S$.

From the equation
$$
f(x_2) = \frac{1}{2} \int_{0}^{2\pi} (x \cdot \alpha_j, \beta_j) \cdot (x \cdot \alpha_j, \beta_j)
$$
.

\nTheorem 1: A scalar function $V(z)$ belongs to the classes S_1 with two conditions hold:

\n(i) $V(z) \in S$.

\n(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$ $\left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S, \text{ respectively} \right)$.

\nProof: First, we consider the class $S_{+} \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} \bigcup_{j=1}^{m} f(z) \bigcup_{j$

well known theorem (see *[71)* gives us

$$
V(z) = c \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) f(t) dt , \qquad (2)
$$

 $<$ ∞ . Moreover, the representation (2) is unique. It is not hard to see that

(i)
$$
V(z) \in S
$$
.
\n(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S, \text{ respectively} \right)$.
\n**Proof:** First we consider the class $S = \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \bigcup_{j=1}^{m} (x_j, \beta_j) \bigcup_{j$

Since (ii), we get.

$$
\alpha_{j} = 2 \qquad j \qquad \alpha_{j} \qquad (3)
$$
\n
$$
e (ii), we get.
$$
\n
$$
\prod_{j=1}^{m} \frac{\beta_{j} - z}{\alpha_{j} - z} V(z) = c_{1} \exp \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^{2}} \right) f_{1}(t) dt,
$$
\n
$$
\sum_{j=1}^{m} \alpha_{j} \frac{\beta_{j} - z}{\alpha_{j} - z} V(z) = c_{1} \exp \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^{2}} \right) f_{2}(t) dt,
$$
\n
$$
V(z) = c_{2} \exp \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^{2}} \right) f_{2}(t) dt,
$$
\n
$$
c_{1} = 2 \qquad \text{and} \qquad c_{2} = 2 \qquad \text{and} \qquad c_{3} = 2 \qquad \text{and} \qquad c_{4} = 2 \qquad \text{and} \qquad c_{5} = 2 \qquad \text{and} \qquad c_{6} = 2 \qquad \text{and} \qquad c_{7} = 2 \qquad \text{and} \qquad c_{8} = 2 \qquad \text{and} \qquad c_{9} = 2 \qquad \text{and} \qquad c_{1} = 2 \qquad \text{and} \qquad c_{1} = 2 \qquad \text{and} \qquad c_{2} = 2 \qquad \text{and} \qquad c_{3} = 2 \qquad \text{and} \qquad c_{4} = 2 \qquad \text{and} \qquad c_{5} = 2 \qquad \text{and} \qquad c_{6} = 2 \qquad \text{and} \qquad c_{7} = 2 \qquad \text{and} \qquad c_{8} = 2 \qquad \text{and} \qquad c_{9} = 2 \qquad \text{and} \qquad c_{1} = 2 \qquad \text{and} \qquad c_{1} = 2 \qquad \text{and} \qquad c_{1} = 2 \qquad \text{and} \qquad c_{2} = 2 \qquad \text{and} \qquad c_{3} = 2 \qquad \text{and} \qquad c_{4} = 2 \qquad \text{and} \qquad c_{
$$

in an analogous way, where $c_1 > 0$ and the function $f_i(t)$ has the same properties as $f(t)$. Using (3) and (4), we obtain

$$
V(z) = c_2 \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) f_2(t) dt,
$$

re

$$
f_2(t) = \begin{cases} f_1(t) & \text{for } t \in \mathbb{R} \setminus \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \\ f(t) = 1 & \text{for } t \in \mathbb{L} \setminus \mathbb{R} \end{cases}
$$

where

and (4), we obtain
\n
$$
V(z) = c_2 \exp \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) f_2(t) dt
$$
\n
$$
f_2(t) = \begin{cases} f_1(t) & \text{for } t \in \mathbb{R} \setminus \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \\ f_1(t) - 1 & \text{for } t \in \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \end{cases}
$$

Because of the uniqueness of the representation (2) it follows

in an analogous way, where
$$
c_1 > 0
$$
 and the function $f_1(t)$ has the same properties as $f(t)$. Using
\n(3) and (4), we obtain
\n
$$
V(z) = c_2 \exp \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f_2(t) dt,
$$
\nwhere
\n
$$
f_2(t) = \begin{cases} f_1(t) & \text{for } t \in \mathbb{R} \setminus \bigcup_{j=1}^m (\alpha_j, \beta_j) \\ f_1(t) - 1 & \text{for } t \in \bigcup_{j=1}^m (\alpha_j, \beta_j) \end{cases}.
$$
\nBecause of the uniqueness of the representation (2) it follows
\n
$$
V(z) = c \exp \int_{\mathbb{R} \setminus \bigcup_{j=1}^m (\alpha_j, \beta_j)} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f(t) dt,
$$
\nwhere $c > 0, 0 \le f(t) \le 1$ a.e. on $\mathbb{R} \setminus \bigcup_{j=1}^m (\alpha_j, \beta_j)$. By a well known theorem (see [7]), the relation
\n(5) implies that $V(z)$ is holomorphic and positive on all intervals (α_j, β_j) .
\nNow assume that $V(z) \in S_x \bigcup_{j=1}^m (\alpha_j, \beta_j) \bigcup_{j=1}^m F_1(t) dt$,
\ninclusion $((\beta_1 - z)/(\alpha_1 - z)) V(z) \in S_x \bigcup_{j=1}^m (\alpha_j, \beta_j) \bigcup_{j=1}^m F_1(t) dt$,
\n $\frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha_j - z} \right) f_1(t) dt$

(5) implies that $V(z)$ is holomorphic and positive on all intervals (α_j, β_j) . where $c > 0$, $0 \le f(t) \le 1$ a.e. on $\mathbb{R} \setminus \bigcup_{j=1}^{m} (\alpha_j, \beta_j)$. By a well known theorem (see [7]), the relation (5) implies that $V(z)$ is holomorphic and positive on all intervals (α_j, β_j) .
Now assume that $V(z) \in S_x \big$

we get

et
\n
$$
\frac{\operatorname{Im} V_{1}(\xi)}{\operatorname{Im} \xi} = \frac{|\alpha_{1} - z|^{2}}{\beta_{1} - \alpha_{1}} \frac{\operatorname{Im} V(z)}{\operatorname{Im} z} \ge 0,
$$
\n
$$
\frac{W(E) \le P_{1} \le P_{1}}{\operatorname{Im} V(E)} \le P_{2} \text{ find } W(E) \le 0.
$$

hence, $V_1(\xi) \in R$, i.e., $V_1(\xi)$ is an R-function. It is not hard to see that $z \in (\alpha, \beta)$, implies $\xi \in (-\infty, 0)$, and since $V(z) \in S$ ₄ $[\alpha, \beta, \bar{j}]$, it follows $V(\xi) \in S$. Now a theorem of M.G. Krein (see [8]) gives *us a Fig. ii* $\chi(\mathbf{z}) \in \mathcal{S}_+$ ($\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$), it issues $\mathbf{v}_1(\mathbf{z}) \in \mathcal{S}$. Now a theorem of M.O. Krein (see Loj) gives $\mathcal{S}_+^n(\mathbf{z}) \in R$. Thus $((\beta_1 - z)/(\alpha_1 - z))V(z) \in R$. Since $((\beta_1 - z)/(\alpha_1 - z))V(z) \$ obtain $((β, -z)/(α, -z))V(z) \in S$. *k*
 *k*₁ *k*₂ *k*₂ = $\frac{|\alpha_1 - z|^2}{\beta_1 - \alpha_1}$
 *k*₁ *k*₂ $\in \beta_1 - \alpha_1$
 *k*₂ $\in \mathbb{Z}$ *k*₂ $\in \mathbb{Z}$ *k*₂ $\in \mathbb{Z}$ *k*₂ $\in \mathbb{Z}$
 *k*₂ $\in \mathbb{Z}$ *k*₂ $\in \mathbb{Z}$ *k*₂ $\in \mathbb{Z}$
 *k*₂ \frac Let $V_1(\zeta) \in R$, i.e., $V_1(\zeta)$ is an R-function

ince $V(z) \in S$, $[(\alpha_1, \beta_1)]$, it follows $V_1(\zeta) \in R$. Thus $((\beta_1 - z)/(\alpha_1 - z))V(z)$
 $n((\beta_1 - z)/(\alpha_1 - z))V(z) \in S$.

We will show that the implication
 $\prod_{j=1}^k \frac{\beta_j - z}{\alpha_j - z} V$ Im ξ β_1 -
 J:e, $V_1(\xi) \in R$, i.e., since $V(z) \in S$, $V_1(\xi) \in R$. Thus ((
 $\int \ln \left((\beta_1 - z) / (\alpha_1 - z) \right)$
 $\int \frac{k}{\beta_1 - z} \frac{\beta_j - z}{\alpha_j - z} V(z)$
 J: $\int \frac{1}{\beta_1 - z} \frac{\beta_j - z}{\alpha_j - z} V(z)$
 Jie. In fact, it is a

We will show that the implication

$$
\prod_{j=1}^k \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \ (1 \leq k \leq m) \implies \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S
$$

is true. In fact, it is not hard to see that

$$
\prod_{j=1}^k \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_{+}[\alpha_{k+1}, \beta_{k+1}].
$$

Now by analogous arguments as above we get

$$
\frac{\beta_{k+1}-z}{\alpha_{k+1}-z}\prod_{j=1}^k\frac{\beta_j-z}{\alpha_j-z}\,V(z)=\prod_{j=1}^{k+1}\frac{\beta_j-z}{\alpha_j-z}\,V(z)\in S\,.
$$

Thus the first part of the theorem is proved.

Now let $V(z) \in S_{-}[\bigcup_{i=1}^{m}(\alpha_i,\beta_i)]$. We will show that the ()-part of (ii) is true. It is not hard to see (cf. [6]) that $V(z) \in R$ if and only if $-V(z)^{-1} \in R$. Thus, the relation $V(z) \in S_{-}[(\alpha, \beta, \beta)]$ implies $-V(z)^{-1} \in R$ and $-V(z)^{-1} > 0$, $z \in (\alpha_1, \beta_1)$. Setting $\xi = (\beta_1 - z)/(\alpha_1 - z)$ and $V_1(\xi) = -V(z)^{-1}$,
we get
 $\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)}\right) \in R$, hence $-\left(\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)}\right)\right)^{-1} = \frac{\alpha_1 - z}{\beta_1 - z}V(z) \in R$. we get by analogous arguments as above we get
 $\frac{\beta_{k+1} - z}{\beta_{k+1} - z}$ $\prod_{j=1}^k \frac{\beta_j - z}{\alpha_j - z} V(z) = \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S$.

the first part of the theorem is proved.

Now let $V(z) \in S$. $\bigcup_{j=1}^m (\alpha_j, \beta_j) \big]$. We will by analogous
 $\frac{\beta_{k+1} - z}{\alpha_{k+1} - z}$ $\prod_{j=1}^{k} \frac{\beta_j}{\alpha_j}$

the first part

Now let $V(z) \in$
 e (cf. [6]) tha

es $-V(z)^{-1} \in I$

et
 $\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)}\right)$
 $\frac{\alpha_1 - z}{\beta_1 - z} V(z)$ *a* ive get
 $\frac{-z}{-z}V(z) \in S$

proved.

We will show

only if $-V(z)$
 $z \in (\alpha_1, \beta_1)$. Se
 $\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)}\right)$

it follows $\frac{\alpha_1}{\beta_1}$

it follows $\frac{\alpha_1}{\beta_1}$ Thus the first part of the theorem is proved.

Now let $V(z) \in S_{-}[\bigcup_{j=1}^{m}(\alpha_{j}, \beta_{j})]$. We will show that

to see (cf. [6]) that $V(z) \in R$ if and only if $-V(z)^{-1} \in \text{implies } -V(z)^{-1} \in R$ and $-V(z)^{-1} > 0$, $z \in (\alpha_{1}, \beta_{1})$. Setti

$$
\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)} \right) \in R, \text{ hence } -\left(\frac{\beta_1 - z}{\alpha_1 - z} \left(-\frac{1}{V(z)} \right) \right)^{-1} = \frac{\alpha_1 - z}{\beta_1 - z} V(z) \in R.
$$

 S. Using an analogous induction method as in the first part of the proof we obtain the ()-part of (ii).

Now assume (i) and (ii)/()-part. Consider the function $-V(z)^{-1} \in R$ and use analogous arguments as in the proof of the sufficiency in the first part. This gives us that $-V(z)^{-1}$ is holomorphic and positive on all intervals (α_j,β_j) . Thus the theorem is proved \blacksquare $V(z)$ ℓ . ℓ , ℓ
 $\frac{z}{z}V(z) \ge 0$ if $z \in \mathbb{R}$

in the first part of

sume (i) and (ii) ℓ

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in the proof of th

in 2: A function
 $\log s$ to the class
 $\log t \ge 0 \le S$.
 $\frac{\beta_j - z}{\alpha_j - z}V(z) \in S$
 w assume (i) and
 s as in the proof

and positive on
 orem 2: A funct
 b, belongs to the
 $V(z) \in S$.
 $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)$

of: Let $V(z) \in S$

Theorem 2: *A function V(z), whose values are operators in a finite-dimensional Hubert* $space$ E, belongs to the classes $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ if and only if the following two conditions hold: *(i)* $V(z) \in S$.

 \sqrt{m} (ii) $\prod_{i=1}^{m} \frac{\beta_i - z}{\alpha_i - z} V(z) \in S$ $\left(\prod_{i=1}^{m} \frac{\alpha_i - z}{\beta_i - z} \right)$ 2: *A* function $V(z)$, whose
 ngs to the classes S_{\pm} $\bigcup_{j=1}^{m}$
 $\bigcup_{j=1}^{m}$
 $\frac{z}{z}$
 $\bigcup_{j=1}^{m}$ $\frac{\alpha_j - z}{\beta_j - z}$ $V(z)$ ϵ S , respectively

Proof: Let $V(z) \in S_x[\bigcup_{i=1}^m (\alpha_i, \beta_i)]$. Considering the scalar function $(V(z)f, f) \in S$ and using Theorem I we get

(ii)
$$
\prod_{j=1}^{m} \frac{b_j - z}{\alpha_j - z} V(z) \in S \left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S, \text{ re}
$$

\n**Proof:** Let $V(z) \in S_x \left[\bigcup_{j=1}^{m} (\alpha_j, \beta_j) \right]$. Considering
\norem 1 we get
\n
$$
\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} (V(z)f, f) = \left(\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)f, f \right) \in S.
$$

 $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} (V(z)f, f) = \left(\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)f, f \right) \in S.$

Hence, the operator-valued function $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)$ belongs to S.

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The sufficiency of conditions (i) and (ii) is trivial. The proof for the class $S_{-}[\bigcup_{i=1}^{m}(\alpha_{i},\beta_{i})]$ is analogous. Thus the theorem is proved \blacksquare

Definition: We will say that a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space *E,* belongs to the class **2.** *2.**V(z)**2.* *****2.**2. V(z) 2.**2. V(z) <i>2. 2. V(z) <i>2. 2. V(z) and (I) z E, belongs to the class* $S_{+}[\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})] \cap S_{-}[\bigcup_{j=1}^{n}(c_{j},d_{j})]$
2. $V(z)$ *6.**C L d*

 $S_{\text{L}}\left[\bigcup_{i=1}^m(\alpha_i,\beta_i)\right]\cap S_{\text{L}}\left[\bigcup_{i=1}^n(c_i,d_i)\right]$

if the following three conditions hold:

1. $V(z) \in S$.

3. $V(z)$ is holomorphic and negative on the intervals (c_k, d_k) ($k = 1, ..., n$).

Theorem 2 immediately implies the following

Theorem 3: A function $V(z)$, whose values are operators in a finite-dimensional Hilbert 2. $V(z)$ is holomorphic and positive on the intervals (α_j, β_j) ($j = 1, ..., m$).

3. $V(z)$ is holomorphic and negative on the intervals (c_k, d_k) ($k = 1, ..., n$).
 Theorem 2 immediately implies the following
 Theorem 3: *A fu two conditions hold:*

- *1. V(z) € S.*
- $\frac{\beta_j z}{\alpha_j z}$ $\prod_{k=1}^n \frac{c_k z}{d_k z} V(z) \in S$. $V(z) \in S$.
 $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z}$

§ 2 Realizable operator-valued functions of the class $S_{+}[\bigcup_{i=1}^{m}(\alpha_{i},\beta_{i})]$

Let *A* be a closed Hermitian operator in a Hilbert space \mathfrak{H} , whose defect numbers are finite two conditions hold:

1. $V(z) \in S$.

2. $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} \prod_{k=1}^{n} \frac{c_k - z}{d_k - z} V(z) \in S$.

§ 2 Realizable operator-valued functions of the class $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$

Let *A* be a closed Hermitian operator in adjoint operator. Clearly, $\overline{\mathfrak{D}(A^*)}$ = $\mathfrak{H}(A^*)$ (where the closure is taken in \mathfrak{H}). We set \mathfrak{H}_+ = $\mathfrak{D}(A^*)$ and introduce the scalar product $(f, g)_+ = (f, g) + (A^*f, A^*g) (f, g \in \mathfrak{H}_+)$. We consider the rigged Hilbert space $\mathfrak{H}_{-} \subset \mathfrak{H} \subset \mathfrak{H}_{-}$ (cf. [2]). Let A be a closed terminal operator in a thiort space sy, whose detected
and coincide. This operator can be considered as acting from $\mathfrak{H}_0 = \overline{2(A)}$ is
adjoint operator. Clearly, $\overline{2(A^*)} = \mathfrak{H}$ (where the closure

the following two conditions

1. $T \supseteq A$, $T^* \supseteq A$ (*A* is closed and Hermitian)

2. -i is a regular point of *^T*

are fulfilled.

We will say that a closed and densely defined operator *T* in \hat{y} belongs to the *class* Ω_A if following two conditions

1. $T \supset A$, $T^* \supset A$ (*A* is closed and Hermitian)

2. -i is a regular point of *T*

fulfill A bounded operator *A*l: $\mathfrak{H}_2 \rightarrow \mathfrak{H}_2$ (i.e., *A*l $\in [\mathfrak{H}_2, \mathfrak{H}_2]$) is called a *biextension* of the Hermitian operator *A* if *Al* \supset *A* and *Al*^{*} \supset *A*. Identifying the dual space of \mathfrak{H}_2 w By $A \rightarrow A$ and $A^* \rightarrow \mathfrak{H}$. (A is closed and Hermitian)
 $A \rightarrow A$, $T^* \rightarrow A$ (A is closed and Hermitian)
 $A \rightarrow A$ are $A \rightarrow \mathfrak{H}$. (i.e., $A^{\dagger} \in [\mathfrak{H}_+, \mathfrak{H}_-]$) is called a *biextension* of the Her-
 $A \rightarrow A$ bounded o Let *Ts A•* Then Al is called a (')-extension of Tif We will say that a closed and densely

bollowing two conditions

1. $T \supset A$, $T^* \supset A$ (*A* is closed and Her

2. -i is a regular point of *T*

ulfilled.

A bounded operator $A! : S_{1} \rightarrow S_{1}$ (i.e.

an operator A if $A \supset$

of Al (cf. [10, 11]). A selfadjoint biextension is called a *strong* biextension if $\hat{A} = \hat{A}^*$ (cf. [10,11]).

$$
A \supset T \supset A, \ A^{\bullet} \supset T^{\bullet} \supset A. \tag{6}
$$

Moreover, if $A\mathbf{I}_R = (A\mathbf{I} + A\mathbf{I}^*)/2$ is a strong selfadjoint biextension, then Al is called a *correct* ()-extension of *T.*

By definition, the class Λ_A denotes the set of all operators $T \in \Omega_A$ such that *A* coincides with the maximal common Hermitian part of *T* and *T.*

Definition: The operator colligation

Stieltjes Type Operator-Valued Functions	187		
Definition: The operator colligation	A^1	K	J
alled <i>rigged</i> if the following four conditions hold:			
1. $J = J^* = J^{-1}$ (dim $E < \infty$).	2. K is a bounded linear operator from E into S_{λ} .		
3. A^1 is a correct (\bullet)-extension of $T \in \Lambda_A$, and			
Im $A^1 = (A^1 - A^1)/2i = KJ^*K^*$.	(8)		
4. The ranges of K and Im A^1 coincide.	2. $\mathbb{F}^T \times \mathbb{F}^*$.	(9)	
1. $W_{\Theta}(z) = I - 2iK^* (A^1 - zI)^{-1} KJ$	(9)		
1. $\mathbb{F}^T \times \mathbb{F}^*$ (where $\mathbb{F}^T \times \mathbb{F}^*$ is the operator \mathbb{F}^T (where $\mathbb{F}^T \times \mathbb{F}^*$ is the operator \mathbb{F}^T is the operator product of \mathbb{F}^T .			
1. $\mathbb{F}^T \times \mathbb{F}^*$ (where $\mathbb{F}^T \times \mathbb{F}^*$ is the operator product of \mathbb{F}^T (where \mathbb{F}^T is the vector product of \mathbb{F}^T (where \mathbb{F}^T is the vector product of \mathbb{F}^T .)	3. \mathbb{F}^T (where \mathbb{F}^T is the vector product of \mathbb{F}^T (where \mathbb{F}^T is the vector product		

is called *rigged* if the following four conditions hold:

1. $J = J^* = J^{-1}$ (dim $E < \infty$).

2. K is a bounded linear operator from E into $S₂$.

3. *Al* is a correct $(*)$ -extension of $T \in \Lambda_A$, and

$$
Im A! = (A! - A!)^2 / 2i = K J^* K^*.
$$
 (8)

4. The ranges of *K* and Im Al coinside.

The operaror-valued function

$$
W_{\Omega}(z) = I - 2iK''(A^{i} - zI)^{-1}KJ
$$
\n(9)

is called a *Livsic type characteristic function* of the colligation Θ .

Furthermore, we introduce the function

$$
V_{\Theta}(z) = K^{\bullet}(A|_{R} - zI)^{-1}K. \tag{10}
$$

It is well known (cf. [3,9]) that the functions $V_\Theta(z)$ and $W_\Theta(z)$ are associated by the relations

called *rigged* if the following four conditions hold:

\n1.
$$
J = J^* = J^{-1}
$$
 (dim $E < \infty$).

\n2. *K* is a bounded linear operator from *E* into \mathfrak{H}_{2-} .

\n3. *A* is a correct (\bullet)-extension of $T \in \Lambda_A$, and

\nIm $A = (AI - AI^*)/2i = KJ^*K^*$.

\n4. The ranges of *K* and Im *A*l coinside.

\nThe operator-valued function

\n $W_{\Theta}(z) = I - 2iK^*(AI - zI)^{-1}KJ$

\ncalled a *Livsic type characteristic function* of the collision Θ .

\nFurthermore, we introduce the function

\n $V_{\Theta}(z) = K^*(A|_{R} - zI)^{-1}K$.

\nwell known (cf. [3, 9]) that the functions $V_{\Theta}(z)$ and $W_{\Theta}(z)$ are associated by the relations

\n $V_{\Theta}(z) = i(W_{\Theta}(z) + I)^{-1}(W_{\Theta}(z) - I)J$ and $W_{\Theta}(z) = (I + iV_{\Theta}(z)J)^{-1}(I - iV_{\Theta}(z)J)$.

\n(11) consider the conservative system (cf. [9])

We consider the conservative system (cf. [9])

The operator-valued function
\n
$$
W_{\Theta}(z) = I - 2iK^*(AI - zI)^{-1}KJ
$$
 (9)
\ncalled a *Livsic type characteristic function* of the collision Θ .
\nFurthermore, we introduce the function
\n $V_{\Theta}(z) = K^*(A|_{R} - zI)^{-1}K$. (10)
\nwell known (cf. [3, 9]) that the functions $V_{\Theta}(z)$ and $W_{\Theta}(z)$ are associated by the relations
\n $V_{\Theta}(z) = i(W_{\Theta}(z) + I)^{-1}(W_{\Theta}(z) - I)J$ and $W_{\Theta}(z) = (I + iV_{\Theta}(z)J)^{-1}(I - iV_{\Theta}(z)J)$. (11)
\nconsider the conservative system (cf. [9])
\n $(AI - zI)x = KJ\varphi$.
\n $\varphi_{+} = \varphi_{-} - 2iK^*x$,
\n $\operatorname{res}X \in \mathfrak{H}_{+} \varphi_{+} \in E, \varphi_{-}$ is the so-called *input vector*, φ_{+} is the output vector, and x is the inner
\ne. It is not hard to see that the transfer function $\Pi(z)$ of such a system (i.e., $\varphi_{+} = \Pi(z)\varphi_{-}$)

where $x \in \mathfrak{H}_+$, $\varphi_+ \in E$, φ_- is the so-called *input vector*, φ_+ is the output vector, and x is the inner state. It is not hard to see that the transfer function $\Pi(z)$ of such a system (i.e., $\varphi_+ = \Pi(z)\varphi_-\$) coincides with the operator function $W_Q(z)$. If $J \neq I$, then the system is called a *crossing* system and if $J = I$, it is called a scattering system (cf. [1]). In the following we will write a conservative system Θ in the form of a rigged operator colligation. *V*(*x*) $\Phi_+ \in \mathcal{L}$, $\varphi_+ \in E$, φ_- is the so-called *input vector*, φ_+ is the output vector, and *x* is the inner *e*. It is not hard to see that the transfer function $\Pi(z)$ of such a system (i.e., $\varphi_+ = \$

Definition: A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space *E,* is called *realizable* if it can be represented as

$$
V(z) = V_{\Theta}(z) = K^{\bullet}(A^j_R - zI)^{-1}K = i(W_{\Theta}(z) + I)^{-1}(W_{\Theta}(z) - I),
$$
\n(13)

where Θ is a conservative scattering system of the form (7).

Theorem 4: Let V(z) be a realizable function, whose values are operators in a finite-di mensional Hilbert space E, i.e., $V(z) = K^{\bullet}(A_R - zI)^{-1}K$. Let $A > 0$ and let (α, β) be an arbi*trary interval of the positive semi-axis. Then* $V(z)$ *belongs to the class* $S_{+}[(\alpha, \beta)]$ *if and only if the following two conditions hold:* **(b) (d) (d) (d) (d)** *a c c, i c <i>c d c <i>d c d f d d d d g d g d g d g d g d g d g g g g g*

1. $A|_{R} \ge 0$.

2. For an arbitrary set $\{z_i\}_{i=1}^P$ of non-real complex numbers such that $z_i \neq \overline{z}_1$ and for all $\varphi_i \in N_{z_i}$ (where N_{z_i} *is the deficiency space of A) it holds*

$$
\sum_{i,\,l=1}^P \bigl(B(z_i, z_l)\varphi_i, \varphi_l\bigr) \geq 0,\tag{15}
$$

 $13*$.

where

V. E. TSEKANOVSKI
\n
$$
B(\lambda, \mu) = \frac{\beta - \alpha}{(\alpha - \lambda)(\alpha - \overline{\mu})} A_R + \frac{\alpha \beta - \beta(\lambda + \overline{\mu}) + \lambda \overline{\mu}}{(\alpha - \lambda)(\alpha - \overline{\mu})} I.
$$
\n(16)

Proof: Assume that the conditions (14) and (15) hold. We will show that $V(z) \in S_{+}[(\alpha, \beta)].$ Since $V(z)$ is realizable, there exists a conservative scattering system

$$
B(\lambda, \mu) = \frac{\beta - \alpha}{(\alpha - \lambda)(\alpha - \overline{\mu})} Al_R + \overline{}
$$

Proof: Assume that the condition

$$
eV(z)
$$
 is realizable, there exist

$$
\Theta = \begin{pmatrix} Al & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- \end{pmatrix}
$$

such that $V(z) = V_0(z) = K^*(A_R - zI)^{-1}K$. The operator Al is a (.)-extension of some densely defined closed operator *T, i.e.,* condition (6) holds, where *A* is the common maximal Hermitian part of T and T^{*}. Let N_z be the deficiency space of the Hermitian operator *A* and let $\{z_i\}_{i=1}^P$ be an arbitrary set of non-real complex numbers such that $z_i \neq \overline{z_i}$. Moreover, let $\varphi_i \in N_{z_i}$. According to [11], there exists a vector $h_i \in E$ such that **Proof:** Assume that the conditions (14) and (15) hold. We will show that $V(z) \in S_x[(\alpha, \beta)]$.
 $eV(z)$ is realizable, there exists a conservative scattering system
 $\theta = \begin{pmatrix} A & K & J \\ \mathfrak{F}_1 \subset \mathfrak{F} \subset \mathfrak{F} \subset \mathfrak{F}_2 \end{pmatrix}$ ω - $V_{\Theta}(z) = K^{\bullet}$
 z - $V_{\Theta}(z) = K^{\bullet}$
 Z - Let N_z be 1
 Zi - Let N_z be 1
 Zi - *Zi* - *Zi* - *Zi* - *Zi* - *Zi*
 Z_i - *Z_j* - *Zj*

$$
\varphi_i = (A_{R} - zI)^{-1} K h_i \quad (i = 1, ..., p). \tag{17}
$$

Set $w_i = (\beta - z_i)/(\alpha - z_i)$. We will prove the inequality

ing to [11], there exists a vector
$$
h_i \in E
$$
 such that
\n
$$
\varphi_i = (A_{R} - zI)^{-1}Kh_i \quad (i = 1, ..., p).
$$
\n
$$
w_i = (\beta - z_i) / (\alpha - z_i).
$$
 We will prove the inequality
\n
$$
\sum_{i, l=1}^{P} \left(\frac{w_i V(z_i) - \overline{w_l} V(\overline{z_l})}{z_i - \overline{z_l}} h_i, h_l \right) \ge 0.
$$
\n(18)

In fact,

ng to [11], there exists a vector
$$
h_i \in E
$$
 such that
\n
$$
\rho_i = (A_R - zI)^{-1}Kh_i \quad (i = 1, ..., p).
$$
\n
$$
v_i = (\beta - z_i) / (\alpha - z_i).
$$
\nWe will prove the inequality
\n
$$
\sum_{i, l=1}^{p} \left(\frac{w_i V(z_i) - \overline{w_i} V(\overline{z_i})}{z_i - \overline{z_i}} h_i, h_l \right) \ge 0.
$$
\n
$$
\sum_{i, l=1}^{p} \left(\frac{w_i V(z_i) - \overline{w_i} V(\overline{z_i})}{z_i - \overline{z_i}} h_i, h_l \right)
$$
\n
$$
= \sum_{i, l=1}^{p} \left(\frac{w_i (A_R - z_i I)^{-1} - \overline{w_i} (A_R - \overline{z_i} I)^{-1}}{z_i - z_i} K_l, Kh_l \right)
$$
\n
$$
= \sum_{i, l=1}^{p} \left(\frac{(A_R - \overline{z_i} I)^{-1} (w_i (A_R - \overline{z_i} I) - \overline{w_i} (A_R - z_i I)) (A_R - z_i I)^{-1}}{z_i - \overline{z_i}} K_h, Kh_l \right)
$$
\n
$$
= \sum_{i, l=1}^{p} \left(\frac{(A_R - \overline{z_i} I)^{-1} (w_i (A_R - \overline{z_i} I) - \overline{w_i} (A_R - z_i I)) (A_R - z_i I)^{-1}}{z_i - \overline{z_i}} K_h, Kh_l \right)
$$
\n
$$
= \sum_{i, l=1}^{p} \left(\frac{\beta - \alpha}{(z_i - \overline{z_i})(\alpha - \overline{z_i})} A_R + \frac{\alpha \beta - \beta(z_i + \overline{z_i}) + z_i \overline{z_i}}{(\alpha - z_i)(\alpha - \overline{z_i})} I \right) \varphi_i, \varphi_l \right)
$$
\n
$$
= \sum_{i, l=1}^{p} (B(z_i, z_i) \varphi_i, \varphi_l) \ge 0.
$$
\n
$$
= \sum_{i, l=1}^{p} (B(z_i, z_i) \varphi_i, \varphi_l) \ge 0.
$$
\n
$$
= \sum_{i, l=1}^{p} (B(z_i, z_i) \varphi_i
$$

Setting $p = 1$, $z_1 = z$, $h_1 = h$, we obtain from (17) and (18)

$$
\left(\frac{\frac{\beta-z}{\alpha-z}V(z)-\frac{\beta-\overline{z}}{\alpha-\overline{z}}V(\overline{z})}{z-\overline{z}}h,h\right)\geq 0\quad\text{and hence}\quad\frac{\text{Im}\left(\frac{\beta-z}{\alpha-z}V(z)\right)}{\text{Im}\,z}\geq 0,
$$

i.e., the operator-valued function $((\beta - z)(\alpha - z))V(z)$ is an operator-valued R-function. Now condition (14) implies that $V(z)$ ε S (cf. [4]). Since for $z < 0$ we have (β - $z)(α$ - $z)$ ≥ 0 it holds $((\beta - z)(\alpha - z))V(z) \in S$. By Theorem 2 we get $V(z) \in S$. $[(\alpha, \beta)]$.

Now assume $V(z) \in S₁(\alpha, \beta)$]. Thus $V(z) \in S$ and hence $Al_R \ge 0$ (cf. [4]). It remains to prove (15). In fact, by Theorem 2 the condition $V(z) \in S_{+}[(\alpha, \beta)]$ implies that $((\beta - z)(\alpha - z))V(z) \in S$. Thus, the operator-valued function $((\beta - z)(\alpha - z))V(z)$ has a representation of the form Stieltjes Type Operator-Valued Functions 189

assume $V(z) \in S_{+}[(\alpha, \beta)]$. Thus $V(z) \in S$ and hence $Al_R \ge 0$ (cf. [4]). It remains to prove

act, by Theorem 2 the condition $V(z) \in S_{+}[(\alpha, \beta)]$ implies that $((\beta - z)(\alpha - z))V(z) \in S$

$$
\frac{\beta-z}{\alpha-z}V(z)=\gamma+\int\limits_0^\infty\frac{1}{t-z}d\sigma(t),\qquad \qquad (19)
$$

where $\gamma \ge 0$,o(*t*) is a non-decreasing operator-valued function in *E* such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$. Stieltjes Type Operator-Valued Functions 189

Now assume $V(z) \in S_x[(\alpha, \beta)]$. Thus $V(z) \in S$ and hence $Al_R \ge 0$ (cf. [4]). It remains to prove

(15). In fact, by Theorem 2 the condition $V(z) \in S_x[(\alpha, \beta)]$ implies that $((\beta - z)(\alpha$ Then according to [11] there exist vectors $h_i \in E$ such that (17) holds. Setting $w_i = (\beta - z_i)(\alpha - z_i)$ we obtain $\frac{P}{Z}V(z) = \gamma + \int_{0}^{1} \frac{1}{t-z} d\sigma(t),$
 $\gamma \ge 0, \sigma(t)$ is a non-decreasing operator-valued function
 $P_{ij}P_{j=1}^P$ be an arbitrary set of non-real complex number

according to [11] there exist vectors $h_i \in E$ such that (17 Sume $V(z) \in S_{+}[(\alpha, \beta)]$

by Theorem 2 the

leerator-valued funct
 $T(z) = \gamma + \int_{0}^{\infty} \frac{1}{t - z} d\sigma$
 $\sigma(t)$ is a non-decreas

be an arbitrary set of
 $\sigma(y) = \frac{W(z_1) - W_1 V(\overline{z_1})}{z_1 - \overline{z_1}}$
 σ
 σ the proof of the sh Let $\{z_i\}_{i=1}^p$ be an arbitrary set of non-real complex numbers such that $z_i \neq \overline{z_1}$. Let $\varphi_i \in N_{z_i}$.

$$
\sum_{i, l=1}^p \Bigl(\frac{w_i\,V(z_i)-\overline{w}_l\,V(\overline{z}_l)}{z_i-\overline{z}_l}\,h_i,h_l\Bigr)=\sum_{i, l=1}^p \Biggl(\,\int\limits_0^\infty \frac{1}{(t-z_i)(t-\overline{z}_l)}d\sigma(t\,)h_i,h_l\Bigr)\geq 0.
$$

It is clear from the proof of the sufficiency part that the inequalities (15) and (18) are equivalent. Thus, the above inequality yields the needed result \blacksquare

Remark:If there is no gap, i.e., if $\alpha = \beta$, the inequality (15) holds trivially and we obtain the results of $[4]$.

Theorem 5: *Let V(z) be a realizable function, whose values are operators in a finite-dimensional Hilbert space E, i.e., V(z)* = $K^{\bullet}(A|_{R} - zI)^{-1}K$. Let (α, β) be an arbitrary interval *of the positive semi-axis. Then* $V(z)$ belongs to the class $S[\alpha, \beta]$ *if and only if the following two conditions hold:*

1. $A|_R \geq 0$.

2. For an arbitrary set $\{z_i\}_{i=1}^P$ of non-real complex numbers such that z_i *** z_j and for all $\varphi_i \in N_{z_i}$ (where N_{z_i} is the deficiency space of A) it holds ons nota:
 \therefore 0.

an arbitrary set $\{z_i\}_{i=1}^P$ of non-real conserved N_{z_i} is the deficiency space of A) is
 $\{(z_i, z_j)\varphi_i, \varphi_j) \ge 0,$
 $\frac{\alpha - \beta}{(\beta - \lambda)(\beta - \overline{\mu})}$ $A\big|_R + \frac{\alpha\beta - \alpha(\lambda + \overline{\mu}) + \lambda}{(\beta - \lambda)(\beta - \overline{\mu})}$

Th

$$
\sum_{i,l=1}^p \bigl(B(z_i,z_l)\varphi_i,\varphi_l\bigr)\geq 0,
$$

where

$$
r e
$$

\n
$$
B(\lambda, \mu) = \frac{\alpha - \beta}{(\beta - \lambda)(\beta - \overline{\mu})} A_{R} + \frac{\alpha \beta - \alpha(\lambda + \overline{\mu}) + \lambda \overline{\mu}}{(\beta - \lambda)(\beta - \overline{\mu})} I.
$$
\n(20)

Proof: This theorem can be proved in the same way as Theorem 4 (set $w_i = (\alpha - z_i)(\beta - z_i)$)

Note further that in the case $\alpha = \beta$ (i.e., if there are no gaps) Theorem 5 is an extension of results of [4]. Moreover, it is not hard to see that the above used method allows us to obtain analogous results for the classes

 $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)]$ and $S_{\pm}[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)] \cap S_{\pm}[\bigcup_{j=1}^{n}(c_j,d_j)].$

In fact, we have the following

Theorem *6: Let V(z) be a realizable function, whose values are operators in a finite -dimensional Hilbert space E, i.e.,* $V(z) = K^*(A_R - zI)^{-1}K$ *. Let* (α_j, β_j) ($j = 1, ..., m$) and (c_k, d_k) $(k = 1, \ldots, n)$ be arbitrary mutually disjoint intervals of the positive semi-axis. Then $V(z)$ belongs *to the class* $S_x[\bigcup_{j=1}^m (\alpha_j, \beta_j)] \cap S_x[\bigcup_{k=1}^n (c_k, d_k)]$ if and only if the following two conditions *hold:*

1. $A|_{R} \ge 0$.

2. For an arbitrary set $\{z_j\}_{i=1}^P$ of non-real complex numbers such that z_j \neq \overline{z}_l and for all **E. TSEKANOVSKII**
 1. $A|_R \ge 0$.
 2. For an arbitrary set $\{z_j\}_{j=1}^P$ of non-real complex r
 E. N_{z_j} (where N_{z_j} is the deficiency space of A) it holds

$$
\sum_{i, l=1}^p (B(z_i, z_l)\varphi_i, \varphi_l) \geq 0,
$$

where

V. E. TSEKANOVSKI
\n1.
$$
A|_R \ge 0
$$
.
\n2. For an arbitrary set $\{z_j\}_{i=1}^p$ of non-real complex numbers such that $z_i \ne \overline{z}_1$ and for all N_{z_i} (where N_{z_i} is the deficiency space of A) it holds
\n
$$
\sum_{i,j=1}^p (B(z_i, z_j)\varphi_i, \varphi_j) \ge 0,
$$
\n
\n
$$
B(\lambda, \mu) = \frac{w(\lambda) - w(\overline{\mu})}{\lambda - \overline{\mu}} A|_R + \frac{\lambda w(\overline{\mu}) - \overline{\mu}w(\lambda)}{\lambda - \overline{\mu}}
$$
\n
$$
w(\lambda) = \prod_{j=1}^m \frac{\beta_j - \lambda}{\alpha_j - \lambda} \prod_{k=1}^n \frac{c_k - \lambda}{d_k - \lambda}.
$$
\n(22)

and

$$
\lambda - \mu \qquad \lambda - \mu
$$
\n
$$
w(\lambda) = \prod_{j=1}^{m} \frac{\beta_j - \lambda}{\alpha_j - \lambda} \prod_{k=1}^{n} \frac{c_k - \lambda}{d_k - \lambda}
$$
\n(22)\nSome subclasses of realizable Stieltjes type operator-valued functions with gaps result of M. G. Krein (see [8]) each Stieltjes type function $V(z)$, whose values are opens in a finite-dimensional Hilbert space E , can be represented in the form

\n
$$
V(z) = \gamma + \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)},
$$
\n
$$
V(z) = \gamma + \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)},
$$
\n
$$
V(z) = \frac{\gamma^2}{2} \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)},
$$
\n
$$
V(z) = \frac{\gamma^2}{2} \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)^2} \, dt
$$
\n
$$
V(z) = \frac{\gamma^2}{2} \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)^2} \, dt
$$
\n
$$
V(z) = \frac{\gamma^2}{2} \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)^2} \, dt
$$
\nUsing the following equations: $z = \frac{\gamma^2}{2} \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)^2} \, dt$.

§ 3 Some subclasses of realizable Stieltjes type operator-valued functions with gaps

By a result of M. G. Krein (see *[81)* each Stieltjes type function V(z), whose values are operators in a finite-dimensional Hubert space *E,* can be represented in the form

$$
V(z) = \gamma + \int_{0}^{\infty} \frac{d\sigma(t)}{(t-z)},
$$
\n(23)

where $\gamma \ge 0$, $\sigma(t)$ is a non-decreasing operator-valued function in *E* such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$. According to [5], we introduce the following notion.

Definition: We will say that a Stieltjes type function $V(z)$, whose values are operators in a finite-dimensional Hilbert space *E*, belongs to the class $S(R)$, if $\gamma f = 0$ for all *f* of the subclass *E* result of M. G. Krein (see [8]) each Stieltjes
 F is in a finite-dimensional Hilbert space E, can
 $V(z) = \gamma + \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)},$
 F $F(z) = \int_{0}^{\infty} \frac{d\sigma(t)}{(t - z)^2}$
 F $F(z) = \int_{0}^{\infty} \frac{d\sigma(t)}{dt}$
 F $\int_{0}^{\$ **Definition:** We will say that a Stieltjes type function $V(z)$, whose values are operators in
ite-dimensional Hilbert space E, belongs to the class $S(R)$, if $\gamma f = 0$ for all f of the sub-
s
 $E_{\infty}^{\perp} = \left\{ f \in E : \int_{0}^{\in$

$$
E_{\infty}^{\perp} = \left\{ f \in E : \int_{0}^{\infty} (d\sigma(t) f, f)_E \leq \infty \right\}.
$$
 (24)

As it was proved in [5], each operator-valued function $V(z) \in S(R)$ can be realized by a conservative scattering system Θ , i.e., it holds (13).

Definition: Following [5], we introduce the following subclasses of $S(R)$:

(i) The class $S^{\circ}(R)$ consisting of all $V(z) \in S(R)$ such that

$$
\int_{0}^{\infty} (d\sigma(t)f, f) = \infty \quad (f \in E, f \neq 0).
$$
\n(25)

(ii) The class $S^1(R)$ consisting of all $V(z) \in S(R)$ such that $\gamma = 0$ and

$$
E_{\infty}^{\perp} = \left\{ f \in E : \int (d\sigma(t)f, f)_{E} < \infty \right\}.
$$
\n(24)
\nAs it was proved in [5], each operator-valued function $V(z) \in S(R)$ can be realized by a
\nseervative scattering system Θ , i.e., it holds (13).
\n**Definition:** Following [5], we introduce the following subclasses of $S(R)$:
\n(i) The class $S^{\circ}(R)$ consisting of all $V(z) \in S(R)$ such that
\n
$$
\int_{0}^{\infty} (d\sigma(t)f, f) = \infty \quad (f \in E, f \neq 0).
$$
\n(ii) The class $S^{\circ}(R)$ consisting of all $V(z) \in S(R)$ such that $\gamma = 0$ and
\n
$$
\int_{0}^{\infty} (d\sigma(t)f, f) < \infty \quad (f \in E)
$$
\n(26)
\nare representation (23).
\n(iii) The class $S^{\circ}(R)$ consisting of all $V(z) \in S(R)$ such that $E_{\infty}^{\perp} \neq \{0\}$ and $E_{\infty}^{\perp} \neq E$.

in the representation (23).

(iii) The class $S^{\circ*}(R)$ consisting of all $V(z) \in S(R)$ such that $E_{\infty}^{\perp} * \{0\}$ and $E_{\infty}^{\perp} * E$.

It is not hard to see that

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 S(R) = $S^0(R) \cup S^1(R) \cup S^0(R)$. (27)
 Definition: We introduce the following subclasses of $S(R)$, $S^0(R)$, $S^1(R)$ and $S^{01}(R)$.

(i) The class $S_{\star}[R, \bigcup_{i=1}^{m}(\alpha_i, \beta_i$ **Definition:** We introduce the following subclasses of $S(R)$, $S^0(R)$, $S^1(R)$ and $S^{01}(R)$.
(i) The class $S_{\pm}[R, \bigcup_{i=1}^m (\alpha_i, \beta_i)]$ consisting of all $V(z) \in S(R)$ such that $V(z)$ is holomor-Stieltjes Type Operator-Valued Functions 19

(*R*) = $S^0(R) \cup S^1(R) \cup S^0(R)$. (27)
 efinition: We introduce the following subclasses of $S(R)$, $S^0(R)$, $S^1(R)$ and $S^{01}(R)$.

(i) The class $S_2[R, \bigcup_{j=1}^m (\alpha_j, \beta_j)]$ consi phic and positive (negative) on all intervals (α_i, β_i) .

(ii) The class $S_1^0[R, \bigcup_{i=1}^m (\alpha_i, \beta_i)]$ consisting of all $V(z) \in S^0(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_i, β_i) .

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(27)

(a) Stephend to see that

(a) The class $S_{\pm}[R, \bigcup_{j=1}^{S_0}(G_j, \beta_j)]$ consisting of all $V(z) \in S(R)$, $S^{0}(R)$, $S^{1}(R)$ and $S^{01}(R)$.

(i) The class $S_{\pm}[R, \bigcup_{j=1}^{S_1}($ phic and positive (negative) on all intervals (α_j,β_j) .

(iv) The class $S_2^{\text{o}}\{R, \bigcup_{i=1}^m (\alpha_i, \beta_i)\}$ consisting of all $V(z) \in S^{\text{o}}(R)$ such that $V(z)$ is holomorphic and positive (negative) on all intervals (α_i, β_i) .

Let Θ be a conservative scattering system of the form (7) such that $V_\Theta(z) = V(z)$ and let A and T be the operators of (6) . Then $(cf, [5])$

 $\overline{\mathfrak{D}(A)}$ = \mathfrak{S}_1 , $\mathfrak{D}(T)$ + $\mathfrak{D}(T^*)$ if $V(z) \in S^{\circ}_{+}[R, \bigcup_{i=1}^{m}(\alpha_i, \beta_i)]$, $\overline{(A)}$ \uparrow \mathfrak{H} **,** $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ if $V(z) \in$ **(b)** be the operators of (6). Then (cf. [5
 (A) = $\hat{\mathfrak{g}}$, $\mathfrak{D}(T)$ + $\mathfrak{D}(T^*)$ if $V(z) \in$
 (A) + $\hat{\mathfrak{g}}$, $\mathfrak{D}(T)$ = $\mathfrak{D}(T^*)$ if $V(z) \in$
 (A) + $\hat{\mathfrak{g}}$, $\mathfrak{D}(T)$ + $\mathfrak{D}(T^*)$ if $V(z) \$

Theorem *7: A function V(z), whose values are operators in a finite-dimensional Hilbert* space E, belongs to the class $S^{\circ}_{+}[R,(\alpha,\beta)]$ if and only if the following two conditions hold: *(i)* $V(z) \in S^{\circ}(R)$. *) if V
(z), wh
 $S_+^0[R, ($
 $\frac{\alpha - z}{\beta - z}$
ndition *v (z)* \in *V(z)* \in *V(z)* \in *V(z)* \in *S*²[*R, U_j^m₁(* α *_j,* β *_j)],

<i>V(z)* \in *S*²[*R, U_j^m₁(* α *_j,* β *_j)],

<i>V(z)* \in *S*²¹[*R, U_j^m₁(* α *_j,* β *_j)].

<i>(z)* \in *S*

(ii)
$$
\frac{\beta-z}{\alpha-z}V(z)\in S^{\circ}(R)\left(\frac{\alpha-z}{\beta-z}V(z)\in S^{\circ}(R),\,respectively\right).
$$
 (28)

Proof: Assume that the conditions (28) hold. Since $S^{\circ}(R) \in S$, we have $V(z) \in S$ _{*i*} R , (α, β) ⁷] by Theorem 2 and, hence, $V(z) \in S^0$ [R,(α, β] because $V(z) \in S^0(R)$. Conversely, assume that $V(z) \in S^{\circ}_{+}[R, (\alpha, \beta)]$. Then, clearly, $V(z) \in S^{\circ}(R)$. It remains to show the first inclusion of (ii). It is well known that **Theorem** 7: A function $V(z)$, whose values are operators in a finite-dimensional Hilbert
 Jet all in $V(z)$ $\in S^0$ *F*, (α, β) *J* if and only if the following two conditions hold:

(i) $V(z) \in S^0$ *F*),

(ii) $\frac{\beta - z$ ii) $\frac{\beta - z}{\alpha - z}V(z) \in S^0(R) \left(\frac{\alpha - z}{\beta - z}V(z) \in S^0(R), respectively \right)$. (28)
 roof: Assume that the conditions (28) hold. Since $S^0(R) \in S$, we have $V(z) \in S_{\pm}[R, (\alpha, \beta)]$

neorem 2 and, hence, $V(z) \in S_{\pm}^0[R, (\alpha, \beta)]$ because $V(z) \in S^0(R$ heorem 2 and, hence, $V(z) \in S_z^0[R, (\alpha, \beta)]$ because $V(z) \in S^0(R)$. Conversely, assume that
 $\int_0^{\infty} (dS(x)f, f) = \lim_{\eta \to \infty} (\eta \operatorname{Im} V(\eta)f, f)$, (29)

well known that
 $\int_0^{\infty} (dS(x)f, f) = \lim_{\eta \to \infty} (\eta \operatorname{Im} V(\eta)f, f)$, (29)

re $\sigma(f)$

When known that

\n
$$
\int_{0}^{\infty} (d\sigma(t) f, f) = \lim_{n \to \infty} (\eta \ln V(i\eta) f, f),
$$
\n(29)

\nand (29) imply

\n
$$
\lim_{n \to \infty} (\eta \ln V(i\eta) f, f) = \infty.
$$
\n(30)

\n
$$
\lim_{n \to \infty} (\eta \ln V(i\eta) f, f) = \infty.
$$
\n(31)

\n
$$
\lim_{n \to \infty} (n \ln V(i\eta) f, f) = \infty.
$$
\n(32)

\n
$$
\lim_{n \to \infty} V(z) = K^*(A|_{R} - zI)^{-1}K, \text{ we obtain}
$$
\n
$$
\lim_{n \to \infty} V(z) = \lim_{n \to \infty} zK^*(A|_{R} - \bar{z}I)^{-1}(A|_{R} - zI)^{-1}K.
$$
\n(31)

\n
$$
f_{\eta} = (A|_{R} - i\eta I)^{-1}Kf.
$$
\n(32)

\n
$$
\lim_{n \to \infty} n^{2}(f, f) = \lim_{n \to \infty} n^{2}((A|_{R} - i\eta I)^{-1}Kf)(A|_{R} - i\eta I)^{-1}Kf.
$$
\n(33)

where $\sigma(t)$ is the operator-valued measure of the representation (23) (cf. [6]). The definitive of $S^{\circ}(R)$ and (29) imply *ure* $g(t)$ is the operator-valued measure of the representation (23) (cf. [6]). The definitive
 $g'(R)$ and (29) imply
 $\lim_{n \to \infty} (\eta \ln V(i\eta) f, f) = \infty.$ (30)
 $\lim_{n \to \infty} V(z) = K^*(A|_R - zI)^{-1}K$, we obtain
 $\lim_{n \to \infty} V(z) = \lim_{n \$

$$
\lim_{n \to \infty} (\eta \operatorname{Im} V(i\eta) f, f) = \infty. \tag{30}
$$

Since $V(z) = K^{\bullet}(A|_{R} - zI)^{-1}K$, we obtain

$$
\operatorname{Im} V(z) = \operatorname{Im} z K^*(A|_{R} - \bar{z}I)^{-1}(A|_{R} - zI)^{-1}K. \tag{31}
$$

Set

$$
f_n = (A|_{R} - i\eta I)^{-1} Kf. \tag{32}
$$

From (30) - (32) it follows

$$
f_{\eta} = (A|_{R} - i\eta I)^{-1} Kf.
$$
\n(32)
\n
$$
f_{\eta} = (A|_{R} - i\eta I)^{-1} Kf.
$$
\n(33)
\n
$$
\lim_{\eta \uparrow \infty} \eta^{2}(f_{\eta}, f_{\eta}) = \lim_{\eta \uparrow \infty} \eta^{2}((A|_{R} - i\eta I)^{-1} Kf, (A|_{R} - i\eta I)^{-1} Kf)
$$
\n(33)

$$
=\lim_{\eta\uparrow\infty}\eta^2\bigl(K^*(A|_R+i\eta I)^{-1}(A|_R-i\eta I)^{-1}Kf,f\bigr)=\lim_{\eta\uparrow\infty}\bigl(\eta\mathop{\rm Im}\nolimits V(i\eta)f,f\bigr)=\infty.
$$

We will show that

V. E. TSEKANOVSKII
\n
$$
= \lim_{\eta \uparrow \infty} \eta^2 (K^*(A|_R + i\eta I)^{-1} (A|_R - i\eta I)^{-1} Kf, f) = \lim_{\eta \uparrow \infty} (\eta \ln V(i\eta) f, f) = \infty.
$$
\nwill show that
\n
$$
\lim_{\eta \uparrow \infty} (\eta \ln \frac{\beta - i\eta}{\alpha - i\eta} V(i\eta) f, f) = \infty.
$$
\n(34)
\n
$$
\text{act, setting } z_i = z_i = i\eta \text{ in the inequality (18) and regarding the considerations of the proof}
$$

In fact, setting z_i = z_j = in in the inequality (18) and regarding the considerations of the proof of this inequality we obtain

$$
\lim_{\eta \uparrow \infty} \eta^2 (K^* (A|_R + i\eta I)^{-1} (A|_R - i\eta I)^{-1} Kf, f) = \lim_{\eta \uparrow \infty} (\eta \ln V(i\eta) f, f) = \infty.
$$
\nwill show that\n
$$
\lim_{\eta \uparrow \infty} \left(\eta \ln \frac{\beta - i\eta}{\alpha - i\eta} V(i\eta) f, f \right) = \infty.
$$
\n(3.101112) (10.112) (11.122) (12.132) (13.122) (14.122) (15.122) (16.122) (17.122) (19.122)

We mention that we have also used Theorem 4 ($A\ell_R \ge 0$) and (33).

Finally, assume that $V(z) \in S^{0}[R, (\alpha, \beta)]$. We will show that $((\alpha - z)/(\beta - z))V(z) \in S^{0}(R)$. Setting $w = (\alpha - i\eta)(\beta - i\eta)$ and using (13) we get

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$$
\vee
$$
 E. TSEKANOVSEII
\n
$$
= \lim_{\eta \to 0} n^{2} (K^{*}(A_{R} + i\eta I)^{-1} (A_{R} - i\eta I)^{-1} Kf, f) = \lim_{\eta \to \infty} (\eta \ln V(i\eta)f, f) = \infty.
$$
\nWe will show that
\n
$$
\lim_{\eta \to \infty} \int n^{2} (K^{*}(A_{R} + i\eta I)^{-1} (A_{R} - i\eta I)^{-1} Kf, f) = \lim_{\eta \to \infty} (n \ln V(i\eta)f, f) = \infty.
$$
\n10.154.1, setting $z_{i} = z_{i} = i\eta$ in the inequality (18) and regarding the considerations of the proof of this inequality we obtain
\n
$$
\lim_{\eta \to \infty} (\eta \ln \frac{\beta - i\eta}{\alpha + i\eta} V(i\eta)f, f) = \lim_{\eta \to \infty} \frac{n^{2}(\beta - \alpha)}{\alpha^{2} + \eta^{2}} (A_{R} f_{\eta}, f_{\eta}) + \frac{n^{2} \alpha \beta}{\alpha^{2} + \eta^{2}} (f_{\eta}, f_{\eta}) + \frac{n^{4}}{\alpha^{2} + \eta^{2}} (f_{\eta}, f_{\eta})
$$
\n
$$
\geq \lim_{\eta \to \infty} \frac{n^{4}}{\alpha^{2} + \eta^{2}} (f_{\eta}, f_{\eta}) = \lim_{\eta \to \infty} \frac{n^{2}}{\alpha^{2} + \eta^{2}} (f_{\eta}, f_{\eta}) + \frac{n^{4}}{\alpha^{2} + \eta^{2}} (f_{\eta}, f_{\eta}) = \infty.
$$
\nWe mention that we have also used Theorem 4 (A_R ≥ 0) and (33).
\nFinally, assume that $V(z) \in \mathcal{S}^{c}[R, (\alpha, \beta)]$. We will show that $((\alpha - z)/(\beta - z))V(z) \in S^{c}(R)$
\nSetting $w = (\alpha - i\eta)\beta - i\eta$ and using (13) we get
\n
$$
(\text{Im } WV(i\eta)f, f) = \frac{1}{2} [((w K^{*}(A_{R} - i\eta I)^{-1} K - \overline{w} K^{*}(A_{R} + i\eta I)^{-1} K)f, f]
$$
\n
$$
= \frac
$$

where f_{η} has the form (32). But $(A_R f_{\eta}, f_{\eta}) = \left(\frac{\text{Im}\, \text{Im}\,V(\text{Im})}{\eta} f f\right)$. In fact,

$$
-\left(\frac{2i}{2i}ARI_{\eta}, I_{\eta}\right) + \frac{2i}{2} \eta(I_{\eta}, I_{\eta})
$$
\n
$$
= \frac{\eta(\alpha - \beta)}{|\beta - i\eta|^{2}} (A_{R}f_{\eta}, I_{\eta}) + \frac{\eta(\alpha\beta + \eta^{2})}{|\beta - i\eta|^{2}} (I_{\eta}, I_{\eta}),
$$
\nref I_{η} has the form (32). But $(A_{R}f_{\eta}, I_{\eta}) = \left(\frac{\text{Im}\,i\eta V(i\eta)}{\eta} f, f\right)$. In fact,
\n
$$
\left(\frac{\text{Im}\,i\eta V(i\eta)}{\eta} f, f\right) = \left(\frac{i\eta V(i\eta) + i\eta V(-i\eta)}{2i\eta} f, f\right)
$$
\n
$$
= \left(\frac{i\eta K^{*}(A_{R} - i\eta I)^{-1}K + i\eta K^{*}(A_{R} + i\eta I)^{-1}K}{2i\eta} f, f\right)
$$
\n
$$
= \left(\frac{(A_{R} + i\eta I)^{-1}(i\eta(A_{R} + i\eta I) + i\eta(A_{R} - i\eta I))(A_{R} - i\eta I)^{-1}}{2i\eta} Kf, Kf\right)
$$
\n
$$
= (A_{R}f_{\eta}, I_{\eta}).
$$
\n
$$
\left(\frac{\text{Im}\,i\eta V(i\eta)}{\eta} f, f\right) = (\gamma f, f) + \int_{0}^{\infty} \frac{t}{t^{2} + \eta^{2}} (d\sigma(t) f, f).
$$
\ng Lebesgue's Dominated Convergence Theorem (cf. [6]), it follows
\n
$$
\lim_{\eta \uparrow \infty} (A_{R}f_{\eta}, I_{\eta}) = \lim_{\eta \uparrow \infty} \left(\frac{\text{Im}\,i\eta V(i\eta)}{\eta} f, f\right) = (\gamma f, f) < \infty.
$$
\n(36)
\n(35) and (36) imply

Regarding (23) we obtain

$$
= (A|_{R} f_{\eta}, f_{\eta}).
$$

arding (23) we obtain

$$
\left(\frac{\text{Im}\,\text{in}\,V(\text{in})}{\eta}f, f\right) = (\gamma f, f) + \int_{0}^{\infty} \frac{f}{t^{2} + \eta^{2}}(d\sigma(t)f, f).
$$

Using Lebesgue's Dominated Convergence Theorem (cf. [61), it follows

$$
\left(\frac{\text{Im}\,\text{in}\,V(\text{in})}{\eta}\,f,f\right) = (\gamma f,f) + \int_{0}^{\infty} \frac{t}{t^2 + \eta^2} \big(d\sigma(t)f,f\big).
$$
\ng Lebesgue's Dominated Convergence Theorem (cf. [6]), it follows
\n
$$
\lim_{\eta \to \infty} (A\vert_R f_\eta, f_\eta) = \lim_{\eta \to \infty} \left(\frac{\text{Im}\,\text{in}\,V(\text{in})}{\eta}\,f,f\right) = (\gamma f,f) < \infty.
$$
\n(36)
\n(35) and (36) imply
\n
$$
\lim_{\eta \to \infty} \left(\eta \text{Im}\,\frac{\alpha - \text{in}}{\beta - \text{in}}\,V(\text{in})f,f\right) = \lim_{\eta \to \infty} \left(\frac{\eta^2(\alpha - \beta)}{\beta^2 + \eta^2}(A\vert_R f_\eta, f_\eta) + \frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2}(f_\eta, f_\eta) + \frac{\eta^2}{\beta^2 + \eta^2}\eta^2(f_\eta, f_\eta)\right)
$$

Now (35) and (36) imply

$$
\lim_{\eta \uparrow \infty} \left(\frac{35}{100} \sin \frac{(\pi \cdot \eta \cdot \eta)}{\pi^2} \right) = \lim_{\eta \uparrow \infty} \left(\frac{\eta^2 (\alpha - \beta)}{\beta^2 + \eta^2} (A_{R} f_{\eta}, f_{\eta}) + \frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) + \frac{\eta^2}{\beta^2 + \eta^2} \eta^2 (f_{\eta}, f_{\eta}) \right)
$$
\n
$$
\lim_{\eta \uparrow \infty} \left(\frac{\eta \left(\frac{\alpha - \beta}{\beta - \eta} \right)}{\alpha^2 + \beta^2} (A_{R} f_{\eta}, f_{\eta}) + \frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) + \frac{\eta^2}{\beta^2 + \eta^2} \eta^2 (f_{\eta}, f_{\eta}) \right)
$$

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\n
$$
\geq \lim_{n \to \infty} \left(\frac{n^2(\alpha - \beta)}{\beta^2 + \eta^2} (A^{\dagger}R f_{\eta}, f_{\eta}) + \frac{n^2}{\beta^2 + \eta^2} \eta^2 (f_{\eta}, f_{\eta}) \right) \geq \infty.
$$
 (37)

Thus the theorem is proved \blacksquare

Theorem 8: *A function V(z), whose values are operators in a finite -dimensional Hilbert* space E, belongs to the class S^1 [R, (α, β)] *if and only if the following two conditions hold:*

- (i) $V(z) \in S^1(R)$.
- (ii) $\frac{\beta-z}{\alpha-z}V(z) \in S^{\vee}(R) \left(\frac{\alpha-z}{\beta-z}V(z) \in S^{\vee}(R) \right)$, *respectively*

Proof: The sufficiency is obvious (compare the proof of Theorem 7). Now assume that *V(z)* $\in S^1$ [R,(α,β)]. Clearly, $V(z)$ \in S^1 (R). We will show that $((β - z)/(α - z))V(z)$ \in S^1 (R). Since $V(z)$ is realizable, the relation (13) holds. In this relation, the operator A_{R} is a bounded linear operator from S_+ into S_- . Let *R* be the (isometric) Riesz-Berezanskii operator, which arises in a natural way in the therory of nested Hubert spaces (cf. [21).The operator *R* has the pro-**Proof:** The sufficiency is obvious (compare the proof of $\in S^1$, R , (α, β)). Clearly, $V(z) \in S^1$ (*R*). We will show that (($V(z)$ is realizable, the relation (13) holds. In this relation, tho operator from \mathfrak{H}_2 (i) $V(z) \in S^{1}(R)$.

(ii) $\frac{\beta - z}{\alpha - z}V(z) \in S^{1}(R)$ $\left(\frac{\alpha - z}{\beta - z}V(z) \in S^{1}(R)\right)$, *Proof*: The sufficiency is obvious (compare the proof of Theorem 7). Now assume that $[R, (\alpha, \beta)]$. Clearly, $V(z) \in S^{1}(R)$. We will show that

$$
(A|_{R}f_{\eta}, f_{\eta}) = |(\mathbb{R}A_{R}f_{\eta}, f_{\eta})_{+}| \leq ||\mathbb{R}A|_{R}|| ||f_{\eta}||^{2} = ||\mathbb{R}A|_{R}|| (||f_{\eta}||^{2} + ||A^{*}f_{\eta}||^{2})
$$
\n
$$
= ||\mathbb{R}A|_{R}|| (||f_{\eta}||^{2} + \eta^{2}||Pf_{\eta}||^{2}) \leq ||\mathbb{R}A|_{R}|| (1 + \eta^{2})||f_{\eta}||^{2},
$$
\n(38)

where *P* is the orthoprojector of δ onto $\overline{\mathcal{D}(A)}$ and the operator *A* is the maximal common Herwhere *P* is the orthoprojector of \mathfrak{H} onto $\overline{\mathfrak{D}(A)}$ and the operator *A* is the maximal common Hermitian part of the operators *T* and *T*^{*} that arise realizing the operator-valued function *V(z)* as a tran a transfer function of the conservative scattering system (7). Furthermore, as it was noted in the proof of Theorem 7, we have us
 $\| (\|f_{\eta}\|^2 + \|A^*f_{\eta}\|)$
 $+\eta^2) \|f_{\eta}\|^2$,

ator A is the max

the operator-valu

(7). Furthermore
 $+\frac{\eta^2 \alpha \beta}{\alpha^2 + \eta^2} (f_{\eta}, f_{\eta})$

28) implies $\lim_{n \to \infty}$ imal common
ed function $V($, as it was not
+ $\frac{\eta^4}{\alpha^2 + \eta^2} (f_\eta, f_\eta)$
nlm⁸ = ⁱⁿ - $V(i)$ In a natural way in the theory of its tell filled in Spaces (cf. [2]). The operator A has the properties $(f, g) = (Rf, Bg) + (Rf, g) = (f, fg)$ ($f, g \in \mathfrak{H}$). Thus
 $|(A_R f_n, f_n)| = |(\mathbf{R} A_R f_n, f_n)_+| \leq ||\mathbf{R} A_R|| ||f_n||^2 = ||\mathbf{R} A_R|| (||f_n||^$

proof of Theorem 7, we have
\n
$$
\lim_{n \to \infty} \left(\eta \operatorname{Im} \frac{\beta - i\eta}{\alpha - i\eta} V(i\eta) f, f \right) = \lim_{n \to \infty} \left(\frac{\eta^2 (\beta - \alpha)}{\alpha^2 + \eta^2} (A \cdot R f \eta, f \eta) + \frac{\eta^2 \alpha \beta}{\alpha^2 + \eta^2} (f \eta, f \eta) + \frac{\eta^4}{\alpha^2 + \eta^2} (f \eta, f \eta) \right)
$$

 \leq ∞. Now assume *V*(*z*) ε S¹[R,(α, β]. We will show that $((α - z)/(β - z))V(z)$ ε S¹(R). Using (37) we get $\int \frac{1}{\pi} \frac{\ln m}{1 + \infty}$
 f_{η}, f_{η} \leq
 $\frac{1}{\pi} \lim_{\eta + \infty}$
imates the operators T
tion of the con-
heorem 7, we had
 $\frac{\beta - i\eta}{\alpha - i\eta} V(i\eta) f, f$
 $\int f, f$ = $\lim_{\eta + \infty} \eta^2 (f \cos \theta)$
 $\lim_{\beta - i\eta} V(i\eta) f, f$
ain the last esti-= $\lim_{\eta \to \infty} \left(\frac{\eta^2(\beta - \alpha)}{\alpha^2 + \eta^2} (A|_R f_{\eta}) \right)$
 π, f_{η} : ∞ , the realization
 $R, (\alpha, \beta)]$. We will show
 $\leq \lim_{\eta \to \infty} \left(\frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) + \right)$

mates we have used the

we get
\n
$$
\lim_{\eta \uparrow \infty} \left(\eta \operatorname{Im} \frac{\alpha - i \eta}{\beta - i \eta} V(i \eta) f, f \right) \le \lim_{\eta \uparrow \infty} \left(\frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) + \frac{\eta^2}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) \right) < \infty.
$$

In order to obtain the last estimates we have used the fact that $A_R \geq 0$.

The following theorem is an immediate consequence of Theorems 7 and (8).

Theorem *9: A function V(z), whose values are operators in a finite -dimensional Hilbert space E, belongs to the class* $S^{\circ\mathsf{L}}_+ [R, (\alpha, \beta)]$ *if and only if the following two conditions hold:*

(i)
$$
V(z) \in S^{\circ}(\mathbb{R})
$$
.

(ii)
$$
\frac{\beta-z}{\alpha-z}V(z)\in S^{o_1}(R)\left(\frac{\alpha-z}{\beta-z}V(z)\in S^{o_1}(R),\text{ respectively}\right)
$$
.

Combining the results of Theorems $7 - 9$ and regarding (27) we obtain the following

Theorem 10: *A function V(z), whose values are operators in a finite-dimensional Hubert* space E, belongs to the class $S_{+}[R, (\alpha, \beta)]$ if and only if the following two conditions hold:

(i) $V(z) \in S(R)$.

(ii) $\frac{\beta-z}{\alpha-z}V(z)\in S(R)$ $\left(\frac{\alpha-z}{\beta-z}V(z)\in S(R)$, *respectively*

Theorem 11: *A function V(z), whose values are operators in a finite-dimensional Hubert* **Theorem 10:** *A function* $V(z)$, whose values are operators in a finite-dimensional Hilbert
space E, belongs to the class $S_{\pm}[R, (\alpha, \beta)]$ if and only if the following two conditions hold:
(i) $V(z) \in S(R)$.
(ii) $\frac{\beta - z}{\alpha$ **Theorem 11:** *A function V(z), whose val*
 ie E, belongs to the class $S_1^{\circ} [R, \bigcup_{j=1}^m (\alpha_j, \beta_j)]$

(i) $V(z) \in S^{\circ}(R)$.

(ii) $\prod_{j=1}^m \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R) \left(\prod_{j=1}^m \frac{\alpha_j - z}{\beta_j - z} \right)$ α - 2
 Form 11: A
 J, belongs to
 $V(z) \in S^{\circ}(P)$
 $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V$
 of: The sufficient (ii) $\frac{u^2 - 2}{\alpha - 2}V(z) \in S(R)$ $\left(\frac{\alpha - 2}{\beta - 2}V(z) \in S(R)\right)$, respectively).
 Theorem 11: A function $V(z)$, whose values are operators in a finite-dimensional space *E*, belongs to the class $S_1^2[R, \bigcup_{j=1}^m(\alpha_j, \beta_j)]$

(i) $V(z) \in S^{\circ}(R)$.

(ii)
$$
\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^{\circ}(R) \left(\prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S^{\circ}(R), \text{ respectively} \right).
$$

Proof: The sufficiency of the conditions is easy to prove. Since $S^0(R) \subset S$, we have $V(z) \in S$ $S_{+} \left[\bigcup_{i=1}^{m} (\alpha_i, \beta_i) \right]$ because of Theorem 2. But since $V(z) \in S^{\circ}(R)$, we obtain $V(z) \in S^{\circ}[R] \cup \bigcup_{i=1}^{m} (\alpha_i, \beta_i)$.

The necessity is proved with aid of mathematical induction. For n = 1 the result was proved

(i) V(z) € S°(R).

Theorem 7. Now assume that
\n(i)
$$
V(z) \in S^0(R)
$$
.
\n(ii)
$$
\prod_{j=1}^P \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0(R)
$$
.

We will show that this fact remains true for $n = p + 1$. Assume that $V(z) \in S^o(F, \bigcup_{i=1}^{p+1} (\alpha_i, \beta_i)$. Then, clearly, $V(z) \in S^{\circ}_{+}[R, \bigcup_{j=1}^{p} (\alpha_j, \beta_j)]$ and hence in Theorem 7. Now assume that for $m = p$ from $V(z) \in S^0[R, \bigcup_{j=1}^P (\alpha_j, \beta_j) \big]$ it fol

(i) $V(z) \in S^0(R)$.

(ii) $\prod_{j=1}^P \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0(R)$.

We will show that this fact remains true for $n = p + 1$. Assume that $V(z) \$

(i) $V(z) \in S^{\circ}(R)$. *P*13. - z* (i) $V(z) \in S^0$
(ii) $\prod_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z}$ $\prod_{j=1}^{\lfloor T/2 \rfloor} \frac{Z}{\alpha_j - Z} V(z) \in S^{\circ}(R).$

$$
\prod_{j=1}^P \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0 \left[R, (\alpha_{p+1}, \beta_{p+1}) \right].
$$

Hence by Theorem 7,

$$
\sum_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0 \left[R, (\alpha_{p+1}, \beta_{p+1}) \right].
$$
\n
\n
$$
\sum_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0 \left[R, (\alpha_{p+1}, \beta_{p+1}) \right].
$$
\n
\n
$$
\sum_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) = \sum_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0(R).
$$
\n
\n
$$
\sum_{j=1}^{p} \frac{\beta_j - z}{\alpha_j - z} V(z) = \sum_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S^0 \left[R, \bigcup_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} \right].
$$

An analogous proof works in the case $V(z) \in S^{\circ}[R, \bigcup_{i=1}^{n}(\alpha_i, \beta_i)]$

It is not hard to see that for the classes

$$
S_{\pm}^{1}[R,\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})], S_{\pm}^{01}[R,\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})] \text{ and } S_{\pm}[R,\bigcup_{j=1}^{m}(\alpha_{j},\beta_{j})]
$$

analogous results hold. Combining the above stated theorems we get the following

Theorem 12: A function $V(z)$, whose values are operators in a finite-dimensional Hilbert *space E, belongs to the class* $S_{+}[R, \bigcup_{i=1}^{m}(\alpha_i, \beta_i)] \cap S_{-}[\bigcup_{k=1}^{n}(c_k, d_k)]$ if and only if the following *two conditions hold:*

(i) $V(z) \in S(R)$.

(ii)
$$
\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} \prod_{k=1}^{n} \frac{c_k - z}{d_k - z} V(z) \in S(R).
$$

Note that analogous results can be for

Note that analogous results can be formulated for the classes $S_{\tau}^{\alpha}[R, \bigcup_{i=1}^{m}(\alpha_i, \beta_i)] \cap S_{\tau}^{\alpha}[\bigcup_{k=1}^{m}(\epsilon_k, d_k)],$ $S_{+}^{1}[R, \bigcup_{j=1}^{m}(\alpha_{j}, \beta_{j})] \cap S_{-}^{1}[\bigcup_{k=1}^{m}(c_{k}, d_{k})],$ $S^{\circ 1}_{\uparrow}[R,\bigcup_{i=1}^{m}(\alpha_i,\beta_i)] \cap S^{\circ 1}_{\uparrow}[\bigcup_{k=1}^{m}(\overline{c}_k,\overline{d}_k)].$ *i*],
.].
rator i
D(A)

Definition: Let *A* be a symmetric operator in a Hilbert space $\mathbf{\hat{s}}$. The interval (α, β) is called a *gap* of the operator *A* if

$$
\left\| Af - \frac{\alpha + \beta}{2} f \right\| \ge \frac{\beta - \alpha}{2} \| f \| \quad \text{for all } f \in \mathfrak{D}(A). \tag{39}
$$

Theorem 13: *Let V(z) b a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., V*(*z*) = *K* *($A|_R$ - *zI*)⁻¹ K , where (6) holds. Let $\overline{\mathfrak{D}(A)}$ = \mathfrak{H} and A ≥ 0 . Let (α, β) *be an arbitrary interval of the positive semi-axis. Then* $V(z) \in S_+[\alpha, \beta]$ *and* (α, β) *is a gap of the operator A if and only if the following two conditions hold:*

 (i) *A* $_{\rm I\!P}$ \geq 0.

(ii)
$$
(A|_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\beta}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\beta}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi) \ge 0 \ \forall \varphi \in \mathfrak{H}_+.
$$
 (40)

Proof: Assume that (40) holds. Let $\{z_j\}_{j=1}^P$ be an arbitrary set of non-real complex numbers such that $z_i + z_j$. Let N_{z_k} be the deficiency space of the operator A and $\varphi_i \in N_{z_j}$. Set φ
 $= \sum_{i=1}^{5} (\alpha - z_i)^{-1} \varphi_i$. Since $A^* \varphi_i = z_i \varphi_i$, we obtain from (40)
 $(A_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\beta}{\$

(i)
$$
A|_{R} \ge 0
$$
.
\n(ii) $(A|_{R}\varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(A^*\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(\varphi, A^*\varphi) + \frac{1}{\beta - \alpha}(A^*\varphi, A^*\varphi) \ge 0$
\n**Proof:** Assume that (40) holds. Let $\{z_i\}_{i=1}^P$ be an arbitrary set of non-real co-
\nbers such that $z_i \neq z_i$. Let N_{z_k} be the deficiency space of the operator A and φ_i to
\n $= \sum_{i=1}^P (\alpha - z_i)^{-1} \varphi_i$. Since $A^*\varphi_i = z_i \varphi_i$, we obtain from (40)
\n $(A|_{R}\varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(A^*\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(\varphi, A^*\varphi) + \frac{1}{\beta - \alpha}(A^*\varphi, A^*\varphi)$
\n $= \sum_{i, l=1}^P \frac{1}{(\alpha - z_i)(\alpha - \overline{z_i})}(A_R\varphi_i, \varphi_l) + \frac{\alpha \beta}{\beta - \alpha} \sum_{i, l=1}^P \frac{1}{(\alpha - z_i)(\alpha - \overline{z_i})}(\varphi_i, \varphi_l)$
\n $- \frac{\beta}{\beta - \alpha} \sum_{i, l=1}^P \frac{1}{(\alpha - z_i)(\alpha - \overline{z_i})}(A^*\varphi_i, \varphi_l) - \frac{\beta}{\beta - \alpha} \sum_{i, l=1}^P \frac{1}{(\alpha - z_i)(\alpha - \overline{z_i})}(\varphi_i, A^*\varphi_l)$
\n $+ \frac{1}{\beta - \alpha} \sum_{i, l=1}^P \frac{1}{(\alpha - z_i)(\alpha - \overline{z_i})}(A^*\varphi_i, A^*\varphi_l)$
\n $= \frac{1}{\beta - \alpha} \sum_{i, l=1}^P (\frac{\beta - \alpha}{(\alpha - z_i)(\alpha - \overline{z_i})} A_R + \frac{\alpha \beta - \beta(z_i + \overline{z_i$

where $B(\lambda,\mu)$ has the form (16). Thus Theorem 4 yields $V(z) \in S₊[{\alpha,\beta}]$.

Now we will show that (α, β) is a gap of *A*. In fact, if the vector φ of the inequality (40) belongs to $\mathfrak{D}(A)$, then with regard to the inclusions $A^* \supset A$ and $A|_R \supset A$ we obrain

$$
(AR\varphi,\varphi)+\frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi)-\frac{\beta}{\beta-\alpha}(A*\varphi,\varphi)-\frac{\beta}{\beta-\alpha}(\varphi,A*\varphi)+\frac{1}{\beta-\alpha}(A*\varphi,A*\varphi)
$$

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\n=
$$
(A\varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) + \frac{2\beta}{\beta - \alpha}(A\varphi, \varphi) + \frac{1}{\beta - \alpha}(A\varphi, A\varphi)
$$

\n= $\frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) + \frac{1}{\beta - \alpha}(A\varphi, A\varphi) - \frac{\alpha + \beta}{\beta - \alpha}(A\varphi, \varphi) \ge 0$.
\nimplies
\n $(\alpha + \beta)(A\varphi, \varphi) \le \alpha \beta(\varphi, \varphi) + (A\varphi, A\varphi)$.
\nconditions (39) and (41) are equivalent. In fact, if (39) ho
\n $(A, \alpha + \beta) \ge (A, \alpha + \beta) \ge (A, \alpha + \beta) \ge (B, \alpha)^2$.

This implies

$$
(\alpha + \beta)(A\varphi, \varphi) \leq \alpha \beta(\varphi, \varphi) + (A\varphi, A\varphi).
$$
\n(41)

But conditions (39) and (41) are equivalent. In fact, if (39) holds, we get

$$
\left((A - \frac{\alpha + \beta}{2}I)\varphi, \left(A - \frac{\alpha + \beta}{2}I\right)\varphi\right) \geq \left(\frac{\beta - \alpha}{2}\right)^2(\varphi, \varphi),
$$

hence

$$
(A\phi,A\phi) \text{ - } (\alpha \text{ + } \beta)(A\phi,\phi) \text{ + } \left(\!\frac{\alpha \text{ + } \beta}{2}\!\right)^{\!2}\!\!\left(\phi,\phi\right) \text{ } \geq \left(\!\frac{\beta-\alpha}{2}\!\right)^{\!2}\!\!\left(\phi,\phi\right)
$$

and $(\alpha + \beta)(A\varphi, \varphi) \leq \alpha \beta(\varphi, \varphi) + (A\varphi, \varphi)$, i.e., (41). It is not hard to see that the converse conclusion is also true. Thus, the interval (α, β) is a gap of the operator A.

Now let $V(z) \in S₊[(\alpha, \beta)]$ and the interval (α, β) be a gap of the operator *A*. Then by Theorem 4 it holds (15). As it was proved above, this yields the inequality (40) for all vectors φ of the form

$$
\varphi = \sum_{i=1}^{p} (\alpha - z_i)^{-1} \varphi_i,
$$
 (42)

where $\{z_i\}_{i=1}^P$ be an arbitrary set of non-real complex numbers such that $z_i \neq z_I$ and φ_i is an Now let $V(z) \in S_+[(\alpha, \beta)]$ and the interval (α, β) be a gap of the operator *A*. Then by Theo-
rem 4 it holds (15). As it was proved above, this yields the inequality (40) for all vectors φ of
the form
 $\varphi = \sum_{i=1}^{P$ arbitrary vector of the deficiency space N_{z_i} . Let $\mathfrak{H}_1 = \bigvee_{z \in \mathbb{Z}} N_z$, where the closure is taken with respect to the metric of the space \mathfrak{H} . Then, clearly, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where the subs and \mathfrak{H}_2 are invariant subspaces of the operator *A* and the operator $A_2 = A|\mathfrak{g}_2|$ is selfadjoint. Thus, $A = A_1 \oplus A_2$, where $A_1 = A|_{\mathfrak{D}_1}$. It is easy to see that $A^* = A_1^* \oplus A_2$. It follows that each vector $\varphi \in \mathfrak{H}_+$ can be representeds in the form $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in \overline{\mathfrak{D}(A_1^*)}$ and $\varphi_2 \in \mathfrak{D}(A_2)$. Since the operators A_R and A^* are continuous operators from \mathfrak{H}_+ into \mathfrak{H}_- , we can extend the inequality (40) from all vecotrs of the form (42) to all vectors $\varphi \in \overline{\mathfrak{D}(A_i^*)}$. It is easy to see that an arbitrary selfadjoint extension \widetilde{A} of the operator A has the form $\widetilde{A} = \widetilde{A}_1 \oplus A_2$, wherte \widetilde{A}_1 is a selfadjoint extension of the operator A_{1} in the space \mathfrak{H}_{1} . Since by assumption the interval (α,β) is a gap of the operator *A*, it is also a gap of the operator A_2 . Thus, for the operator A_2 it holds (39) and hence (41), as it was shown above. Setting φ = $\phi_\texttt{1}$ + $\phi_\texttt{2}$ in (40) we obtain *(ARp, p)* $+ \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) - \frac{\beta}{\beta-\alpha}(A^*\varphi,\varphi) + \frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi) + \frac{\alpha\beta}{\$ are invariant subspaces
 $A_1 = A_1 \oplus A_2$, where $A_1 = A_2$
 $\rho \in \mathfrak{H}_+$ can be represente

the operators A_R and A^* a

ty (40) from all vecotrs of

rary selfadjoint extension

int extension of the operator A , it i

IDENTIFY: Seraagonit extension A of the operator A has the form
$$
A = A_1 \oplus A_2
$$
, where A_1 = $A_1 \oplus A_2$, where A_1 is a disjoint extension of the operator A_1 in the space \mathfrak{F}_1 . Since by assumption the interval (gap of the operator A, it is also a gap of the operator A_2 . Thus, for the operator A_2 it is and hence (41), as it was shown above. Setting $\varphi = \varphi_1 + \varphi_2$ in (40) we obtain

\n
$$
(A_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\beta}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\beta}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi)
$$

\n
$$
= (A_R \varphi_1, \varphi_1) + \frac{\alpha \beta}{\beta - \alpha} (\varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (A^* \varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (\varphi_1, A^* \varphi_1) + \frac{1}{\beta - \alpha} (A^* \varphi_1, A^* \varphi_1)
$$

\n
$$
+ (A_R \varphi_2, \varphi_2) + \frac{\alpha \beta}{\beta - \alpha} (\varphi_2, \varphi_2) - \frac{\beta}{\beta - \alpha} (A^* \varphi_2, \varphi_2) - \frac{\beta}{\beta - \alpha} (\varphi_2, A^* \varphi_2) + \frac{1}{\beta - \alpha} (A^* \varphi_2, A^* \varphi_2)
$$

\n
$$
= \left((A_R \varphi_1, \varphi_1) + \frac{\alpha \beta}{\beta - \alpha} (\varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (A^* \varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (\varphi_1, A^* \varphi_1) + \frac{1}{\beta - \alpha} (A^* \varphi_1, A^* \varphi_1) \right)
$$

\n

We note that the last inequality holds since the terms in the big brackets are non-negative \blacksquare

Theorem 14: *Let V(z) be a realizable operator-valued function in a finite-dimensional* **Hubert space E, i.e.,** $V(z)$ be a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., $V(z) = K^*(A|_R - zI)^{-1}K$, where (6) holds. Let $\overline{2O(A)} = S_0$ and $A \ge 0$. Let (α, β) be an arbitrary in (α, β) be an arbitrary interval of the positive semi-axis. Then $V(z) \in S$ ₁ (α, β) and (α, β) *is a gap of the operator A if and only if the following two conditions hold:*

(i) $A \geq 0$.

(ii)
$$
-(A|_{R}\varphi,\varphi)+\frac{\alpha\beta}{\beta-\alpha}(\varphi,\varphi)-\frac{\alpha}{\beta-\alpha}(A^{\bullet}\varphi,\varphi)-\frac{\alpha}{\beta-\alpha}(\varphi,A^{\bullet}\varphi)+\frac{1}{\beta-\alpha}(A^{\bullet}\varphi,A^{\bullet}\varphi)\geq 0 \,\forall \varphi \in \mathfrak{H}_{+}.
$$
 (43)

Proof: Assume that (43) holds. Let $\{z_i\}_{i=1}$ be an arbitrary set of non-real complex numbers such that z_i *** z_1 . Let N_z , be the feficiency space of the operator *A* and $\varphi_i \in N_{z_i}$. Setting $\varphi = \sum_{i=1}^{p} (\alpha - z_i)^{-1} \varphi_i$ in (43), we obtain with regard to $A^* \varphi_i = z_i \varphi_i$ **Proof:** Assume that (43) holds. Let $\{z_i\}_{i=1}$ be an arbitrary set of non such that $z_i \neq z_j$. Let N_{z_j} be the feficiency space of the operator A $z_i^p = 1$ ($\alpha - z_i$)⁻¹ φ_i in (43), we obtain with regard to $A^* \$

e operator A if and only if the following two conditions hold:
\ni)
$$
A l \ge 0
$$
.
\nii) $-(A|_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi) \ge 0$
\n**Proof:** Assume that (43) holds. Let $\{z_j\}_{j=1}$ be an arbitrary set of non-real c
\nsuch that $z_j \neq z_j$. Let N_{z_j} be the efficiency space of the operator A and $\varphi_j \in$
\n $\int_{i=1}^P (\alpha - z_j)^{-1} \varphi_j$ in (43), we obtain with regard to $A^* \varphi_j \stackrel{?}{=} z_j \varphi_j$
\n $(A|_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi)$
\n $= -\sum_{i,j=1}^P \frac{1}{(\beta - z_j)(\beta - \overline{z_j})} (A|_R \varphi_j, \varphi_j) + \frac{\alpha \beta}{\beta - \alpha} \sum_{i,j=1}^P \frac{1}{(\beta - z_j)(\beta - \overline{z_j})} (\varphi_j, \varphi_j)$
\n $- \frac{\alpha}{\beta - \alpha} \sum_{i,j=1}^P (\frac{1}{(\beta - z_j)(\beta - \overline{z_j})} (A^* \varphi_i, \varphi_j) - \frac{\alpha}{\beta - \alpha} \sum_{i,j=1}^P (\frac{1}{(\beta - z_j)(\beta - \overline{z_j})} (\varphi_i, A^* \varphi_j)$
\n $+ \frac{1}{\beta - \alpha} \sum_{i,j=1}^P (\frac{\alpha - \beta}{(\beta - z_j)(\beta - \overline{z_j})} A|_R + \frac{\alpha \beta - \alpha(z_j + \overline{z_j}) + z_j \overline{z_j}}{(\beta - z_j)(\beta - \overline{z_j})} I|_{\var$

where $B(\lambda,\mu)$ has the form (20). This implies the inclusion $V(z) \in S[\alpha,\beta]$ by Theorem 5.

We will now show that (α, β) is a gap of the operator *A*. In fact, if the vector φ in the inequality (43) belongs to $\mathfrak{D}(A)$, then with regard to the inclusions $A^* \supset A$ and $A \mid R \supset A$ we obtain

$$
= \frac{1}{\beta - \alpha} \sum_{i, l = 1}^{P} (B(z_i, z_l)\varphi_i, \varphi_l) \ge 0,
$$

re $B(\lambda, \mu)$ has the form (20). This implies the inclusion $V(z) \in S \cdot [(\alpha, \beta)]$
We will now show that (α, β) is a gap of the operator A. In fact, if the
ity (43) belongs to $\mathfrak{D}(A)$, then with regard to the inclusions $A^* \supset A$ an
 $-(A\vert_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi)$

$$
= -(A\varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{2\alpha}{\beta - \alpha} (A\varphi, \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi)
$$

$$
= \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) + \frac{1}{\beta - \alpha} (A\varphi, A\varphi) - \frac{\alpha + \beta}{\beta - \alpha} (A\varphi, \varphi) \ge 0.
$$

This yields $(\alpha + \beta)(A\varphi, \varphi) \leq \alpha\beta(\varphi, \varphi) + (A\varphi, A\varphi)$. Thus, the relation (41) is true for the operator A. As it was shown above, this implies that the interval (α, β) is a gap of A. The necessity part can be proved in an analogous way as the necessity part of Theorem **131**

As a corollary of Theorems 13 and 14 we obtain the following general

Theorem 15: *Let V(z) be a realizable operator-valued function in a finite-dimensional Hilbert space E, i.e., V(z)* = $K^*(A_R - zI)^{-1}K$, where (6) holds. Let $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $A \ge 0$. Let (α_i, β_i) $(j = 1,..., m)$ and (c_k, d_k) $(k = 1,..., n)$ two arbitrary sets of mutually disjoint intervals of the po*sitive semi-axis. Then* $V(z) \in S_{+}\left[\bigcup_{j=1}^{m}(\alpha_j,\beta_j)\right] \cap S_{-}\left[\bigcup_{k=1}^{n}(c_k,d_k)\right]$ and all intervals (α_j,β_j) and

(i) Al^:O.

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\nsitive semi-axis. Then
$$
V(z) \in S_x[U_{j=1}^m(\alpha_j, \beta_j)] \cap S_x[U_{k=1}^n(c_k, d_k)]
$$
 and all intervals (α_j, β_j) and
\n (c_k, d_k) are gaps of the operator A if and only if the following tree conditions hold:
\n(i) $Al \ge 0$.
\n(ii) $(Al_R \varphi, \varphi) + \frac{\alpha_j \beta_j}{\beta_j - \alpha_j}(\varphi, \varphi) - \frac{\beta_j}{\beta_j - \alpha_j}(A^* \varphi, \varphi) - \frac{\beta_j}{\beta_j - \alpha_j}(\varphi, A^* \varphi) + \frac{1}{\beta_j - \alpha_j}(A^* \varphi, A^* \varphi) \ge 0$
\nfor each $\varphi \in S_x$ and all $j = 1,..., m$.
\n(iii) $-(Al_R \varphi, \varphi) + \frac{c_k d_k}{d_k - c_k}(\varphi, \varphi) - \frac{c_k}{d_k - c_k}(A^* \varphi, \varphi) - \frac{c_k}{d_k - c_k}(\varphi, A^* \varphi) + \frac{1}{d_k - c_k}(A^* \varphi, A^* \varphi) \ge 0$
\nfor each $\varphi \in S_x$ and all $k = 1,..., n$.

for each $\varphi \in \mathfrak{H}_+$ *and all j* = 1,..., *m.*

$$
\text{(iii)} \; \text{-} (A\!l_R \varphi, \varphi) + \frac{c_k d_k}{d_k - c_k} (\varphi, \varphi) - \frac{c_k}{d_k - c_k} (A^* \varphi, \varphi) - \frac{c_k}{d_k - c_k} (\varphi, A^* \varphi) + \frac{1}{d_k - c_k} (A^* \varphi, A^* \varphi) \ge 0
$$

for each $\varphi \in \mathfrak{H}_+$ *and all* $k = 1, ..., n$.

Remark: Theorems 14 and 15 were obtained in collaboration with E. R. Tsekanovskii.

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