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Approximate Solution of Bisingular Integro-Differential Equations

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This paper gives necessary and sufficient conditions for the applicability of collocation and Galerkin methods to bisingular integro-differential equations with continuous coefficients. They are obtained by a local principle for paraalgebras.

Key words: *Bisingular integro-differential equation, approximation method, paraalgebra, local principle*

AMS subject classification: 45E05, 47D30, 65J10

1. Introduction

When solving bisingular integro-differential equations by collocation and Galerkin methods one naturally asks whether the approximate solutions exist, are uniquely determined and converge to the exact solution. These problems were studied in [2],[6] for Toeplitz and singular integral operators by means of Banach algebra techniques. The integro-differential operator treated here acts from one Banach space E_1 into another Banach space E_2, where $E_1 \neq E_2$. Thus, there is no multiplication operation in the set $\mathcal{L}(E_1, E_2)$ of all bounded linear operators. This necessitates the consideration of special paraalgebras which allows us to reduce the original problem of the applicability of collocation and Galerkin methods to the investigation of the invertibility of certain elements in a quotient paraalgebra $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. This problem can be solved using a local principle for paraalgebras (cf. [3]) generalizing the well-known local principle of Gohberg-Krupnik [5]. We note that some results on the approximate solution of pseudodifferential equations are already contained in [9].

2. The concept of paraalgebras

We suppose that the reader is familiar with the theory of Banach algebras, especially with the local principle proposed in [5]. The modifications for the case of paraalgebras will be given in the sequel. For convenience, we restrict ourselves to the case of paraalgebras of operators. (The reader is referred for further details and for the general case to [10] and [3].)

Definition 2.1: (a) Let E_i be a Banach space and let A_i be a subalgebra of $\mathcal{L}(E_i) := \mathcal{L}(E_i, E_i), i = 1, 2$. Further let S_1 and S_2 be closed subspaces of $\mathcal{L}(E_1, E_2)$ and $\mathcal{L}(E_2, E_1)$, respectively. If for any operators $A \in S_1$, $B \in S_2$, $C \in A_1$, $D \in A_2$ we have $AB \in A_2$, $BA \in A_1$, $DA, AC \in S_1$, $BD, CB \in S_2$, then the system

$$\mathcal{P} = \begin{pmatrix} A_1 & \begin{matrix} S_1 \\ S_2 \end{matrix} & A_2 \end{pmatrix}$$

is called a *paraalgebra* of operators. It is called a *paraalgebra with identities* if A_i contains the identity operator on E_i, $i = 1, 2$. The elements of $A_1 \cup A_2 \cup S_1 \cup S_2$ are called *the elements of the paraalgebra* $\mathcal{P}$.

(b) A *two-sided ideal* of a paraalgebra $\mathcal{P}$ is a paraalgebra

$$\mathcal{J} = \begin{pmatrix} A_1' & \begin{matrix} S_1' \\ S_2' \end{matrix} & A_2' \end{pmatrix}$$

with $\mathcal{J} \subset \mathcal{P}$ such that for any two elements $A \in \mathcal{J}$, $B \in \mathcal{P}$ for which the operation AB or BA is performable, the product AB or BA belongs to $\mathcal{J}$. It can be verified that in this case

$$\mathcal{P}/\mathcal{J} := \begin{pmatrix} A_1/A_1' & \begin{matrix} S_1/S_1' \\ S_2/S_2' \end{matrix} & A_2/A_2' \end{pmatrix}$$

is a paraalgebra again. It is called *the quotient-paraalgebra of* $\mathcal{P}$ *with respect to* $\mathcal{J}$.

(c) Let $M^{(i)}$ be a localizing class in $A_i, i = 1, 2$ (cf.[5]). They are said to *commute with respect to an element* $A \in S_1$ if

(i) for each $C \in M^{(1)}$ there exists a $D \in M^{(2)}$ such that $AC = DA$,

(ii) for each $D \in M^{(2)}$ there exists a $C \in M^{(1)}$ such that $AC = DA$.

Two elements $A, A' \in S_1$ are called $\{M^{(1)}, M^{(2)}\}$*-equivalent* if

$$\inf_{C \in M^{(1)}} \|(A - A')C\| = \inf_{D \in M^{(2)}} \|D(A - A')\| = 0.$$

An element $A \in S_1$ is called $\{M^{(1)}, M^{(2)}\}$*-invertible* if there exist $C \in M^{(1)}$, $D \in M^{(2)}$ and $B \in S_2$ such that

$$BAC = C \qquad \text{and} \qquad DAB = D.$$

Theorem 2.2 (Local principle for paraalgebras, cf.[3, Theorem 3.1]): *Let* $\{M_\omega^{(i)}\}_{\omega \in \Omega}$ *be a covering system of localizing classes in* A_i $(i = 1, 2)$ *commuting for each* $\omega \in \Omega$ *with respect to an element* $A \in S_1$. *Further let* A *be* $\{M_\omega^{(1)}, M_\omega^{(2)}\}$*-equivalent to* $A_\omega \in S_1$ *for each* ω. *Then* A *is invertible if and only if* A_ω *is* $\{M_\omega^{(1)}, M_\omega^{(2)}\}$*-invertible for each* $\omega \in \Omega$.

Now we proceed to the construction of a paraalgebra which can be related to approximation methods for certain classes of operator equations in a pair of Banach spaces. Let

X, Y, Z, V be Banach spaces. We will denote the strong convergence of an operator sequence $\{A_n\}_{n=1}^\infty$ to A by $A_n \to A$ as $n \to \infty$. Assume that $\{P_n^\mu\}_{n=1}^\infty, \mu \in \{X, Y, Z, V\}$, are operator sequences defined on μ, where

(i) $(P_n^\mu)^2 = P_n^\mu$

(ii) $P_n^\mu \to I_\mu$ (the identity operator on μ) as $n \to \infty$.

Analogously to [7],[12], we assume that we are given operator sequences $\{W_n^Y\}_{n=1}^\infty$ and $\{W_n^V\}_{n=1}^\infty$ on Y and V, respectively, which satisfy

(iii) $(W_n^Y)^2 = P_n^Y$, $(W_n^V)^2 = P_n^V$

(iv) $W_n^Y P_n^Y = W_n^Y$, $W_n^V P_n^V = W_n^V$

(v) the operators $W_n^Y, W_n^V, (W_n^Y)^*, (W_n^V)^*$ converge weakly to zero as $n \to \infty$.

(vi) $(P_n^Y)^* \to I_{Y^*}$, $(P_n^V)^* \to I_{V^*}$ as $n \to \infty$.

Further, denote by $\mathcal{C}^Y$ the set of all sequences $\{C_n\}_{n=1}^\infty$, $C_n : \operatorname{im} P_n^Y \to \operatorname{im} P_n^Y$ for which there exist operators $C, \tilde{C} \in \mathcal{L}(Y)$ such that

$$C_n P_n^Y \to C \ , \ W_n^Y C_n W_n^Y \to \tilde{C},$$

$$C_n^*(P_n^Y)^* \to C^* \ , \ (W_n^Y C_n W_n^Y)^*(P_n^Y)^* \to \tilde{C}^*$$

as $n \to \infty$. Suppose that there is an invertible operator $B \in \mathcal{L}(X, Y)$ being subject to the condition

$$BP_n^X = P_n^Y BP_n^X , \ n = 1, 2, \ldots .$$

Now define the Banach spaces $\mathcal{A}^{XY}, \mathcal{A}^{YX}, \mathcal{A}^X, \mathcal{A}^Y$ as follows:

$$\begin{aligned}
\mathcal{A}^{XY} &= \left\{\{A_n\}_{n=1}^\infty : A_n = P_n^Y C_n B P_n^X ; \{C_n\}_{n=1}^\infty \in \mathcal{C}^Y\right\} \\
\mathcal{A}^{YX} &= \left\{\{A_n\}_{n=1}^\infty : A_n = P_n^X B^{-1} C_n P_n^Y ; \{C_n\}_{n=1}^\infty \in \mathcal{C}^Y\right\} \\
\mathcal{A}^{X} &= \left\{\{A_n\}_{n=1}^\infty : A_n = P_n^X B^{-1} C_n B P_n^X ; \{C_n\}_{n=1}^\infty \in \mathcal{C}^Y\right\} \\
\mathcal{A}^{Y} &= \mathcal{C}^Y .
\end{aligned}$$

The operations in these spaces are defined in a natural way, and the norm is given by $\|\{A_n\}\| = \sup_n \|A_n\|$. Assume further that there exists an invertible operator $D \in \mathcal{L}(Z, V)$ with $DP_n^Z = P_n^V DP_n^Z$ for all $n = 1, 2, \ldots$. As above, we define the Banach spaces $\mathcal{A}^{ZV}, \mathcal{A}^{VZ}, \mathcal{A}^Z, \mathcal{A}^V$ with the help of the operator D. Define

$$\tilde{\mathcal{A}}^{XY,ZV} = \mathcal{A}^{XY} \otimes \mathcal{A}^{ZV} + \mathcal{N}^I ,$$

where $\mathcal{A}^{XY} \otimes \mathcal{A}^{ZV}$ denotes the tensorial product of $\mathcal{A}^{XY}$ and $\mathcal{A}^{ZV}$, i.e., the closure with respect to the supremum norm of the set of all sequences of the form

$$\{A_n\}_{n=1}^{\infty} = \left\{ \sum_{k=1}^{m} B_n^{(k)} \otimes D_n^{(k)} \right\}_{n=1}^{\infty},$$

where $m = 1, 2, 3, \ldots, \{B_n^{(k)}\}_{n=1}^{\infty} \in \mathcal{A}^{XY}, \{D_n^{(k)}\}_{n=1}^{\infty} \in \mathcal{A}^{ZV}$, and $\mathcal{N}^I$ is the set of all sequences of operators $N_n^I \in \mathcal{L}(X \otimes Z, Y \otimes V)$ tending uniformly to zero as $n \to \infty$. Similarly, put

$$\tilde{\mathcal{A}}^{YX,VZ} = \mathcal{A}^{YX} \otimes \mathcal{A}^{VZ} + \mathcal{N}^{II},$$

$$\tilde{\mathcal{A}}^{X,Z} = \mathcal{A}^{X} \otimes \mathcal{A}^{Z} + \mathcal{N}^{III},$$

$$\tilde{\mathcal{A}}^{Y,V} = \mathcal{A}^{Y} \otimes \mathcal{A}^{V} + \mathcal{N}^{IV}.$$

Remark 2.3: Notice that $\{A_n\} \in \tilde{\mathcal{A}}^{XY,ZV}$ implies that there exist operators $A \in \mathcal{L}(X,Y) \otimes \mathcal{L}(Z,V)$ and $C, C_1, C_2, C_3 \in \mathcal{L}(Y) \otimes \mathcal{L}(V)$ satisfying the following relations as $n \to \infty$:

$$A_n(P_n^X \otimes P_n^Z) \to A \tag{1}$$

$$C_n^{(1)} := (W_n^Y \otimes P_n^V)A_n(P_n^X \otimes P_n^Z)B^{-1} \otimes D^{-1})(W_n^Y \otimes P_n^V) \to C_1 \tag{2}$$

$$C_n^{(2)} := (P_n^Y \otimes W_n^V)A_n(P_n^X \otimes P_n^Z)B^{-1} \otimes D^{-1})(P_n^Y \otimes W_n^V) \to C_2 \tag{3}$$

$$C_n^{(3)} := (W_n^Y \otimes W_n^V)A_n(P_n^X \otimes P_n^Z)B^{-1} \otimes D^{-1})(W_n^Y \otimes W_n^V) \to C_3 \tag{4}$$

$$[A_n(P_n^X \otimes P_n^Z)(B^{-1} \otimes D^{-1})(P_n^Y \otimes P_n^V)]^*[P_n^Y \otimes P_n^V]^* \to C^* \tag{5}$$

$$(C_n^{(i)})^*[P_n^Y \otimes P_n^V]^* \to C_i^*, \ i = 1, 2, 3. \tag{6}$$

For the sake of brevity we shall assume that the spaces Y, V satisfy $\mathcal{K}(Y \otimes V) = \mathcal{K}(Y) \otimes \mathcal{K}(V)$, where $\mathcal{K}(\mu)$ designates the set of all compact operators on the Banach space μ. Observe that all spaces occuring in Section 4 possess this property. So we can define the ideal of our paraalgebra by means of tensorial techniques. To this end define the sets $\mathcal{J}^{XY} \subset \mathcal{A}^{XY}, \mathcal{J}^{YX} \subset \mathcal{A}^{YX}, \mathcal{J}^{X} \subset \mathcal{A}^{X}, \mathcal{J}^{Y} \subset \mathcal{A}^{Y}$ by

$$\begin{aligned}
\mathcal{J}^{XY} &= \left\{ \{J_n\}_{n=1}^{\infty} : J_n = P_n^Y T B P_n^X + W_n^Y M W_n^Y B P_n^X + N_n^{(1)} \right\} \\
\mathcal{J}^{YX} &= \left\{ \{J_n\}_{n=1}^{\infty} : J_n = P_n^X B^{-1} P_n^Y T P_n^Y + P_n^X B^{-1} W_n^Y M W_n^Y + N_n^{(2)} \right\} \\
\mathcal{J}^{X} &= \left\{ \{J_n\}_{n=1}^{\infty} : J_n = P_n^X B^{-1} P_n^Y T B P_n^X + P_n^X B^{-1} W_n^Y M W_n^Y B P_n^X + N_n^{(3)} \right\} \\
\mathcal{J}^{Y} &= \left\{ \{J_n\}_{n=1}^{\infty} : J_n = P_n^Y T P_n^Y + W_n^Y M W_n^Y + N_n^{(4)} \right\},
\end{aligned}$$

where $N_n^{(i)}$ are operators on the adequate spaces satisfying $\|N_n^{(i)}\| \to 0$ as $n \to \infty$ $(i = 1,2,3,4)$ and T, M run through $\mathcal{K}(Y)$. Similarly, we define $\mathcal{J}^{ZV} \in \mathcal{A}^{ZV}, \mathcal{J}^{VZ} \in \mathcal{A}^{VZ}, \mathcal{J}^{Z} \in \mathcal{A}^{Z}, \mathcal{J}^{V} \in \mathcal{A}^{V}$.

Now let

$$\begin{aligned}
\tilde{\mathcal{J}}^{XY,ZV} &= \mathcal{J}^{XY} \otimes \mathcal{J}^{ZV} + \mathcal{N}^{I} \quad , \quad \tilde{\mathcal{J}}^{X,Z} = \mathcal{J}^{X} \otimes \mathcal{J}^{Z} + \mathcal{N}^{III} \\
\tilde{\mathcal{J}}^{YX,VZ} &= \mathcal{J}^{YX} \otimes \mathcal{J}^{VZ} + \mathcal{N}^{II} \quad , \quad \tilde{\mathcal{J}}^{Y,V} = \mathcal{J}^{Y} \otimes \mathcal{J}^{V} + \mathcal{N}^{IV} ,
\end{aligned}$$

where $\mathcal{N}^{I,II,III,IV}$ are as above. Finally, we introduce one more notation.

Definition 2.4: Given a sequence of projections $\{P_n^{(i)}\}_{n=1}^{\infty}$ on the Banach space $E_i, i = 1,2$. For $A \in \mathcal{L}(E_1, E_2)$ let $A_n \in \mathcal{L}(\mathrm{im} P_n^{(1)}, \mathrm{im} P_n^{(2)})$ be the restriction of $P_n^{(2)} A$ to $\mathrm{im} P_n^{(1)}$. We denote by $\Pi\{P_n^{(1)}, P_n^{(2)}\}$ the set of all operators A for which

(i) $A_n P_n^{(1)} \to A$ as $n \to \infty$

(ii) A_n is invertible for all sufficiently large n , say $n \geq n_0$

(iii) $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$

(iv) $P_n^{(2)} \to I_{E_2}$ as $n \to \infty$.

Remark 2.5: The importance of the set $\Pi\{P_n^{(1)}, P_n^{(2)}\}$ can be illustrated by the following: If $A \in \Pi\{P_n^{(1)}, P_n^{(2)}\}$, then for all $y \in E_2$ the sequence $\{x_n\}_{n=n_0}^{\infty}$, where $x_n \in \mathrm{im} P_n^{(1)}$ is the (unique) solution of $A_n x_n = P_n^{(2)} y$, converges to an element $x \in E_1$ which satisfies $Ax = y$.

3. General theorem

As in [4, Theorem 1.2], we prove

Lemma 3.1: *The sets*

$$\tilde{\mathcal{A}} = \begin{pmatrix} \tilde{\mathcal{A}}^{X,Z} & \begin{matrix} \tilde{\mathcal{A}}^{XY,ZV} \\ \tilde{\mathcal{A}}^{YX,VZ} \end{matrix} & \tilde{\mathcal{A}}^{Y,V} \end{pmatrix} \text{ and } \tilde{\mathcal{J}} = \begin{pmatrix} \tilde{\mathcal{J}}^{X,Z} & \begin{matrix} \tilde{\mathcal{J}}^{XY,ZV} \\ \tilde{\mathcal{J}}^{YX,VZ} \end{matrix} & \tilde{\mathcal{J}}^{Y,V} \end{pmatrix}$$

are a paraalgebra with identities and a closed two-sided ideal in $\tilde{\mathcal{A}}$, *respectively.*

Denote by $\{A_n\}\check{}\ (\in \tilde{\mathcal{A}}/\tilde{\mathcal{J}})$ the coset containing the sequence $\{A_n\}$. The next theorem states a criterion for $A \in \Pi\{P_n^X \otimes P_n^Z, P_n^Y \otimes P_n^V\}$ in terms of the invertibility of certain elements in a quotient-paraalgebra.

Theorem 3.2: *Let* $A \in \mathcal{L}(X \otimes Z, Y \otimes V)$ *be an operator for which* $\{A_n\}_{n=1}^{\infty} \in \mathcal{A}^{XY,ZV}$ *and* $A_n(P_n^X \otimes P_n^Z) \to A$ *as* $n \to \infty$, *where* A_n *is defined according to Definition 2.4 (with* $P_n^{(1)} = P_n^X \otimes P_n^Z, P_n^{(2)} = P_n^Y \otimes P_n^V$*). For* $A \in \Pi\{P_n^X \otimes P_n^Z, P_n^Y \otimes P_n^V)$ *it is necessary and* *sufficient that the operators A, C_1, C_2, C_3 from Remark 2.3 are invertible and the coset $\{A_n\}^\cdot$ is invertible in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$.*

Proof: We shall give a proof for the sufficiency part only. Assume that the coset $\{A_n\}^\cdot$ is invertible in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. Then there exists a sequence $\{B_n\}_{n=1}^\infty \in \tilde{\mathcal{A}}^{YX,VZ}$ such that

$$\begin{aligned} B_nA_n = {} & P_n^X \otimes P_n^Z + (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} P_n^V)T(P_n^Y B P_n^X \otimes P_n^V D P_n^Z) \\ & + (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} P_n^V) M (W_n^Y B P_n^X \otimes P_n^V D P_n^Z) \\ & + (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} W_n^V) R (P_n^Y B P_n^X \otimes W_n^V D P_n^Z) \\ & + (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} W_n^V) S (W_n^Y B P_n^X \otimes W_n^V D P_n^Z) + N_n, \end{aligned}$$

where $T, M, R, S \in \mathcal{K}(Y \otimes V)$ and $\|N_n\| \to 0$ as $n \to \infty$. Since the operators $A \in \mathcal{L}(X \otimes Z, Y \otimes V)$, $C_1, C_2, C_3 \in \mathcal{L}(Y \otimes V)$ are invertible we can define a sequence $\{B_n'\}_{n=1}^\infty$ by

$$\begin{aligned} B_n' = B_n & - (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} P_n^V) T (B \otimes D) A^{-1} (P_n^Y \otimes P_n^V) \\ & - (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} P_n^V) M C_1^{-1} (W_n^Y \otimes P_n^V) \\ & - (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} W_n^V) R C_2^{-1} (P_n^Y \otimes W_n^V) \\ & - (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} W_n^V) S C_3^{-1} (W_n^Y \otimes W_n^V) \end{aligned}$$

and calculate the product $B_n' A_n$:

$$\begin{aligned} B_n' A_n = {} & P_n^X \otimes P_n^Z + (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} P_n^V) T (B \otimes D) A^{-1} \\ & \times [A - (P_n^Y \otimes P_n^V) A_n (P_n^X \otimes P_n^Z)](P_n^X \otimes P_n^Z) \\ & + (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} P_n^V) M C_1^{-1} \\ & \times [C_1 - (W_n^Y \otimes P_n^V) A_n (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} P_n^V)](W_n^Y B P_n^X \otimes P_n^V D P_n^Z) \\ & + (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} W_n^V) R C_2^{-1} \\ & \times [C_2 - (P_n^Y \otimes W_n^V) A_n (P_n^X B^{-1} P_n^Y \otimes P_n^Z D^{-1} W_n^V)](P_n^Y B P_n^X \otimes W_n^V D P_n^Z) \\ & + (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} W_n^V) S C_3^{-1} \\ & \times [C_3 - (W_n^Y \otimes W_n^V) A_n (P_n^X B^{-1} W_n^Y \otimes P_n^Z D^{-1} W_n^V)](W_n^Y B P_n^X \otimes W_n^V D P_n^Z) + N_n'. \end{aligned} \tag{7}$$

By virtue of (1) - (6), we derive from (7) that

$$B_n' A_n = P_n^X \otimes P_n^Z + N_n'', \tag{8}$$

where $\|N_n''\| \to 0$ as $n \to \infty$. Hence, the operators $A_n : \mathrm{im}(P_n^X \otimes P_n^Z) \to \mathrm{im}(P_n^Y \otimes P_n^V)$ are left invertible for all sufficiently large n. Analogously, we find a sequence $\{B_n''\}_{n=1}^\infty$ with

$$A_n B_n'' = P_n^Y \otimes P_n^V + N_n''' \quad , \quad \|N_n'''\| \to 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

Now the proof follows immediately from the relations (8),(9) ∎

Next we proceed to a result about the invertibility in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. Therefore we introduce the paraalgebras

$$\tilde{\mathcal{J}}^{1,ZV} = \begin{pmatrix} \mathcal{A}^X \otimes \mathcal{J}^Z + \mathcal{N}^{III} & \begin{matrix} \mathcal{A}^{XY} \otimes \mathcal{J}^{ZV} + \mathcal{N}^{I} \\ \mathcal{A}^{YX} \otimes \mathcal{J}^{VZ} + \mathcal{N}^{II} \end{matrix} & \mathcal{A}^Y \otimes \mathcal{J}^V + \mathcal{N}^{IV} \end{pmatrix}$$

$$\tilde{\mathcal{J}}^{XY,2} = \begin{pmatrix} \mathcal{J}^X \otimes \mathcal{A}^Z + \mathcal{N}^{III} & \begin{matrix} \mathcal{J}^{XY} \otimes \mathcal{A}^{ZV} + \mathcal{N}^{I} \\ \mathcal{J}^{YX} \otimes \mathcal{A}^{VZ} + \mathcal{N}^{II} \end{matrix} & \mathcal{J}^Y \otimes \mathcal{A}^V + \mathcal{N}^{IV} \end{pmatrix}.$$

Note that they are both ideals in $\tilde{\mathcal{A}}$. So we can consider the quotient-paraalgebras $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{XY,2}$. The corresponding cosets containing the sequence $\{A_n\}_{n=1}^{\infty} \in \tilde{\mathcal{A}}$ will be denoted by $\{A_n\}_1^{\cdot}$ and $\{A_n\}_2^{\cdot}$, respectively.

Remark 3.3: Observe that the quotient-paraalgebras $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{XY,2}$ are smaller than $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$, since $\tilde{\mathcal{J}}$ is properly contained in both $\tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{J}}^{XY,2}$. Therefore one can expect that, in special situations, the question of invertibility in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{XY,2}$ is simpler to be investigated than in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. Actually, this is the case for the paraalgebras considered in Section 4. There the invertibility in the smaller paraalgebras is tackled with the local principle (Theorem 2.2). This, together with Lemma 3.4, will solve the problem of invertibility in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$.

Lemma 3.4: *Let $\{A_n\}_{n=1}^{\infty} \in \tilde{\mathcal{A}}$. The coset $\{A_n\}^{\cdot}$ is invertible in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$ if and only if the cosets $\{A_n\}_1^{\cdot}$ and $\{A_n\}_2^{\cdot}$ are invertible in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{XY,2}$, respectively.*

Proof: Since $\tilde{\mathcal{J}} \subset \tilde{\mathcal{J}}^{1,ZV}$ and $\tilde{\mathcal{J}} \subset \tilde{\mathcal{J}}^{XY,2}$, the invertibility of $\{A_n\}_1^{\cdot}$ and $\{A_n\}_2^{\cdot}$ follows from the invertibility of $\{A_n\}^{\cdot}$. For the proof of the reverse implication suppose that $\{A_n\}_1^{\cdot}$ and $\{A_n\}_2^{\cdot}$ are invertible. Then there exist sequences $\{B_n^{(1)}\}_{n=1}^{\infty}$, $\{B_n^{(2)}\}_{n=1}^{\infty} \in \tilde{\mathcal{A}}$ such that

$$B_n^{(1)} A_n = P_n^X \otimes P_n^Z + \sum_{k=1}^{m_1} (F_n^{(k)} \otimes T_n^{(k)}) + N_n^{(1)}, \tag{10}$$

$$B_n^{(2)} A_n = P_n^X \otimes P_n^Z + \sum_{j=1}^{m_2} (M_n^{(j)} \otimes G_n^{(j)}) + N_n^{(2)}, \tag{11}$$

where $\{F_n^{(k)}\} \in \mathcal{A}^X$, $\{T_n^{(k)}\} \in \mathcal{J}^Z$ $(k = 1, 2, \ldots, m_1)$, $\{M_n^{(j)}\} \in \mathcal{J}^X$, $\{G_n^{(j)}\} \in \mathcal{A}^Z$ $(j = 1, 2, \ldots, m_2)$ and $\|N_n^{(i)}\| \to 0$ as $n \to \infty$ $(i = 1, 2)$. From (10),(11) we get

$$(B_n^{(1)} + B_n^{(2)} - B_n^{(1)} A_n B_n^{(2)}) A_n = P_n^X \otimes P_n^Z - \sum_{k=1}^{m_1} \sum_{j=1}^{m_2} (F_n^{(k)} M_n^{(j)} \otimes T_n^{(k)} G_n^{(j)}) + N_n^{(3)}.$$

Since

$$\left\{ \sum_{k=1}^{m_1} \sum_{j=1}^{m_2} (F_n^{(k)} M_n^{(j)} \otimes T_n^{(k)} G_n^{(j)}) + N_n^{(3)} \right\}_{n=1}^{\infty} \in \tilde{\mathcal{J}} ,$$

$\{A_n\}^{\cdot}$ is left invertible in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. The right invertibility can be shown in a similar way ∎

4. Approximate solution of bisingular integro-differential equations

Let $\Gamma = \{t \in \mathbf{C} : |t| = 1\}$ be the unit circle with the center at the origin of coordinates. It is known [5] that the operators S_1, S_2 defined by

$$(S_1\varphi)(t_1,t_2) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau,t_2)d\tau}{\tau - t_1} \quad , \quad (S_2\varphi)(t_1,t_2) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t_1,\tau)d\tau}{\tau - t_2}$$

are bounded on $L_2(\Gamma\times\Gamma)$. Denote by $P^{++}, P^{+-}, P^{-+}, P^{--}$ the projections

$$P^{\pm\pm} = \frac{1}{4}(I \pm S_1)(I \pm S_2).$$

Define the derivative of a function $f(t) = \sum_{k=-\infty}^{+\infty} f_k t^k$ by

$$\frac{d}{dt}\Big(\sum_{k=-\infty}^{+\infty} f_k t^k\Big) = \sum_{\substack{k=-\infty \\ k\neq -1}}^{+\infty} (k+1) f_{k+1} t^k.$$

In a similar way, we define partial derivatives $\frac{\partial^{r+p}x}{\partial t_1^r \partial t_2^p}$ of functions x on $\Gamma\times\Gamma$, and we denote them for short by $x^{(r,p)}$ $(r, p = 0, 1, 2, \dots)$.

Let m, q be fixed positive integers and consider the following bisingular integro-differential equation

$$\begin{aligned} Kx \equiv\ & a_{mq}P^{++}x^{(m,q)} + b_{mq}P^{+-}x^{(m,q)} + c_{mq}P^{-+}x^{(m,q)} + d_{mq}P^{--}x^{(m,q)} \\ & + \sum_{r=0}^{m-1}\sum_{p=0}^{q-1}\{a_{rp}P^{++}x^{(r,p)} + b_{rp}P^{+-}x^{(r,p)} + c_{rp}P^{-+}x^{(r,p)} + d_{rp}P^{--}x^{(r,p)}\} = f \end{aligned} \tag{12}$$

with the condition

$$\int_\Gamma\int_\Gamma x(t_1,t_2)t_1^{-r-1}t_2^{-p-1}dt_1dt_2 = 0 \qquad (r = 0,1,\dots,m-1;\ p = 0,1,\dots,q-1), \tag{13}$$

where $a_{rp}, b_{rp}, c_{rp}, d_{rp}$ are bounded functions on $\Gamma\times\Gamma$ $(r = 0, 1, \dots, m;\ p = 0, 1, \dots, q)$ and $f \in L_2(\Gamma\times\Gamma)$. Let $\varphi \in L_2(\Gamma\times\Gamma)$. Denote by φ_{jk} $(j, k = 0, \pm 1, \dots)$ its Fourier-coefficients:

$$\varphi_{jk} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}, e^{i\eta})e^{-ij\theta}e^{-ik\eta}d\theta d\eta.$$

We shall seek an approximate solution to equation (12) in the form

$$\begin{aligned} x_n(t_1,t_2) = & \sum_{k=m}^{n+m}\sum_{j=q}^{n+q} x_{kj}t_1^k t_2^j + \sum_{k=m}^{n+m}\sum_{j=-n}^{-1} x_{kj}t_1^k t_2^j \\ & + \sum_{k=-n}^{-1}\sum_{j=q}^{n+q} x_{kj}t_1^k t_2^j + \sum_{k=-n}^{-1}\sum_{j=-n}^{-1} x_{kj}t_1^k t_2^j\,, \end{aligned} \tag{14}$$

where the coefficients x_{kj} are determined by the following system of linear algebraic equations:

$$
\begin{aligned}
& \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{(m+k)!(q+j)!}{k!j!} a_{l-k,s-j}^{(m,q)} x_{m+k,q+j} \\
+ & \sum_{k=0}^{n} \sum_{j=q+1}^{q+n} (-1)^{q} \frac{(m+k)!(j-1)!}{k!(j-q-1)!} b_{l-k,s+j}^{(m,q)} x_{m+k,q-j} \\
+ & \sum_{k=m+1}^{n+m} \sum_{j=0}^{n} (-1)^{m} \frac{(k-1)!(q+j)!}{(k-m-1)!j!} c_{l+k,s-j}^{(m,q)} x_{m-k,q+j} \\
+ & \sum_{k=m+1}^{n+m} \sum_{j=q+1}^{n+q} (-1)^{m+q} \frac{(k-1)!(j-1)!}{(k-m-1)!(j-q-1)!} d_{l+k,s+j}^{(m,q)} x_{m-k,q-j} \\
+ & \sum_{r=0}^{m-1} \sum_{p=0}^{q-1} \left\{ \sum_{k=m-r}^{n+m-r} \sum_{j=q-p}^{n+q-p} \frac{(r+k)!(p+j)!}{k!j!} a_{l-k,s-j}^{(r,p)} x_{r+k,p+j} \right. \\
+ & \sum_{k=m-r}^{n+m-r} \sum_{j=p+1}^{n+p} (-1)^{p} \frac{(r+k)!(j-1)!}{k!(j-p-1)!} b_{l-k,s+j}^{(r,p)} x_{r+k,p-j} \\
+ & \sum_{k=r+1}^{n+r} \sum_{j=q-p}^{n+q-p} (-1)^{r} \frac{(k-1)!(p+j)!}{(k-r-1)!j!} c_{l+k,s-j}^{(r,p)} x_{r-k,p+j} \\
+ & \left. \sum_{k=r+1}^{n+r} \sum_{j=p+1}^{n+p} (-1)^{r+p} \frac{(k-1)!(j-1)!}{(k-r-1)!(j-p-1)!} d_{l+k,s+j}^{(r,p)} x_{r-k,p-j} \right\} = f_{l,s} \quad , \\
& \qquad l,s = -n,-n+1,\ldots,n.
\end{aligned}
\tag{15}
$$

Here $f_{jk}, a_{jk}^{(r,p)}, b_{jk}^{(r,p)}, c_{jk}^{(r,p)}, d_{jk}^{(r,p)}$ $(r = 0,1,\ldots,m;\ p = 0,1,\ldots,q;\ j,k = 0,\pm1,\ldots)$ are the Fourier coefficients of the functions $f, a_{rp}, b_{rp}, c_{rp}, d_{rp}$, respectively. In the following we shall investigate the solvability of the system (15) and the convergence of the sequence $\{x_n\}$ of approximate solutions to the exact solution of problem (12),(13).

We introduce some necessary notations. Denote by $H_2^q = H_2^q(\Gamma)$ the set of $q-$times differentiable functions φ which possess absolutely continuous derivatives $\varphi^{(j)}$ $(j = 0,1,\ldots, q-1)$ and satisfy

$$\int_{\Gamma} \varphi(t) t^{-j-1} dt = 0 \quad (j = 0,1,\ldots,q-1)$$

and for which there exists $\varphi^{(q)} \in L_2(\Gamma)$. Similarly, $H_2^{m,q} = H_2^{m,q}(\Gamma\times\Gamma)$ is the set of functions φ which possess absolutely continuous mixed derivatives $\varphi^{(k,j)}$ $(k = 0,1,\ldots,m-1;\ j = 0,1,\ldots,q-1)$ and satisfy

$$\int_{\Gamma}\int_{\Gamma} \varphi(t_1,t_2) t_1^{-k-1} t_2^{-j-1} dt_1 dt_2 = 0 \quad (k = 0,1,\ldots,m-1;\ j = 0,1,\ldots,q-1)$$

and for which there exists $\varphi^{(m,q)} \in L_2(\Gamma\times\Gamma)$. We consider the following sequence $\{P_n^q\}_{n=1}^{\infty}$ of projections on H_2^q :

$$(P_n^q\varphi)(t) = \sum_{k=q}^{n+q} \varphi_k t^k + \sum_{k=-n}^{-1} \varphi_k t^k, \text{ where } \varphi_k = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) e^{-ik\theta} d\theta \quad (k = 0,\pm1,\ldots).$$

Obviously, $\{P_n\}_{n=1}^{\infty}$ with $P_n = P_n^0$ is a sequence of projections acting on $L_2(\Gamma) = H_2^0(\Gamma)$. It is known that $P_n^* = P_n$ and

$$P_n^q \to I_{H_2^q} \quad , \quad P_n \to I_{L_2} \quad \text{as} \quad n \to \infty.$$

Further we need the projections P^+, P^- being defined on $L_2(\Gamma)$ by

$$(P^+\varphi)(t) = \sum_{k=0}^{\infty} \varphi_k t^k \quad , \quad (P^-\varphi)(t) = \sum_{k=-1}^{-\infty} \varphi_k t^k \quad ,$$

where φ_k are as above.

Let D_t^q denote the operator $(D_t^q\varphi)(t) = \varphi^{(q)}(t)$. According to [11], the operators

$$B = (P^+ + t^m P^-)D_t^m : \; H_2^m \to L_2 \; \text{ and } \; D = (P^+ + t^q P^-)D_t^q : \; H_2^q \to L_2$$

are invertible, and from [1] we can derive the validity of the identities

$$BP_n^m = P_n B P_n^m = P_n B \quad , \quad DP_n^q = P_n D P_n^q = P_n D. \tag{16}$$

Now it can be seen that the linear algebraic system (15) is equivalent to the equation

$$(P_n \otimes P_n)K(P_n^m \otimes P_n^q)x_n = (P_n \otimes P_n)f,$$

where K is defined as in (12).

For a function $f \in L_\infty(\Gamma \times \Gamma)$ put

$$f^{(1)}(t_1, t_2) = f(\bar{t_1}, t_2), \tag{17}$$

$$f^{(2)}(t_1, t_2) = f(t_1, \bar{t_2}), \tag{18}$$

$$f^{(3)}(t_1, t_2) = f(\bar{t_1}, \bar{t_2}). \tag{19}$$

Theorem 4.1 (The Galerkin method): *Let*

$$a_{mq}, b_{mq}, c_{mq}, d_{mq} \in C(\Gamma \times \Gamma), \; a_{rp}, b_{rp}, c_{rp}, d_{rp} \in L_\infty(\Gamma \times \Gamma)$$
$$(r = 0, 1, \ldots, m-1; \; p = 0, 1, \ldots, q-1).$$

For $K \in \Pi\{P_n^m \otimes P_n^q, P_n \otimes P_n\}$ it is necessary and sufficient that the operators

$$K \in \mathcal{L}(H_2^{m,q}(\Gamma \times \Gamma), L_2(\Gamma \times \Gamma)) \text{ and } C_1, C_2, C_3 \in \mathcal{L}(L_2(\Gamma \times \Gamma))$$

are invertible, where

$$\begin{aligned}
C_1 = {} & (P^+ \otimes I)\, a_{mq}^{(1)}\, (P^+ \otimes P^+) + (P^+ \otimes I)\, b_{mq}^{(1)} t_2^{-q}\, (P^+ \otimes P^-) \\
& + (P^- \otimes I)\, c_{mq}^{(1)} t_1^m\, (P^- \otimes P^+) + (P^- \otimes I)\, d_{mq}^{(1)} t_1^m t_2^{-q} (P^- \otimes P^-), \\
C_2 = {} & (I \otimes P^+)\, a_{mq}^{(2)}\, (P^+ \otimes P^+) + (I \otimes P^-)\, b_{mq}^{(2)} t_2^q\, (P^+ \otimes P^-) \\
& + (I \otimes P^+) c_{mq}^{(2)} t_1^{-m} (P^- \otimes P^+) + (I \otimes P^-) d_{mq}^{(2)} t_1^{-m} t_2^q (P^- \otimes P^-), \\
C_3 = {} & (P^+ \otimes P^+)\, a_{mq}^{(3)}\, (P^+ \otimes P^+) + (P^+ \otimes P^-)\, b_{mq}^{(3)} t_2^q\, (P^+ \otimes P^-) \\
& + (P^- \otimes P^+)\, c_{mq}^{(3)} t_1^m\, (P^- \otimes P^+) + (P^- \otimes P^-)\, d_{mq}^{(3)} t_1^m t_2^q\, (P^- \otimes P^-).
\end{aligned}$$

Proof: As in [12], we define on $L_2(\Gamma)$ the operator sequence $\{W_n\}_{n=1}^{\infty}$:

$$(W_n f)(t) = W_n\left(\sum_{k=-\infty}^{+\infty} f_k t^k \right) = f_{-1}t^{-n} + \cdots + f_{-n}t^{-1} + f_n + f_{n-1}t + \cdots + f_0 t^n.$$

This sequence satisfies the relations (iii)-(v) from Section 2, therefore the necessity of the conditions of Theorem 4.1 follows immediately from Theorem 3.2, applied to the case $X = H_2^m(\Gamma)$, $Z = H_2^q(\Gamma)$, $Y = V = L_2(\Gamma)$. The sufficiency can be shown as follows using Theorem 3.2, Lemma 3.4, and Theorem 2.2. At first we describe a system of localizing classes in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1,H_2^q L_2}$ and in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{H_2^m L_2,2}$. Denote by $N_\tau \subset C(\Gamma)$, $\tau \in \Gamma$, the set of real functions f_τ with values in the segment $[0,1]$ and $f_\tau(t) = 1$ for t in some neighborhood of τ. The systems $M_\tau^r \subset \mathcal{A}^{L_2}/\mathcal{J}^{L_2}$, $M_\tau^l \subset \mathcal{A}^{H_2^q}/\mathcal{J}^{H_2^q}$ are defined for $\tau \in \Gamma$ as

$$M_\tau^r = \left\{ \{P_n(P^+ f_\tau P^+ + P^- f_\tau P^-)P_n\}^{\cdot} : f_\tau \in N_\tau \right\}$$

$$M_\tau^l = \left\{ \{P_n^q D^{-1}(P^+ f_\tau P^+ + P^- f_\tau P^-)DP_n^q\}^{\cdot} : f_\tau \in N_\tau \right\}.$$

For each $\tau \in \Gamma$, they are localizing classes in $\mathcal{A}^{L_2}/\mathcal{J}^{L_2}$ and $\mathcal{A}^{H_2^q}/\mathcal{J}^{H_2^q}$, respectively. Indeed, consider the product of two elements $\{P_n^q D^{-1}(P^+ f_\tau^i P^+ + P^- f_\tau^i P^-)DP_n^q\}^{\cdot}$ $(i = 1,2)$ belonging to M_τ^l for some fixed $\tau \in \Gamma$. Using relations (16) and [12:(3.2)] we obtain

$$\begin{aligned}
&P_n^q D^{-1}(P^+ f_\tau^1 P^+ + P^- f_\tau^1 P^-)DP_n^q \cdot P_n^q D^{-1}(P^+ f_\tau^2 P^+ + P^- f_\tau^2 P^-)DP_n^q \\
&\quad = P_n^q D^{-1}(P^+ f_\tau^1 P^+ + P^- f_\tau^1 P^-)P_n(P^+ f_\tau^2 P^+ + P^- f_\tau^2 P^-)DP_n^q \\
&\quad = P_n^q D^{-1}(P^+ f_\tau^1 f_\tau^2 P^+ + P^- f_\tau^1 f_\tau^2 P^-)DP_n^q \qquad (20)\\
&\qquad - P_n^q D^{-1}(P^+ f_\tau^1 P^- f_\tau^2 P^+ + P^- f_\tau^1 P^+ f_\tau^2 P^-)DP_n^q \\
&\qquad - P_n^q D^{-1}(W_n P^+ \tilde{f}_\tau^1 P^- \tilde{f}_\tau^2 P^+ W_n + W_n P^- \tilde{f}_\tau^1 P^+ \tilde{f}_\tau^2 P^- W_n)DP_n^q,
\end{aligned}$$

where $\tilde{f}(t) = f(1/t)$, $t \in \Gamma$. The last two summands belong to the ideal $\mathcal{J}^{H_2^q}$ since $P^+ f P^-$, $P^- f P^+$ are compact for $f \in C(\Gamma)$. Now the assertion follows easily from (20).

Similarly to [12, Theorem 4.1], we can prove that

$$\begin{aligned}
&\Big\{ P_n(aP^+ + bP^-)D_t^q P_n^q \cdot P_n^q D^{-1}(P^+ f_\tau P^+ + P^- f_\tau P^-)DP_n^q \\
&\quad - P_n(af_\tau P^+ + bf_\tau P^-)D_t^q P_n^q \Big\}_{n=1}^{\infty} \in \mathcal{J}^{H_2^q L_2}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\{ P_n(P^+ f_\tau P^+ + P^- f_\tau P^-)P_n \cdot P_n(aP^+ + bP^-)D_t^q P_n^q \\
&\quad - P_n(af_\tau P^+ + bf_\tau P^-)D_t^q P_n^q \Big\}_{n=1}^{\infty} \in \mathcal{J}^{H_2^q L_2},
\end{aligned}$$

where $a, b \in L_\infty(\Gamma)$, $f_\tau \in N_\tau$, $\tau \in \Gamma$. This immediately yield that, for each $\tau \in \Gamma$, the systems M_τ^l and M_τ^r commute (see Definition 2.1(c)) with respect to any element of the form

$$\{P_n(aP^+ + bP^-)D_t^q P_n^q\}^{\cdot},\ a, b \in L_\infty(\Gamma).$$

As a consequence of these considerations we get the following lemmata.

Lemma 4.2: *For each* $\tau \in \Gamma$, *the systems*

$$\left\{\{P_n^m \otimes P_n^q D^{-1}(P^+ f_\tau P^+ + P^- f_\tau P^-) D P_n^q\}_1^{\hat{}},\ f_\tau \in N_\tau\right\}$$

$$\left(\left\{\{P_n \otimes P_n(P^+ f_\tau P^+ + P^- f_\tau P^-) P_n\}_1^{\hat{}},\ f_\tau \in N_\tau\right\}\right)$$

form a left (right) covering system of localizing classes in $\tilde{A}/\tilde{\mathcal{J}}^{1,H_2^q L_2}$, *and they commute with respect to any element of the form*

$$\left\{(P_n \otimes P_n)(aP^{++} + bP^{+-} + cP^{-+} + dP^{--})(D_{t_1}^m \otimes D_{t_2}^q)(P_n^m \otimes P_n^q)\right\}_1^{\hat{}}$$

where $a, b, c, d \in C(\Gamma \otimes \Gamma)$.

Lemma 4.3: *For each* $\tau \in \Gamma$, *the systems*

$$\left\{\{P_n^m B^{-1}(P^+ f_\tau P^+ + P^- f_\tau P^-) B P_n^m \otimes P_n^q\}_2^{\hat{}},\ f_\tau \in N_\tau\right\}$$

$$\left(\left\{\{P_n(P^+ f_\tau P^+ + P^- f_\tau P^-) P_n \otimes P_n\}_2^{\hat{}},\ f_\tau \in N_\tau\right\}\right)$$

form a left (right) covering system of localizing classes in $\tilde{A}/\tilde{\mathcal{J}}^{H_2^m L_2,2}$, *and they commute with respect to any element of the form*

$$\left\{(P_n \otimes P_n)(aP^{++} + bP^{+-} + cP^{-+} + dP^{--})(D_{t_1}^m \otimes D_{t_2}^q)(P_n^m \otimes P_n^q)\right\}_2^{\hat{}}$$

where $a, b, c, d \in C(\Gamma \times \Gamma)$.

Now we continue the proof of Theorem 4.1. Consider an arbitrary fixed $\tau \in \Gamma$ and note that $\{K_n\}_2^{\hat{}} = \{(P_n \otimes P_n) K (P_n^m \otimes P_n^q)\}_2^{\hat{}}$ is locally $\{M_\tau^l, M_\tau^r\}$–equivalent to the element $\{K_n^\tau\}_2^{\hat{}}$, where

$$\begin{aligned} K_n^\tau = \quad & [P_n(a_{mq}(\cdot,\tau)P^+ + c_{mq}(\cdot,\tau)P^-) D_{t_1}^m P_n^m] \otimes [P_n P^+ D_{t_2}^q P_n^q] \\ & + [P_n(b_{mq}(\cdot,\tau)\tau^{-q} P^+ + d_{mq}(\cdot,\tau)\tau^{-q} P^-) D_{t_1}^m P_n^m] \otimes [P_n t_2^q P^- D_{t_2}^q P_n^q] . \end{aligned}$$

Since the operator K is invertible, we infer from [8] the invertibility of the operators

$$K_{01}^\tau,\ K_{11}^\tau,\ K_{02}^\tau,\ K_{12}^\tau :\ H_2^m \to L_2,$$

where

$$K_{01}^\tau = \left(a_{mq}(\cdot,\tau)P^+ + c_{mq}(\cdot,\tau)P^-\right) D_{t_1}^m,$$

$$K_{02}^\tau = \left(P^+ a_{mq}^{(1)}(\cdot,\tau)P^+ + P^- c_{mq}^{(1)}(\cdot,\tau) t_1^{2m} P^-\right) D_{t_1}^m,$$

$$K_{11}^\tau = \left(b_{mq}(\cdot,\tau)P^+ + d_{mq}(\cdot,\tau)P^-\right) D_{t_1}^m,$$

$$K_{12}^{\tau} = \tau^{-q}\left(P^{+}b_{mq}^{(1)}(\cdot,\tau)P^{+} + P^{-}d_{mq}^{(1)}(\cdot,\tau)t_1^{2m}P^{-}\right)D_{t_1}^{m}\,.$$

Therefore, $K_{01}^{\tau}, K_{11}^{\tau} \in \Pi\{P_n^m, P_n\}$ [7]. Then there exists a number $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ the operators

$$K_{01,n}^{\tau}, K_{11,n}^{\tau} : \text{ im } P_n^m \to \text{ im } P_n, \quad \text{where } K_{01,n}^{\tau} = P_n K_{01}^{\tau} P_n^m \quad , \quad K_{11,n}^{\tau} = P_n K_{11}^{\tau} P_n^m \quad ,$$

are invertible and the norms of the inverses are uniformly bounded, i.e.,

$$\|(K_{01,n}^{\tau})^{-1}P_n\| \leq c \quad , \quad \|(K_{01,n}^{\tau})^{-1}P_n\| \leq c \quad , \quad n \geq n_0.$$

Consider the operators $R_n\ : \text{im}(P_n \otimes P_n) \to \text{im}(P_n^m \otimes P_n^q)$ having the form

$$R_n = ((K_{01,n}^{\tau})^{-1}P_n) \otimes (P_n^q D^{-1}P^{+}P_n) + (\tau^q (K_{11,n}^{\tau})^{-1}P_n) \otimes (P_n^q D^{-1}P^{-}P_n)\,.$$

Compute the products $R_n K_n^{\tau}$ and $K_n^{\tau} R_n$:

$$\begin{aligned} R_n K_n^{\tau} &= P_n^m \otimes [(P_n^q D^{-1}P^{+}P_n)(P_n P^{+} D_{t_2}^q P_n^q)] \\ &\quad + P_n^m \otimes [(P_n^q D^{-1}P^{-}P_n)(P_n t_2^q P^{-} D_{t_2}^q P_n^q)] \\ &= P_n^m \otimes [P_n^q D^{-1}(P^{+} + t_2^q P^{-})D_{t_2}^q P_n^q] \\ &= P_n^m \otimes P_n^q \quad , \end{aligned}$$

similarly, $K_n^{\tau} R_n = P_n \otimes P_n$. This shows that, for all fixed $t_2 = \tau$ and for all $n \geq n_0$, the operators

$$K_n^{\tau} : \text{im}(P_n^m \otimes P_n^q) \to \text{im}(P_n \otimes P_n)$$

are invertible (and thus, $\{M_{\tau}^{l}, M_{\tau}^{r}\}$–invertible) and

$$\|(K_n^{\tau})^{-1}(P_n \otimes P_n)\| = \|R_n\| \leq 2c\|D^{-1}\|\,.$$

Using the local principle (Theorem 2.2), we obtain the invertibility of the coset $\{K_n\}_2^{\cdot}$ in the paraalgebra $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{H_2^m L_2, 2}$. Analogously, one can prove the invertibility of $\{K_n\}_1^{\cdot}$ in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}^{1, H_2^q L_2}$. According to Lemma 3.4, these facts yield the invertibility of $\{K_n\}^{\cdot}$ in $\tilde{\mathcal{A}}/\tilde{\mathcal{J}}$. This and the invertibility of A, C_1, C_2, C_3 allow us to apply Theorem 3.2 which finishes the proof ∎

Now we shall study the collocation method for solving the bisingular integro-differential equation (12). Denote by $R = R(\Gamma \times \Gamma)$ the set of functions which are Riemann-integrable on $\Gamma \times \Gamma$. Suppose that $f \in R$. An approximate solution of equation (12) is sought in the form (14), but the unknown coefficients are to be determined by the following system of linear algebraic equations:

$$(Kx_n)(t_j, t_l) = f(t_j, t_l) \quad (j, l = 0, \pm 1, \ldots, \pm n)\,, \tag{21}$$

where $t_j = \exp(\frac{2\pi i}{2n+1}j)$. Introduce the operator

$$L_n : R(\Gamma) \to \operatorname{im} P_n , \quad (L_n f)(t) = \sum_{k=-n}^{n} a_k t^k \quad , \quad a_k = \frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) t_j^{-k} .$$

It is obvious from Remark 2.4 that the solvability of (21) and the convergence of the approximate solutions (14) to the exact solution of (12),(13) is equivalent to the condition

$$K \in \Pi\{P_n^m \otimes P_n^q, L_n \otimes L_n\} .$$

Theorem 4.4 (The collocation method): *Let the conditions of Theorem 4.1 be fulfilled. For the validity of* $K \in \Pi\{P_n^m \otimes P_n^q, L_n \otimes L_n\}$ *it is necessary and sufficient that the operators*

$$K \in \mathcal{L}(H_2^{m,q}(\Gamma \times \Gamma), L_2(\Gamma \times \Gamma)) \text{ and } \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in \mathcal{L}(L_2(\Gamma \times \Gamma))$$

are invertible, where

$$\tilde{C}_1 = a_{mq}^{(1)} P^{++} + b_{mq}^{(1)} t_2^{-q} P^{+-} + c_{mq}^{(1)} t_1^{m} P^{-+} + d_{mq}^{(1)} t_1^{m} t_2^{-q} P^{--} ,$$
$$\tilde{C}_2 = a_{mq}^{(2)} P^{++} + b_{mq}^{(2)} t_2^{q} P^{+-} + c_{mq}^{(2)} t_1^{-m} P^{-+} + d_{mq}^{(2)} t_1^{-m} t_2^{q} P^{--} ,$$
$$\tilde{C}_3 = a_{mq}^{(3)} P^{++} + b_{mq}^{(3)} t_2^{q} P^{+-} + c_{mq}^{(3)} t_1^{m} P^{-+} + d_{mq}^{(3)} t_1^{m} t_2^{q} P^{--} ,$$

and $a_{mq}^{(i)}, b_{mq}^{(i)}, c_{mq}^{(i)}, d_{mq}^{(i)}$ $(i = 1,2,3)$ are defined as in (17) - (19).

Proof: The proof of this assertion runs parallel to that of Theorem 4.1. We take as localizing classes in the paraalgebras $\tilde{A}/\tilde{\mathcal{J}}^{H_2^m L_2, 2}$ and $\tilde{A}/\tilde{\mathcal{J}}^{1, H_2^q L_2}$ the systems

$$\{\{P_n^m B^{-1} L_n f_\tau L_n B P_n^m \otimes P_n^q\}_2^{\cdot}\} \quad , \quad \{\{L_n f_\tau L_n \otimes P_n\}_2^{\cdot}\}$$

and

$$\{\{P_n^m \otimes P_n^q D^{-1} L_n f_\tau L_n D P_n^q\}_1^{\cdot}\} \quad , \quad \{\{P_n \otimes L_n f_\tau L_n\}_1^{\cdot}\} ,$$

respectively, with f_τ running through N_τ , $\tau \in \Gamma$. Now little modifications in the proof of Theorem 4.1 are needed to obtain the assertion ∎

Remark 4.5: It is easily seen that one can state Theorems 4.1 and 4.4 for systems of bisingular integro-differential equations.

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Book reviews

S. REMPEL and B.-W. SCHULZE: **Asymptotics for Elliptic Mixed Boundary Problems. Pseudodifferential and Mellin Operators in Spaces with Conormal Singularity** (Math. Research: Vol. 50). Berlin: Akademie - Verlag 1989, 418 pp.

At the beginning of the 20th century, when great success was being achieved in the formation of the classical theory of partial differential equations, there developed a tendency to start investigations in a whole series of new scientific directions. Among these were, in particular:

a) Investigation of the behaviour of solutions of elliptic equations in the neighbourhood of sets of singular points and the description of removable singularities (possible generalizations of the Liouville, Borel and Bernstein theorems, which are known from potential theory).

b) Investigation of the influence of dimension and smoothness of the carrier (borders of the range of the solution) on the well-posedness of problems for elliptic equations.

c) Extension of the sphere of linear problems, including those of mixed type and others.

Resulting from the investigations in these directions, in the course of more than half a century the foundations were laid for the theory of elliptic equations on open and closed manifolds, respectively.