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On the Continuous Dependence on Parameter of the Solution Set of Differential Inclusions

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We prove a theorem on continuous dependence of solutions of differential inclusions in Banach spaces on parameters and derive from it the first fundamental Bogoliubov type theorem on averaging.

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In recent years the theory of differential inclusions in infinite-dimensional Banach spaces has attracted much attention due to its application to optimal control problems described by partial differential equations. An important question that arises in the study of such inclusions is the continuous dependence of the solution set with respect to a parameter contained in the right-hand side. For finite-dimensional spaces a theorem on continuous dependence of solutions of differential inclusions on parameters has already been established (see [1, 10]). The aim of this note is to extend this theorem to Banach spaces and to derive from it the first fundamental theorem of Bogoliubov on averaging in finite intervals [2] for differential inclusions in standard form in Banach spaces (see [6, 10]).

Throughout the sequel solutions of differential inclusions will always be taken in Carathéodory sense.

Let X be a Banach space with a strictly convex norm $\|\cdot\|$ and the associated metric $\rho(\cdot,\cdot)$. By Comp X (Conv X) we denote the collection of all non-empty compact (convex and compact, respectively) subsets of X endowed with the Hausdorff metric $\alpha(\cdot,\cdot)$, by $\rho(x,A)$ the distance from a point $x \in X$ to a set $A \subset X$, and by $I = [0,T]$ a segment of the real positive axis $\mathbb{R}^+ = [0,+\infty)$, $0 < T \in \mathbb{R}$. By $C(I,X)$ we mean the Banach space of continuous mappings from I to X, equipped with the standard norm. We define the modulus of a set $A \in$ Comp X to be the number $|A| = \alpha(A,\{0\})$. Measurability, strong measurability of mappings and integrals of multivalued mappings are understood as in [8].

Consider the differential inclusion

$$\dot{x}(t) \in F(t,x(t)),\ x(0) = x_0, \tag{1}$$

where $F: I \times X \to \mathrm{Comp}\, X$.

Lemma 1: *Let $F: I \times X \to$ Comp X be a mapping of Carathéodory type (i.e., a mapping such that $F(t,x)$ is strongly measurable in t and continuous in x). Assume, furthermore, that $F(t,x)$ satisfies a Lipschitz condition in x with constant k and that there exists a function $\omega(t)$ integrable on I such that $|F(t,0)| \le \omega(t)$ for all $t \in I$. Then for any strongly measurable mapping*

v: $I \to X$ there exists a solution $x(t)$ of the inclusion (1) *such that, almost everywhere on I,* $\|v(t) - x(t)\| = \rho(v(t), F(t, x(t)))$.

Proof: Define a selector $f(t,x) \in F(t,x)$ by the relation $\|v(t) - f(t,x)\| = \rho(v(t), F(t,x))$. Such a selector exists and is uniquely defined by virtue of the convexity and compactness of $F(t,x)$ and the strong convexity of the norm in X. Furthermore, by Lemma 1.1 in [8], $f(t,x)$ is strongly measurable in t for every fixed x and since $F(t,x)$ is continuous in x, it follows from the theorem on maximum of optimal solutions (see [3]) that $f(t,x)$ is continuous in x for every fixed t.

Following [8] we denote by $U(t)$ the solution of Hukuhara's equation on I associated with the inclusion (1). This solution exists by Corollary 2.2 in [7]. By Lemma 3.1 in [8] the set K of all continuous selectors $x(t)$ of the solution $U(t)$ that satisfy

$$x(t_2) - x(t_1) \in \int_{t_1}^{t_2} G(s, U(s))\, ds \quad (t_1 \le t_2,\ t_1, t_2 \in I),$$

where $G(s,U(s)) = \mathrm{Co}\cup\{F(s,y)\colon y \in U(s)\}$, is a convex, compact set in the space $C(I,X)$ (Co stands for the closed convex hull).

Consider the operator g on K,

$$g(u)(t) = x_0 + \int_0^{\tau} f(s, u(s))\, ds,\ t \in I.$$

From the properties of $f(t,x)$ and the definition of K it is clear that g is a continuous mapping from K into K. By the Schauder-Tikhonov fixed point theorem, there exists an element $x(\cdot) \in K$ such that

$$x(t) = x_0 + \int_0^{\tau} f(s, x(s))\, ds.$$

Obviously, $x(t)$ is a solution of the inclusion (1) satisfying $\|v(t) - x(t)\| = \rho(v(t), F(t, x(t)))$ almost everywhere in I ∎

Consider now a differential inclusion

$$\dot{x}(t) \in F(t, x(t), \lambda),\ x(0) = x_0, \tag{2}$$

where F: $I \times X \times \Lambda \to \mathrm{Comp}\, X$, with Λ being a normed space.

Theorem 1: *Assume that $D \subset X$ is a bounded domain and that the mapping F satisfies the following conditions:*

(i) *F is strongly measurable in t for every fixed (x, λ) and is continuous in x for every fixed (t, λ).*

(ii) *F satisfies a Lipschitz condition in x with some constant k.*

(iii) $|F(t,x,\lambda)| \le M$, $M > 0$, *for all* $(t,x,\lambda) \in I \times X \times \Lambda$.

(iv) *We have, uniformly with respect to* $(t,x) \in I \times D$,

$$\lim_{\lambda \to \lambda_0} \alpha\left(\int_0^{\tau} F(t,x,\lambda)\, dt, \int_0^{\tau} F(t,x,\lambda_0)\, dt \right) = 0,$$

where λ_0 is a limit point of the space Λ.

(v) *There exists a neighbourhood* $\overline{V}(\lambda_0)$ *of the point* λ_0 *such that any solution of the inclusion* (2) *for* $\lambda \in \overline{V}(\lambda_0)$ *lies in D together with its* β*-neighbourhood.*

Then for any $\eta > 0$ *there exists a neighbourhood* $V(\lambda_0) \subset \overline{V}(\lambda_0)$ *of* λ_0 *such that for all* $\lambda \in V(\lambda_0)$ *and any solution* $x(t,\lambda)$ *of the inclusion* (2) *there exists a solution* $x_0(t,\lambda_0)$ *of the inclusion*

$$\dot{x}_0(t) \in F(t,x_0(t),\lambda_0), \; x_0(0) = x_0 \tag{3}$$

satisfying

$$\|x(t,\lambda) - x_0(t,\lambda_0)\| \le \eta; \tag{4}$$

and for any solution $x_0(t,\lambda_0)$ *of the inclusion* (3) *there exists a solution* $x(t,\lambda)$ *of the inclusion* (2) *satisfying* (4).

Proof: In view of Consequence 2.2 in [7] it suffices to prove the proposition for the case where $F(t,x,\lambda) \in \operatorname{Conv} X$ for all $(t,x,\lambda) \in I \times X \times \Lambda$. Divide the interval I into m equal parts by the points t_i ($i = 0,1,\dots, m$). Let $x(t,\lambda)$ be some solution of the inclusion (2). Setting $\overline{x}(t) = x(t_i,\lambda)$ for all $t \in [t_i,t_{i+1}]$ we consider the following two inclusions:

$$\dot{x}_1(t) \in F(t,\overline{x}(t),\lambda), \quad x_1(0) = x_0 \tag{5}$$

$$\dot{\overline{x}}_1(t) \in F(t,\overline{x}(t),\lambda_0), \; \overline{x}_1(0) = x_0. \tag{6}$$

According to Lemma 1 there exists a solution $x_1(t)$ of the inclusion (5) such that

$$\|\dot{x}(t,\lambda) - \dot{x}_1(t)\| = \rho(\dot{x}(t,\lambda), F(t,\overline{x}(t),\lambda)).$$

Let $t \in [t_i,t_{i+1}]$. Since

$$x(t,\lambda) - x_1(t_i) = x(t_i,\lambda) - x_1(t_i) + \int_{t_i}^{t} [\dot{x}(s,\lambda) - \dot{x}_1(s)]\, ds,$$

we can write

$$\begin{aligned}
\|x(t,\lambda) - x_1(t)\| &\le \|x(t_i,\lambda) - x_1(t_i)\| + \int_{t_i}^{t} \|[\dot{x}(s,\lambda) - \dot{x}_1(s)]\|\, ds \\
&\le \|x(t_i,\lambda) - x_1(t_i)\| + \int_{t_i}^{t} \rho(\dot{x}(s,\lambda), F(s,\overline{x}(s),\lambda))\, ds \\
&\le \|x(t_i,\lambda) - x_1(t_i)\| + \int_{t_i}^{t} \alpha(F(s,\overline{x}(s),\lambda), F(s,x(s),\lambda))\, ds \\
&\le \|x(t_i,\lambda) - x_1(t_i)\| + kM\int_{t_i}^{t} (s - t_i)\, ds \\
&\le \|x(t_i,\lambda) - x_1(t_i)\| + kMT^2/2m^2.
\end{aligned}$$

Hence a simple computation yields

$$\|x(t,\lambda) - x_1(t)\| \le kMT^2/2m. \tag{7}$$

From condition (iv) it follows readily that for every $\eta_1 > 0$, m there exists a neighbourhood $V(\lambda_0)$ such that whenever $\lambda \in V(\lambda_0)$, then

$$\alpha\left(\int_{t_i}^{t_{i+1}} F(t,x,\lambda)\, dt, \int_{t_i}^{t_{i+1}} F(t,x,\lambda_0)\, dt \right) \le \eta_1 \quad \text{for all } x \in D.$$

Consequently, if

$$x_1(t) = x_1(t_j) + \int_{t_j}^{t} v_1(s)\,ds,\ v_1(s) \in F(s, x(t_j, \lambda), \lambda),$$

then there exists $\overline{v}_1(s) \in F(s, x(t_j, \lambda), \lambda_0)$ such that

$$\left\| \int_{t_j}^{t_{j+1}} (v_1(s) - \overline{v}_1(s))\,ds \right\| \le \eta_1 .$$

It is clear that $\overline{x}_1(t) = \overline{x}_1(t_j) + \int_0^t \overline{v}_1(s)\,ds\ (t \in (t_j, t_{j+1}])$ is a solution of the inclusion (6) and satisfies the inequality

$$\|x_1(t) - \overline{x}_1(t)\| \le m\,\eta_1 + 2MT/m. \tag{8}$$

By Lemma 1 there exists a solution $x_0(t,\lambda_0)$ of the inclusion (3) such that

$$\begin{aligned}\|\dot{\overline{x}}_1(t) - \dot{x}_0(t,\lambda_0)\| &= \rho(\dot{\overline{x}}_1(t), F(t, x_0(t,\lambda_0),\lambda_0)) \\ &\le \alpha(F(t,\overline{x}(t),\lambda_0), F(t, x_0(t,\lambda_0),\lambda_0)) \\ &\le k\,\|\overline{x}(t) - x_0(t,\lambda_0)\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\dot{\overline{x}}_1(t) - \dot{x}_0(t,\lambda_0)\| &\le k\,\|\overline{x}(t) - x_0(t,\lambda_0)\| \\ &\le k\Big[\|\overline{x}(t) - x(t,\lambda)\| + \|x(t,\lambda) - x_1(t)\| + \|x_1(t) - \overline{x}_1(t)\| + \|\overline{x}_1(t) - x_0(t,\lambda_0)\|\Big].\end{aligned}$$

Taking account of the estimates (7), (8) and the boundedness of F we obtain

$$\|\dot{\overline{x}}_1(t) - \dot{x}_0(t,\lambda_0)\| \le k\,\|x_1(t) - x_0(t,\lambda_0)\| + k(m\eta_1 + kMT^2/2m + 3MT/m).$$

But by Gronwall's lemma

$$\|\overline{x}_1(t) - x_0(t,\lambda_0)\| \le k(m\eta_1 + kMT^2/2m + 3MT/m)\exp(kT). \tag{9}$$

Hence, in view of (7) - (9),

$$\begin{aligned}\|x(t,\lambda) - x_0(t,\lambda_0)\| \le\ & k(m\eta_1 + kMT^2/2m + 3MT/m)\exp(kT) \\ & + m\eta_1 + kMT^2/2m + 3MT/m.\end{aligned} \tag{10}$$

From this it follows that one can find m and η such that $\|x(t,\lambda) - x_0(t,\lambda_0)\| \le \eta$, proving the first assertion of the theorem. The second one can be proved in a similar way ∎

By an argument analogous to that used in [10] we can derive from this theorem the following

Theorem 2: *Let $D \subset X$ be a bounded domain and assume that the right-hand sides of the differential inclusions*

$$\dot{x}(t) \in \varepsilon G(t, x(t)),\quad x(0) = x_0 \tag{11}$$

$$\dot{\overline{x}}(t) \in \varepsilon \overline{G}(\overline{x}(t)),\quad \overline{x}(0) = x_0. \tag{12}$$

satisfy the following conditions:

(i) *The mapping* G: $[0,+\infty)\times X\rightarrow \operatorname{Comp} X$ *is of Carathéodory type and satisfies a Lipschitz condition in* x *with constant* k.

(ii) *We have* $|G(t,x)|\le M$, $M>0$, *for every* $(t,x)\in[0,+\infty)\times X$.

(iii) *We have, uniformly with respect to* $x\in D$,

$$\lim_{T\to+\infty}\alpha\left(\frac{1}{T}\int_0^T G(t,x)\,dt,\ \bar{G}(x)\right)=0.$$

(iv) *Any solution of each of the inclusions* (11), (12) *lies in the interior of* D.

Then for any $\eta>0$ *and* $L>0$ *there exists* $\varepsilon_0>0$ *such that, on every interval* $[0,L\varepsilon^{-1}]$ *with* $0<\varepsilon\le\varepsilon_0$, *for any solution* $x(t)$ *of the inclusion* (11) *there exists a solution* $\bar{x}(t)$ *of the inclusion* (12) satisfying

$$\|x(t)-\bar{x}(t)\|\le\eta \tag{13}$$

and for any solution $\bar{x}(t)$ *of the inclusion* (12) *there exists a solution* $x(t)$ *of the inclusion* (11) *satisfying the estimate* (13).

Remarks: 1. Theorem 1 is Bogoliubov's theorem on averaging in finite intervals for differential inclusions in Banach spaces. For differential inclusions in finite-dimensional spaces this proposition has been established in [6,10]. **2.** Using a lemma of Frankowska (Lemma 1 in [4]), the above results can be extended to the case where the right-hand sides of the inclusions (2),(4) and (11),(12) are closed sets (see [9]).

Example: As an example of application of the above results let us consider a control system described by the partial differential equation

$$\begin{gathered}\partial x/\partial t=\varepsilon\big(\sin t\,\cos(\partial^2x/\partial y)+\cos t\,\sin(\partial x/\partial y)+(\sin\pi y)u(t)\big),\quad y\in(0,1)\\ x(t,0)=x(t,1)=0,\quad t\in[0,L\varepsilon^{-1}],\ L>0,\ \text{and } x(0,y)=\sin\pi y,\end{gathered} \tag{14}$$

where $u(t)\in U=[0,1]$ is the control. A function $u:[0,L\varepsilon^{-1}]\rightarrow U$ is called an *admissible control* if it is measurable on $[0,L\varepsilon^{-1}]$. A function $x(t,y)$ is called a *solution of system* (14), if for every t the function $x(t,\cdot)$ is of class $L_2(0,1)$, possesses generalized derivatives $\partial x/\partial y$, $\partial^2x/\partial y^2$, all of class $L_2(0,1)$, and if $x(t,y)$ satisfies (14) for almost every t. System (14) can be written in the form (11) with state space $X=\{z\in L_2(0,1)\colon z(0)=z(1)=0\}$ and

$$G(t,x)=f(t,A_1x,A_2x)+A_3U,\ x_0(y)=\sin\pi y,$$

where

$$f(t,A_1x,A_2x)=\sin t\,\cos A_1x+\cos t\,\sin A_2x,$$

with

$$A_1x=\partial^2x/\partial y^2,\ A_2x=\partial x/\partial y\ \text{ and }\ (A_3U)(y)=(\sin\pi y)u.$$

Let $D=\{z\in L_2(0,1)\colon \|z\|\le 3L,\ z(0)=z(1)=0\}$. From (14) we have $x(t)\in D$ and since f is a periodic function of period 2π, we obtain

$$\lim_{T\to+\infty}\int_0^T f(t,A_1x,A_2x)\,dt=\frac{1}{2\pi}\int_0^{2\pi}f(t,A_1x,A_2x)\,dt=0$$

uniformly with respect to $x\in D$, and consequently

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T G(t,x)\,dt=A_3U.$$

Now it is easily verified that the right-hand side of the corresponding inclusion (11) satisfies all the conditions of Theorem 2 for G and D just defined. Furthermore, as an approximation of the solution set to system (14) we may take

$$K_\varepsilon = \left\{x : x(t,y) = \sin\pi y + \varepsilon(\sin\pi y)\int_0^t u(s)\,ds,\ u(s) \subset U \text{ for } (t,y) \subset [0, L\varepsilon^{-1}] \times [0,1]\right\}.$$

Clearly, K_ε is a solution set of the system

$$\frac{\partial x}{\partial t} = \varepsilon(\sin\pi y)u(t),\ y \in (0,1),\ u(t) \in U,\ t \in [0, L\varepsilon^{-1}]$$

$$x(t,0) = x(t,1) = 0,\ x(0,y) = \sin\pi y,$$

which is an averaging system for system (14).

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